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A DYNAMICAL SYSTEMS THEORY SOLUTION TO AN ORBITING EARTH-SATELLITE MODEL

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Abstract: Several models of earth satellites that exist in the literature are not verified for real satellite configurations. An analytic procedure is applied to examine the equations representing the pitch librations of a satellite. The veracity of the model and the method of analysis are confirmed for a typical earth-satellite measurements.

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Key Words: Earth-satellite, dynamical system, Floquet theory, iteration

1. Introduction

Attitude orientation can be critical to the missions of the satellites. Librations in these attitude components (angles) which occur naturally in the case of uncontrolled satellites can seriously detract the mission from success. Several of the theoretical models designed to take care of attitude stabilization problems are not supported by real life data. In this paper, the model equations governing the motions of an earth-satellite is analyzed and solved using dynamical systems theory. The coefficients of the matrix of the resulting linearized system

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turn out to be periodic. The primary goal is to develop analytical procedure for determining solutions of the nonlinear equations encountered in the pitch attitude libration of Using Floquet theory, we obtain the analytical solution to the system of highly nonlinear equations through an iterative procedure and test our results for real satellite configurations.

The motion of earth satellites have been modeled under varying special conditions. One of the earlier models formulated by Groves [1] used the rate of rotation of orbital planes combined with higher harmonics to derive the governing equations. These equations were based on Newtonian laws and were not verified for configurations of a typical satellite.

Many of these earlier investigations used numerical methods for solving the equations involved, which are nonlinear and may contain cyclically varying coefficients. The accuracy and stability of the specific numerical methods used therefore greatly impacted on the results obtained. Some research was therefore undertaken to examine these accuracy and stability issues associated with the numerical methods, whilst other investigators proposed hybrid numerical and analytical approaches to solving the representative differential equations. An example of the latter is the proposed Expanded Point Mapping (EPM) procedure of Golat and Flashner [10].

In order to avoid analysis and/or solution of complex nonlinear model equations, several other models and methods of analysis have been proposed. Bhatangar and Mehra [2] employed a synodic coordinate system to determine the equations of an orbiting earth-satellite. They considered the effect of terrestrial, solar, and lunar gravitational forces to the equations of satellite librations. Other model variations can be found in [3] and [4]. In [3] in particular, V. A. Brumberg obtained the motion of an earth-satellite in the general framework of special relativity. This model is derived from considering Newtonian indirect third body perturbations, and is formulated in a geocentric reference system. We consider the basic equation for a satellite in torque-free motion and fit the resulting equations to actual satellite data.

2. Derivation of Governing Equation in Terms of Anomaly

The fundamentals of spacecraft dynamics and orbital mechanics are discussed in detail in, for example [5, 6]. The attitude librations of an earth-satellite is usually defined in terms of the roll, pitch and yaw (Euler angles) [7, 8]. To derive the equations that govern the motion, we proceed as follows: First we make the following assumptions [9]:

1. The earth is spherically symmetrical and is of uniform density and thus it is treated as a point mass.

2. The satellite orbits high enough above the earth, as such, drag force is negligible.

3. There is no maneuvering or significant change in path, hence thrust force is ignored.

4. Other forces such as those due to solar radiation and electromagnetic fields are negligible compared to the earth's gravity.

5. We consider a satellite that is relatively close to the earth, thus the gravitational attraction of the sun and other third bodies are ignored.

The list of symbols and parameters used in our model are given below:

Symbol	**Description**
r	Orbital radius
μ	Gravitational parameters
ϕ	Roll attitude angle
ψ	Yaw attitude angle
θ	Pitch attitude angle
ω_0	Orbital rate
ω_c	Orbital rate corresponding to circular orbit at perigee
e	Eccentricity of the orbit
φ	True anomaly of the satellite
σ	Inertial ratio
F_g	Gravitational force on satellite due to Earth
G	Universal gravitational constant
m_e	Mass of Earth
m	Mass of satellite
r	Distance from satellite to Earth's center
$\hat{\mathbf{r}}$	Unit vector in the direction or the satellite position from the center of Earth

Ignoring relativity effects which represent the departure of the motion from Newtonian Laws, we derive the equation that describes the motion of the earth-satellite as follows (see for example : Starting with Newton's Universal Law of gravitation, we have:

$$F_g = -\frac{Gm_e m}{r^2}\hat{\mathbf{r}} \tag{1}$$

By Newton's second Law of motion, we substitute for F_g in equation (1) to obtain:

$$m\ddot{\mathbf{r}} = -\frac{\mu m}{r^3}\hat{\mathbf{r}} \tag{2}$$

where $\mu = Gm_e$ is a gravitational parameter. The orbital equation of motion is, thus, given by:

