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THREE-DIMENSIONAL CONFORMALLY FLAT FINSLER SPACES

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Abstract: The purpose of the present paper is to obtained the scalar curvature of a three-dimensional conformally flat Finsler space.

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1. Introduction

As for as we know [1, 2, 3, 4] h-scalar curvature of Finsler spaces have hardly been determined in concrete form. R. Yoshikama and K. Okubo in his paper [5] give the concrete form of the scalar curvature of two-dimensional conformally flat Finsler space and some examples of Finsler spaces which have a positive definite scalar or a negative definite scalar curvature, he has also proved in the same paper that in two-dimensional conformally flat Finsler spaces are Landsberg spaces they are Berwald spaces.

The purpose of the present paper is to study three-dimensional conformally flat Finsler spaces as for concrete form of scalar curvature is concern we could not work out in three-dimensional as it has been done in two-dimensional even then we have found out two components out of three of scalar curvature in three-dimensional Finsler spaces.

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2. Preliminaries

Let $(x, y) = (x^i, y^i)$ be a local coordinate system of the total space of the tangent bundle TM of a three-dimensional differentiable manifold M. We consider a conformal change $L(x, y) \longrightarrow \bar{L}(x, y) = e^{\sigma(x)} L(x, y)$ of a three-dimensional Finsler space $F^3 = (M^3, L(x, y))$ with the fundamental function L(x, y), where $\sigma(x)$ is a scalar function of position x^i alone, called the conformal factor. We shall denote the Finsler space with the changed fundamental function $\bar{L}(x, y)$ by $\bar{F}^3 = (\bar{M}^3, \bar{L}(x, y))$ and quantities of $\bar{F}^3$ by aperline, the following change of important quantities as well known [6] and use the following notations $F = \frac{L^2}{2}, \bar{F} = \frac{\bar{L}^2}{2}$.

$$\bar{F} = e^{2\sigma(x)} F, \quad \bar{g}_{ij} = e^{2\sigma(x)} g_{ij}, \quad \bar{g} = e^{4\sigma} g, \quad \bar{g}^{ij} = e^{-2\sigma} g^{ij} \tag{1}$$

The relation between the Moors frame $(l_i,\ m_i,\ n_i)$ of the space (M, L) and the Moors frame $(\bar{l}_i,\ \bar{m}_i,\ \bar{n}_i)$ of the space $(M, \bar{L})$ are written in the following form:

$$\bar{l}_i = e^{\sigma} l_i, \quad \bar{m}_i = e^{\sigma} m_i, \quad \bar{n}_i = e^{\sigma} n_i, \tag{2}$$

$$\bar{l}^i = e^{-\sigma} l^i, \quad \bar{m}^i = e^{-\sigma} m^i, \quad \bar{n}^i = e^{-\sigma} n^i, \tag{3}$$

$$\bar{C}_{ijk} = e^{2\sigma} C_{ijk}, \quad \bar{C}^i_{jk} = C^i_{jk}, \tag{4}$$

$$\bar{H} = H, \quad \bar{J} = J, \quad \bar{I} = I.$$

We put,

$$G_i = \frac{[(\frac{\partial^2 F}{\partial x^r \partial y^i}) y^r - \frac{\partial F}{\partial x^i}]}{2} \quad \text{and} \quad G^i = g^{ij} G_j.$$

Since the space (M, L) is locally Minkowski space, $G^i = 0$ in a rectilinear coordinate system from this formula and $\bar{F}(x, y) = e^{2\sigma} F(y)$.

$$\bar{G}^i = e^{2\sigma} F(2\sigma_r l^r l_i - \sigma_i),$$

where, we put $\sigma_i = \frac{\partial \sigma}{\partial x^i}$. Because of

$$\bar{g}^{ir} = e^{-2\sigma} g^{ir} = e^{-2\sigma}(l^i l^r + m^i m^r + n^i n^r).$$

The following relation is obtained

$$\bar{G}^i = F(l^i l^r - m^i m^r - n^i n^r)\sigma_r$$

Putting, $l^r \sigma_r = p$, $m^r \sigma_r = q$, $n^r \sigma_r = r$.

We get the formula

$$\bar{G}^i = F(pl^i - qm^i - rn^i) \tag{5}$$

3. The Scalar Curvature $\bar{R}$ of the Finsler Space $(M, \bar{L})$

It is well known that

$$\begin{cases} L\dot{\partial}_j l^i = m^i m_j + n^i n_j \\ L\dot{\partial}_j m^i = -l^i m_j + n^i v_j - Hm^i m_j - In^i n_j + J(n^i m_j + m^i n_j) \\ L\dot{\partial}_j n^i = -l^i n_j - m^i v_j - Jn^i n_j + Jm^i m_j - I(n^i m_j + m^i n_j) \end{cases} \tag{6}$$

$$\begin{cases} L\dot{\partial}_j l_i = m_i m_j + n_i n_j \\ L\dot{\partial}_j m_i = -l_i m_j + n_i v_j - Hm_i m_j - In_i n_j + J(n_i m_j + m_i n_j) \\ L\dot{\partial}_j n_i = -l_i n_j - m_i v_j - Jn_i n_j + Jm_i m_j - I(n_i m_j + m_i n_j) \end{cases} \tag{7}$$

$$\begin{cases} L\dot{\partial}_k p = pm_k + rn_k \\ L\dot{\partial}_k q = -pm_k + rv_k - Hqm_k - Irn_k + J(rm_k + qn_k) \\ L\dot{\partial}_k r = -pn_k - qv_k - Jrn_k + Jqm_k - I(rm_k + qn_k) \end{cases} \tag{8}$$