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\hat{\mathbf{r}} = 0 \tag{3}$$

Following the method in [11], we make a couple more critical assumptions here. First, we consider the satellite as a rigid non-spinning body that has no internal momentum and no passive or active control subsystem. On the other hand, we assume that the earth is a point mass. These assumptions make it possible to consider the earth and orbiting satellite are considered as a two body system. This enables us to describe the attitude dynamics by a system of equations that govern the dynamics of a rigid body. Next, by ignoring the departure of the motion from Newton's law due to relativity effects, the governing equations can be derived using Newtonian universal law of gravitation as:

$$\begin{aligned}
I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z &= M_x \\
I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= M_y \\
I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y &= M_z
\end{aligned} \tag{4}$$

where I_x, I_y, I_z are the non-zero components of the moment of inertia tensor of the body, and M_x, M_y, M_z are components of the external torque vector acting on the body. For an orbiting satellite, the external torque owing to gravity is usually expressed as:

$$\begin{aligned}
M_x &= -\frac{3\mu}{r_c^3}(I_y - I_z)cos^2\theta cos\phi sin\phi \\
M_y &= -\frac{3\mu}{r_c^3}(I_x - I_z)sin\theta cos\theta cos\phi \\
M_z &= -\frac{3\mu}{r_c^3}(I_y - I_x)sin\theta cos\theta sin\phi
\end{aligned} \tag{5}$$

The attitude equations are then obtained by substituting equation (5) in equation (4).

Next, by substituting for the angular velocities and their derivatives, we obtain:

$$I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z = -\frac{3\mu}{r_c^3}(I_y - I_z)cos^2\theta cos\phi sin\phi$$

$$
\begin{aligned}
I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= -\frac{3\mu}{r_c^3}(I_x - I_z)sin\theta cos\theta cos\phi \\
I_z + (I_y - I_x)\omega_x\omega_y &= -\frac{3\mu}{r_c^3}(I_y - I_x)sin\theta cos\theta sin\phi
\end{aligned}
\tag{6}
$$

Pitch is one of the components of spacecraft attitude which is the angular orientation of the spacecraft with respect to some axes in space. The small perturbations in the attitude components are called librations. We can now use the attitude characterization in terms of the Euler angles (Euler 3-2-1 scheme) of the satellite to obtain the equations of the dynamics as:

$$
\begin{aligned}
I_x\ddot{\phi} - I_x\dot{\omega}_0 sin\psi - \omega_0(I_z - I_y + I_x) - (I_z - I_y)(\omega_0^2 + \frac{3\mu}{r_c^3})sin\phi &= 0 \\
I_y\ddot{\theta} + \frac{3\mu}{r_c^3}(I_x - I_z)sin\theta &= I_y\dot{\omega}_0 \\
I_z\dot{\psi} + I_z\dot{\omega}_0 + \omega_0(I_z - I_y + I_x)\dot{\phi} - \omega_0^2(I_x - I_y)sin\psi &= 0
\end{aligned}
\tag{7}
$$

where ϕ is the roll attitude angle, ψ is the yaw attitude angle, ω_c is the orbital rate corresponding to a circular orbit at the perigee and θ is the pitch attitude angle.

From the system of equations (7) we see that the dynamics comprise of a uncoupled pitch mode and a coupled roll-yaw mode. Thus, it is possible to study the two modes independently. Here, we will focus on the pitch attitude. We therefore consider the system made up of the orbital equation (3) and the pitch attitude equation from the system (7) as the model equation. We then transform these equations into state-space form by utilizing the following substitutions:

$$x_1 = \theta \tag{8}$$

$$\dot{x_1} = \dot{\theta} = x_2 \tag{9}$$

$$\alpha_1 = \frac{\omega_c^2}{2}\Big(\frac{1 + ecos\varphi}{1 + e}\Big)^3 \tag{10}$$

$$\alpha_2 = esin\varphi \tag{11}$$

where φ is the true anomaly [13].

The final form of our model equations is obtained by eliminating time t from the system of first order equations that results from the substitutions above. The resulting equation is in terms of the true anomaly φ [14, 15]. By the chain rule of differentiation we have:

$$\dot{x}_i = \frac{dx_i}{dt} = \frac{dx_i}{d\varphi}\frac{d\varphi}{dt} = \dot{\varphi}x_i', i = 1, 2 \tag{12}$$

Since

$$\dot{\varphi} = \frac{\omega_c^2(1 + ecos\varphi\varphi)^2}{(1+e)^{3/2}} \tag{13}$$

we have:

$$x_1' = \frac{\dot{x}_1}{\dot{\varphi}} = \frac{(1+e)^{3/2}}{\omega_c(1+ecos\varphi)^2}x_2$$

$$\begin{aligned}
x_2'(\varphi) &= \frac{(1+e)^{3/2}}{\omega_c(1+ecos\varphi)^2}\Big[-3\alpha_1\sigma sin2x_1 - 4\alpha_1\alpha_2\Big] \\
&= \frac{(1+e)^{3/2}}{\omega_c(1+ecos\varphi)^2}3\alpha_1\omega\Big[-sin2x_1 - \frac{4\alpha_2}{3\sigma}\Big] \\
&= \frac{3\sigma\omega_c(1+ecos\varphi)}{2(1+ecos\varphi)^{3/2}}\Big[-sin2x_1 - \frac{4\alpha_2}{3\sigma}\Big]
\end{aligned}$$

By making the substitution $k_1 = \frac{1}{k_2} = \dfrac{(1+e)^{3/2}}{\omega_c}$, the system of nonlinear equations reduces to:

$$x_1'(\varphi) = \frac{k_1}{(1+ecos\varphi)^2}x_2 \tag{14}$$

$$x_2'(\varphi) = \frac{3\sigma k_2}{2}(1+cos\varphi)(-2sin2x_1 - \frac{4e}{3\sigma}sin\varphi) \tag{15}$$

The two equations (14, 15) making up the system can be expressed as a single vector equation as:

$$x'(\varphi) = F(x,\varphi) \tag{16}$$

where

$$x(\varphi) = \begin{bmatrix} x_1(\varphi) \\ x_2(\varphi) \end{bmatrix}$$

and

$$F(x,\varphi) = \begin{bmatrix} \frac{k_1}{(1+ecos\varphi)^2}x_2 \\ \frac{3\sigma k_2}{2}(1+ecos\varphi)(-2sin2x_1 - \frac{4e}{3\sigma}sin\varphi) \end{bmatrix}.$$

3. Satellite Configuration

In this section, we state the parameters that will be applied in our model. We consider an earth-orbiting axi-symmetric satellite with its axis of symmetry pointed towards the earth. For this configuration, the components of the moments of inertia about the $x-$ and $y-$ axes are equal and of value $200kgm^2$. That is

$$I_x = I_y = 200kgm^2$$

The third component of the moment of inertia is taken as

$$I_z = 10kgm^2$$

The inertia ratio

$$\sigma = \frac{I_x - I_z}{I_y}$$

evaluates to 0.95. For an earth orbiting satellite, the gravitational constant μ is

$$\mu = 3.986 \times 10^5 km^3 s^{-2}$$

The orbit is one with low eccentricity, thus, $e = 0.05$. The altitude at the perigee is taken as 200 km, given a radius of perigee $r_p = 6580km$, while the orbital angular velocity at perigee is $\omega_c = \sqrt{\mu/r_p^3} = 1.183 \times 10^{-3} grad/sec.$

4. Analysis of Model Equations

Substitution of the satellite and orbital parameters into equations (15) and (16) respectively yields the non linear equations:

$$x(\varphi) = F(x, \varphi) \tag{17}$$

where

$$F(x,\varphi) = \left[\begin{array}{c} \frac{9.1\times 10^4}{(1+0.05cos\varphi)^2} x_2(\varphi) \\ 1.296 \times 10^3 (1 + 0.05cos\varphi)(-sin2x_1(\varphi) - 7.0 \times 10^{-1} sin\varphi \end{array} \right]$$

By linearization, the coefficient matrix $A(\varphi)$ now becomes

$$A(\varphi) = \left[\begin{array}{cc} 0 & a_{12} \\ a_{21} & 0 \end{array} \right]$$

where

$$\begin{aligned} a_{12} &= \frac{7.69 \times 10^5}{(1 + 0.05cos\varphi)^2} && (18) \\ a_{21} &= -3.134 \times 10^{-3}(1 + 0.05cos\varphi)cos[sin^{-1}(7.02 \times 10^{-2}sin\varphi)] && (19) \end{aligned}$$