We consider the cartan connection $C\Gamma$ of the space (M, L), for a scalar field s(x, y), we define the h-scalar derivative $(s_{,1}, s_{,2}, s_{,3})$ and v-scalar derivative $(s_{;1}, s_{;2}, s_{;3})$ as follows:

$$s_{|i} = s_{,1} l_i + s_{,2} m_i + s_{,3} n_i$$

$$Ls|_i = s_{;1} l_i + s_{;2} m_i + s_{;3} n_i$$

where, $_|$ denotes h-covariant derivative and $|$ denotes v-covariant derivative, so we must notice that from the definition of p, q and r, we get

$$\sigma_i = pl_i + qm_i + rn_i$$

Now from the definition of (v)h-torsion tensor $\bar{R}^i_{jk}$ of $(M, \bar{L})$, we get

$$\bar{R}^i_{jk} = \frac{\partial \bar{G}^i_j}{\partial x^k} - \frac{\partial \bar{G}^i_k}{\partial x^j} + \bar{G}^r_j \bar{G}^i_{rk} - \bar{G}^r_k \bar{G}^i_{rj} \tag{9}$$

Differentiating (5) by y^j and using (6) and (7), we get

$$\bar{G}^i_j = L[p\delta^i_j + q(l^i m_j - n^i v_j + Hm^i m_j + In^i n_j - J(n^i m_j + m^i n_j)) + r(l^i n_j + m^i v_j + Jn^i n_j - Jm^i m_j + I(n^i m_j + m^i n_j))] \tag{10}$$

$$\frac{\partial \bar{G}^i_j}{\partial x^k} = L[\delta^i_j p_{|k} + q_{|k}(l^i m_j - n^i v_j + Hm^i m_j + In^i n_j - J(n^i m_j + m^i n_j)) + r_{|k}(l^i n_j + m^i v_j + Jn^i n_j - Jm^i m_j + I(n^i m_j + m^i n_j))] \tag{11}$$

Interchanging j and k, we have,

$$\frac{\partial \bar{G}^i_k}{\partial x^j} = L[\delta^i_k p_{|j} + q_{|j}(l^i m_k - n^i v_k + Hm^i m_k + In^i n_k - J(n^i m_k + m^i n_k)) + r_{|j}(l^i n_k + m^i v_k + Jn^i n_k - Jm^i m_k + I(n^i m_k + m^i n_k))] \tag{12}$$

Differentiating (10) by y^k and using the relation (6) and (7), we get

$$\begin{aligned}\bar{G}^i_{jk} = {}& [p(\delta^i_j l_k - l^i m_j m_k + m^i l_j m_k - l^i n_j n_k + n^i l_j n_k - Hm^i m_j m_k \\ & +J\pi_{(ijk)}(m^i m_j m_k) - I\pi_{(ijk)}(n^i n_j m_k) - Jn^i n_j n_k)] + q(\delta^i_j m_k \\ & l^i m_j l_k - m^i l_j l_k + m^i m_j n_k - m^i n_j m_k + H(m^i m_j l_k - m^i l_j m_k \\ & -l^i m_j l_k - m^i l_j l_k - m^i m_j v_k) + J(-m^i m_j l_k - n^i n_j l_k + 2l^i m_j n_k \\ & +2l^i n_j m_k + 3n^i n_j v_k - 3n^i v_j n_k) - H^2 m^i m_j m_k + HI(m^i n_j n_k \\ & -n^i m_j n_k) + HJ\pi_{(ijk)}(m^i m_j n_k) - 2I^2\pi_{(ijk)}(m^i n_j n_k) + \\ & IJ\pi_{(ijk)}(m^i m_j n_k) - IJ(n^i n_j n_k) + J^2(n^i n_j m_k - m^i m_j m_k + \\ & 2n^i m_j n_k - 2m^i n_j n_k) + H_{;2} m^i m_j n_k + H_{;3} n^i m_j n_k - J_{;2} m^i n_j m_k \\ & -J_{;3} n^i m_j n_k + I_{;2} n^i n_j m_k + I_{;3} n^i n_j n_k) + r(\delta^i_j n_k + l^i n_j l_k - n^i l_j l_k \\ & +m^i n_j m_k - n^i m_j n_k + J(n^i n_j l_k - m^i m_j l_k + 2l^i m_j m_k - 2l^i n_j n_k \\ & -3m^i n_j v_k - 3n^i m_j v_k) + I(m^i n_j l_k + m^i l_j n_k + 3n^i n_j v_k - 2m^i m_j v_k \\ & +n^i l_j m_k - l^i n_j n_k - l^i m_j m_k) + HI\pi_{(ijk)}(m^i m_j n_k - 2n^i m_j m_k) + \\ & HJm^i m_j m_k + IJ(n^i n_j m_k + m^i m_j n_k) + J^2\pi_{(ijk)}(m^i m_j m_k - \\ & 3n^i m_j m_k - n^i n_j n_k) + J_{;2}(n^i n_j m_k - m^i m_j m_k) + J_{;3}(n^i n_j n_k - \\ & m^i m_j n_k) + I_{;2}(n^i m_j m_k + m^i n_j m_k) + I_{;3}(n^i m_j n_k + m^i n_j n_k))]\end{aligned} \tag{13}$$