The governing equations reflect the equilibrium state and linearization principles of dynamical systems theory is applicable. Starting with the linearized equation, the coefficient matrix is decomposed into a truncated Fourier series as

$$A(\varphi) \cong a_0 + a_1cos\varphi + b_1sin\varphi + a_2cos2\varphi + b_2sin2\varphi \quad (20)$$

The coefficients a_0, a_1 and a_2 obtained by this analysis are listed below.

$$a_0 = \begin{bmatrix} 0 & 913.029 \\ -3.129 \times 10^{-3} & 0 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 0 & -91.303 \\ -1.566 \times 10^{-4} & 0 \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 0 & 3.367 \\ -3.857 \times 10^{-3} & 0 \end{bmatrix}$$

The coefficients b_1 and b_2 are all null matrices because $A(\varphi)$ is even in φ.

The Fourier coefficients of the Floquet periodic matrix and the Floquet constant matrix are governed by a system of nonlinear indicial and recurrence equations [11]. Their application to a standard real satellite configuration is discussed below.

For this application, we exploit the fact that truncating the Fourier series decomposition of the Floquet matrix at the first harmonic can yield acceptable results [16]. Hence, we let

$$Q(\varphi) = p_0 + p_1cos\varphi + q_1sin\varphi \quad (21)$$

The corresponding equations from the indicial and recurrence relations yield the following system of equations:

$$\begin{aligned} p_0R - a_0p_0 - \frac{1}{2}a_1p_1 - \frac{1}{2}b_1q_1 &= 0 \\ p_1R - a_1p_0 - (a_0 - \frac{1}{2}a_2)p_1 + Iq_1 &= 0 \\ q_1R - b_1p_0 - Ip_1 + (a_0 + \frac{1}{2}a_2)q_1 &= 0 \end{aligned} \quad (22)$$

A fourth equation is obtained from the requirement of the Floquet periodic matrix that

$$Q(0) = I \tag{23}$$

This translates to the equation

$$p_0 + p_1 = I \tag{24}$$

5. Iteration Scheme

The last four equations in the system (22) and equation (23) need to be solved for the four matrices p_0, p_1, q_1 and Q. In this context, we note that the matrices p_0, p_1 and q_1 occur as both variables and coefficients of R in (23). The equations are therefore nonlinear, making them difficult, if not impossible, to solve in closed form. Our approach is therefore based on the development of an iterative scheme. Two different schemes were tried. In the first, equations (23) and (24) were organized as follows: From (24), we express p_0 in terms of p_0, i.e. $p_0 = I - p_1$. We then substitute this into (23) to obtain:

$$\begin{aligned} R - p_1 R - a_0 + (a_0 - \frac{1}{2}a_1)p_1 - \frac{1}{2}b_1 q_1 &= 0 \\ p_1 R - a_1 + (-a_0 + a_1 + \frac{1}{2}a_2)p_1 + (I + \frac{1}{2}b_2)q_1 &= 0 \\ q_1 R - b_1 + (b_1 - I - \frac{1}{2}b_2)p_1 - (a_0 + \frac{1}{2}a_2)q_1 &= 0 \end{aligned} \tag{25}$$

The last two equations are rearranged as

$$\begin{aligned} d_1 p_1 + d_2 q_1 &= a_1 - p_1 R \\ d_3 p_1 - d_4 q_1 &= b_1 - q_1 R \end{aligned} \tag{26}$$

where

$$\begin{aligned} d_1 &= -a_0 + a_1 + \frac{1}{2}a_2 \\ d_2 &= I + \frac{1}{2}b_2 \\ d_3 &= b_1 - I - \frac{1}{2}b_2 \\ d_4 &= -(a_0 + \frac{1}{2}a_2 \end{aligned}$$

Eliminating p_1R between the first two equations of (26) and rearranging yields a third equation

$$R = (a_0 + a_1) - \frac{1}{2}(a_1 + a_2)p_1 - Iq_1 \tag{27}$$

The iteration scheme consists of using previous values of p_1, q_1 and R on the right hand sides of (26) and (27) to obtain current values of same. Starting values thus involved all three matrices. This turned out to be divergent and was thus abandoned. It nevertheless gave valuable clues as to the structure of the matrices involved. In particular, it showed that the matrix had one dominant element in the second column of the first row with the other three elements being zero or essentially zero.

The second iteration scheme exploited this structure to reorganize the equations involved Thus, we assumed that

$$R = ru_{12} \tag{28}$$

where

$$u_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and r is a real constant. When post multiplied into any 2×2 matrix, this matrix $[u_{12}$ has the effect of moving the first column of the matrix into the second column of a new matrix whose first column is a null vector.

Next, we introduce vector variables representing the columns of p_1 and q_1 as follows :

$$\begin{aligned} p_1 &= [x_p \;\; y_p] \\ q_1 &= [x_q \;\; y_q] \end{aligned} \tag{29}$$

Thus, the terms p_1R and q_1R become

$$\begin{aligned} p_1R &= rx_p \\ q_1R &= rx_q \end{aligned} \tag{30}$$

Substitution of (30) and (31) into (27) yields

$$\begin{aligned} d_1x_p + d_2x_p &= a_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ d_1y_p + d_2y_q &= a_1 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} - rx_p \end{aligned} \tag{31}$$

$$
\begin{aligned}
d_3 x_p + d_4 x_q &= b_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\
d_3 y_p + d_4 y_q &= b_1 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} - r x_q
\end{aligned}
$$

From the first and third of these equations, we see that x_p and x_q can be determined independent of r. The second and fourth equations have solutions which depend on r and the vectors x_p and x_q. This requires iteration on r. This scheme proved to be highly convergent, yielding the following results:

$$
\begin{aligned}
p_1 &= \begin{pmatrix} -0.039 & 0 \\ 0 & 0.036 \end{pmatrix} \\
q_1 &= \begin{pmatrix} 0 & -22.813 \\ -0.0042 & 0 \end{pmatrix} \\
R &= \begin{pmatrix} 0 & 846.114 \\ -0.0032 & 0 \end{pmatrix} \\
p_0 &= \begin{pmatrix} 1.039 & 0 \\ 0 & 0.964 \end{pmatrix}
\end{aligned} \tag{32}
$$

6. Results

We obtain the linearized solution in addition to the the full nonlinear system. From linear systems theory, the solution of the equation

$$x'(\varphi) = A(\varphi)x(\varphi) \tag{33}$$

is given by

$$x(\varphi) = \Phi(\varphi)x(0) \tag{34}$$

where $\Phi(\varphi)$ is known as the fundamental matrix. When $A(\varphi)$ is periodic, then, Floquet theory [16, 17, 18] gives the form of the fundamental matrix as

$$\Phi(\varphi) = Q(\varphi)e^{R\varphi} \tag{35}$$

The matrix exponential $e^{R\varphi}$ is determined from the eigenvalues (characteristic exponents) and eigenvectors (characteristic vectors) of the matrix R as

$$e^{R\varphi} = P\Lambda(\varphi)P^{-1} \tag{36}$$

where P is a matrix formed from the characteristic vectors and $\Lambda(\varphi)$ is a matrix derived from the characteristic exponents. The characteristic exponents λ_1 and λ_2 can be assumed to be complex, without loss of generality. The two are complex conjugates for real matrix R, that is:

$$\begin{aligned}\lambda_1 &= \mu + i\omega \\ \lambda_2 &= \mu - i\omega\end{aligned}$$

The matrix $\Lambda(\varphi)$ is given by

$$\Lambda(\varphi)e^{\mu\varphi} = \begin{pmatrix} cos\omega\varphi & sin\omega\varphi \\ -sin\omega\varphi & cos\omega\varphi \end{pmatrix} \tag{37}$$

The characteristic vectors v_1 and v_2 are also complex conjugates and of the form

$$\begin{aligned}v_1 &= v_R + iv_I \\ v_2 &= \mu - iv_I\end{aligned}$$

The matrix P is then given by

$$P = \begin{bmatrix} v_R & v_I \end{bmatrix} \tag{38}$$

Here, R is given by equation (33). It's eigenvalues and eigenvectors are evaluated as

$$\begin{aligned}\lambda_1 &= 1.657i \\ \lambda_2 &= -1.657i \\ v_1 &= \begin{Bmatrix} 1 \\ 0.002i \end{Bmatrix} \\ v_2 &= \begin{Bmatrix} 1 \\ -0.002i \end{Bmatrix}\end{aligned}$$

Hence

$$\begin{aligned}P &= \begin{pmatrix} 1 & 0 \\ 0 & 0.002 \end{pmatrix} \\ \mathit{\Lambda}(\varphi) &= \begin{pmatrix} cos(1.657\varphi) & sin(1.657\varphi) \\ -sin(1.657\varphi) & cos(1.657\varphi) \end{pmatrix}\end{aligned}$$

The Floquet Q matrix is given by

$$Q(\varphi) = p_0 + p_1 cos\varphi + q_1 sin\varphi$$

where p_0, p_1, and q_1 are given by equation (32).