Using the equations (11), (12) and (13) in (9), we get value of $\bar{R}^i_{jk}$ in the component forms. Also,

$$\bar{R}^i_{jk} = \bar{R}_{\alpha\beta\gamma}\bar{e}^i_{\alpha)}\bar{e}_{\beta)j}\bar{e}_{\gamma)k} \tag{14}$$

It is well known in [2], that,

$$\bar{R}_{\beta\gamma\delta} = \gamma_{1\beta\mu}\gamma_{\gamma\delta 2}\bar{R}^{\square}_{\mu 2}$$

$\bar{R}_{212} = \bar{R}^{\square}_{22}$ and $\bar{R}_{313} = \bar{R}^{\square}_{33}$.

Comparing the value $\bar{R}^i_{jk}$ in equation (14) and from the obtain value of $\bar{R}^i_{jk}$, we have

$$\bar{R}^{\square}_{22} = -e^{-2\sigma}(p_{,1} + q_{,2} + Hq_{,1} - Jr_{,1} - p^2 + q^2 + r^2 - 2pHq + pIr + 4HJqr - 2J^2q^2$$

$$+ 3IJqr - H^2q^2 + q^2H_{;2} - 2q_rJ_{;2} + Jpr + J^2qr - HIr^2 - r^2J_{;3}),$$

$$\bar{R}_{33}^{\square} = -e^{-2\sigma}(p_{,1} + r_{,3} + Iq_{,1} + Jr_{,1} - p^2 + q^2 + r^2 - 2pIq - 2pJr - I^2q^2 - HIq^2 + J^2q^2 - q^2J_{;3} - q_rJ_{;3} - J^2r^2 - IJr^2 - I^2r^2 + r^2J_{;3} + q_rI_{;3}).$$

It is not possible to calculate $\bar{R}_{11}^{\square}$ from above because when $\mu = 2 = 1$, then its coefficients becomes zero.

Due to fact we consider a particular condition in which $\bar{R}_{11}^{\square}$. It is well in [2] that the scalar curvature,

$$\bar{R} = 2\bar{R}_{\mu\mu}^{\square} = 2(\bar{R}_{11}^{\square} + \bar{R}_{22}^{\square} + \bar{R}_{33}^{\square}),$$

$$\bar{R} = -2e^{-2\sigma}(2p_{,1} + q_{,2} + r_{,3} + Hq_{,1} + q^2H_{;2} - 2q_rJ_{;2} + Iq_{,1} - q^2J_{;3} - q_rJ_{;3} + r^2J_{;3} + q_rI_{;3} - 2p^2 + 2q^2 + 2r^2 - 2pHq + pIr + 4HJqr - 2J^2q^2 + 3IJqr - H^2q^2 - pJr + J^2qr - HIr^2 - 3pIq - I^2q^2 - HIq^2 + J^2q^2 - J^2r^2 - IJr^2 - I^2r^2) \tag{15}$$

Theorem 1. *The scalar curvature $\bar{R}$ of the three dimensional conformally flat Finsler space $(M, \bar{L})$ where, $\bar{L} = e^{\sigma(x)}L$ is written as in the equation (15) then we have considered $\bar{R}_{11} = 0$, where H, I, J are the main scalars of the space (M, L) and $\sigma_i = pl_i + qm_i + rn_i$.*

Theorem 2. *The scalar curvature $\bar{R}$ of the Finsler space which is conformally to a locally Minkowski space whose main scalars H, I, J are constant and is written in the form,*

$$\bar{R} = -2e^{-2\sigma}(2p_{,1}+q_{,2}+r_{,3}+Hq_{,1}+Iq_{,1}-2p^2+2q^2+2r^2-2pHq+pIr+4HJqr-2J^2q^2+3IJqr-H^2q^2-pJr+J^2qr-HIr^2-3pIq-I^2q^2-HIq^2+J^2q^2-J^2r^2-IJr^2-I^2r^2).$$

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