The fundamental matrix $\Phi(\varphi)$ and hence the linearized solution can now be evaluated from equations (37) and (38). The solution is graphically displayed as a phase-plane diagram in Figure 1 and as an anomaly-history plot in Figure 2, both for the initial conditions.

$$x(0) = \left\{ \begin{array}{c} 0 \\ 1.0 \times 10^{-7} \end{array} \right\}$$

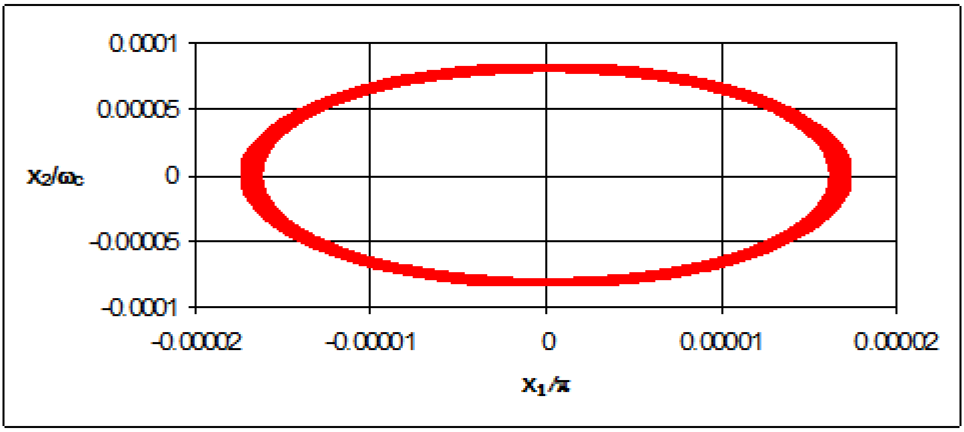

Figure 1: Phase-plane diagram of linear solution

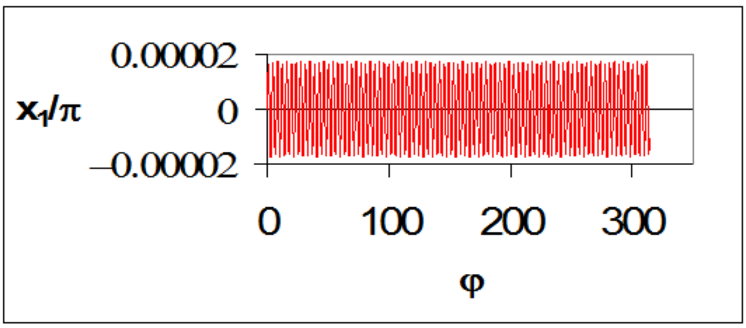

Figure 2: Anomaly-history of linear solution

The nonlinear equation

$$\dot{x} = F(x, \varphi) \tag{39}$$

is expressed in the form

$$\dot{x} = Ax + (F(x, \varphi) - Ax)$$

We thus have the nonlinear non homogeneous equation

$$\dot{x} = Ax + G(x, \varphi) \tag{40}$$

where

$$G(x, \varphi) = F(, \varphi) - Ax = \left\{ \begin{array}{c} 0 \\ z(x, \varphi) \end{array} \right\}. \tag{41}$$

and

$$z(x, \varphi) = 3\sigma k_2(1 + ecos)\varphi\Big(x_1 - \frac{1}{2}sin2x_1 - \frac{2e}{3\sigma}sin\varphi\Big) \tag{42}$$

The x-dependence of $G(x, \varphi)$ can be eliminated under the assumption of low amplitude librations. Equation (41) is then replaced by the linear non-homogeneous equation:

$$\dot{x} = Ax + G(\varphi) \tag{43}$$

with

$$G(\varphi) = \left\{ \begin{array}{c} 0 \\ -4ek_2(1 + ecos\varphi)sin\varphi \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ -4ek_2(sin\varphi + \frac{e}{2}sin2\varphi \end{array} \right\}$$

The general solution of this equation is

$$x(\varphi) = \Phi(\varphi)x_0 + \Phi(\varphi) \int_0^{\varphi} \Phi(s)^{-1}G(s)ds \tag{44}$$

Analytical evaluation of the integral

$$J = \int_0^{\varphi} \Phi(s)^{-1}G(s)ds \tag{45}$$

requires expansion of the integrand into a series of trigonometric functions that are then integrated termwise. This involves extensive algebraic and trigonometric manipulations. An overview of this phase of the analysis is presented below.

We note that the fundamental matrix of the linearized equation is given by

$$\begin{aligned} \Phi(\varphi) &= Q(\varphi)P_x\Lambda(\varphi)P_x^{-1} \\ &= Q(\varphi)M(\varphi) \end{aligned} \tag{46}$$

and that

$$Q(\varphi) = p_0 + p_1 cos\varphi + q_1 sin\varphi$$

Since $Q(\varphi)$ is periodic with period 2π and $M(\varphi)$ is harmonic with frequency of ω, the product is thus harmonic with the multiple frequencies of $\omega, \omega - 1$, and $\omega + 1$. The fundamental matrix therefore has a trigonometric expansion of the form

$$\begin{aligned} \Phi(\varphi) &= e^{\mu\varphi}\Big(f_1 cos\omega\varphi + f_2 sin\omega\varphi + f_3 cos(\omega-1)\varphi + f_4 sin(\omega-1)\varphi \\ &+ f_5 cos(\omega+1)\varphi + f_6 sin(\omega+1)\varphi\Big) \end{aligned} \tag{47}$$

where $f_1, ..., f_6$ are real matrices determined by the characteristic exponents and vectors of $A(\varphi)$, and the Fourier coefficients of $Q(\varphi)$. Expressions for the real matrices $f_1, ..., f_6$ are given by:

$$\begin{aligned} f_1 &= p_0 \\ f_2 &= \begin{bmatrix} -(p_{011}k_1 + p_{012}k_3) & (p_{011}k_2 + (p_{012}k_1) \\ -(p_{021}k_1 + p_{022}k_3) & (p_{021}k_2 + p_{022}k_1) \end{bmatrix} \\ f_3 &= p_1 \\ f_4 &= \begin{bmatrix} -(p_{111}k_1 + p_{112}k_3) & (p_{111}k_2 + p_{112}k_1) \\ -(p_{121}k_1 + p_{122}k_3) & (p_{121}k_2 + p_{122}k_1) \end{bmatrix} \\ f_5 &= f_3 \\ f_6 &= f_4 \end{aligned}$$

The inverse of the fundamental matrix is determined by

$$\Phi^{-1}(\varphi) = \frac{adj\big(\Phi(\varphi)\big)}{det\big(\Phi(\varphi)\big)} \tag{48}$$

One of the intrinsic properties of the fundamental matrix is that

$$det\big(\Phi(\varphi)\big) = det\big(\Phi(0)\big) exp\Big(\int_0^{\phi} tr\big(A(s)\big) ds\Big)$$

Here, $\Phi(0) = I$ and $tr\big(A(\varphi)\big) = 0$. Hence $det\big(\Phi(\varphi)\big) = 1$ and $\Phi^{-1}(\varphi) = adj\big(\Phi(\varphi)\big)$. The inverse matrix thus has a trigonometric series representation of the same form as its parent matrix, that is,

$$\begin{aligned} \Phi^{-1}(\varphi) &= e^{\mu\varphi}\Big(g_1 cos\omega\varphi + g_2 sin\omega\varphi + g_3 cos(\omega-1)\varphi + g_4 sin(\omega-1)\varphi \\ &+ g_5 cos(\omega+1)\varphi + g_6 sin(\omega+1)\varphi\Big) \end{aligned} \tag{49}$$

where $g_i = adj\big(f_i\big), i = 1, ..., 6$.

The vector $G(\varphi)$ is harmonic in φ with the two frequencies. The integrand in equation (45), which is the product of this vector into the inverse of the fundamental matrix is thus harmonic with multiple frequencies of $3-\omega, 2-\omega, 1-\omega, \omega, 1+\omega, 2+\omega$, and $3+\omega$. We therefore write

$$\begin{aligned}
\Phi^{-1}(s)G(s) &= e^{\mu s}\big(J_1 cos(\omega-3)s + J_2 sin(\omega-3)s \\
&+ (J_3 cos(\omega-2)s + J_4 sin(\omega-2)s \\
&+ (J_5 cos(\omega-1)s + J_6 sin(\omega-1)s \\
&+ (J_7 cos\omega s + J_8 sin\omega s \\
&+ (J_9 cos(\omega+1)s + J_{10} sin(\omega+1)s \\
&+ (J_{11} cos(\omega+2)s + J_{12} sin(\omega+2)s \\
&+ (J_{13} cos(\omega+3)s + J_{14} sin(\omega+3)s
\end{aligned} \tag{50}$$

where $j_i, i = 1, ..., 14$ are real-valued vectors.

The integral of equation (45) can now be evaluated by integrating the above series term by term. The non-linear solution of equation (47) is now readily obtained. The solution for the earth-satellite with the given configuration in Section 3 is displayed as a phase-plane plot in Figure 3, and as an anomaly history plot in Figure 4.

7. Conclusion

A method for solving the equations representing the pitch librations of an earth satellite with known configuration was described. Using Fourier series decomposition of the periodic matrix of the nonlinear anomaly equation we have developed a convergent iterative scheme to solve the equation. It is remarkable to note that the results apply to a real earth-satellite configuration. The results compare favorably with available numerical solution [19].

For the linear solution, we invoke Floquets Theory to establish the form of the solution. Unfortunately, the Theory does not provide procedures for the determination of the unknown matrices involved: the periodic matrix and the constant matrix.

This Thesis, thus set out to develop a methodology for dealing with this situation and to demonstrate its feasibility through application to a sample problem. This goal has been achieved.

The methodology is based on Fourier series decomposition of the periodic coefficient matrix, and of the Floquet periodic matrix . It establishes an infinite

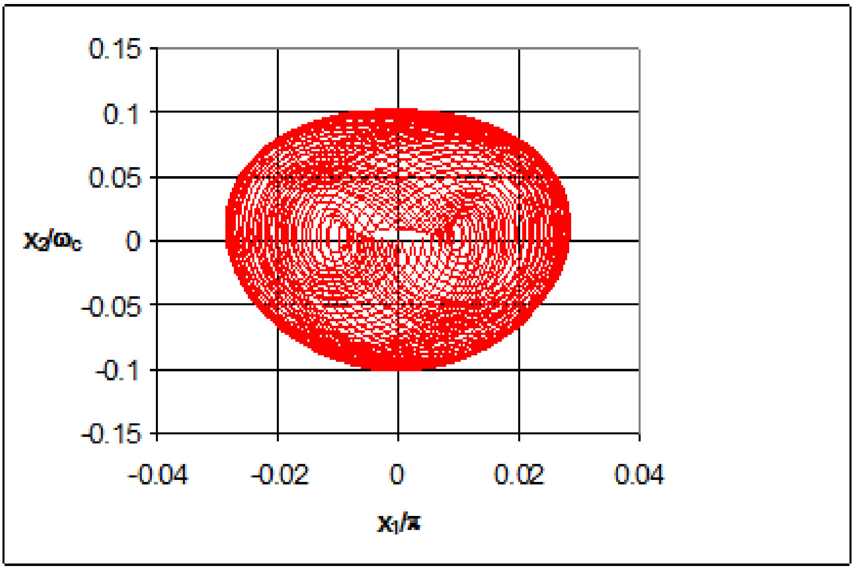

Figure 3: Phase-plane portrait of non-linear response. This compares favorably to available numerical solution in the literature [19]

set of nonlinear equations from which a finite set can be selected consistent with the level of truncation of the matrix decomposition.

The solution of these nonlinear equations represents the major challenge of the methodology. We developed a number of iterative schemes to address this challenge. Only one turned out to be convergent. This required knowledge and use of the structure of the Floquet constant matrix, to convert the matrix equations into vector equations. The structure of the matrix which led to a convergent solution was discovered accidentally. We believe however that a theoretical basis of the structure can be established.

The results for our sample problem were obtained by truncating the Fourier series decomposition of at the first harmonic. We have not tested the validity of the methodology for truncation of the F loquet matrix at higher harmonics. To assess the acceptability of the sample problem results as a solution to the physical problem, we compare our results with published data. Figure 5.1 represents a phase portrait of the solution for same sample problem, presented in reference [32]. These were obtained by numerical integration techniques. Comparison of this with Figure 4.3 shows remarkable agreement both qualitatively and quantitatively. This further substantiates the validity of the methodology

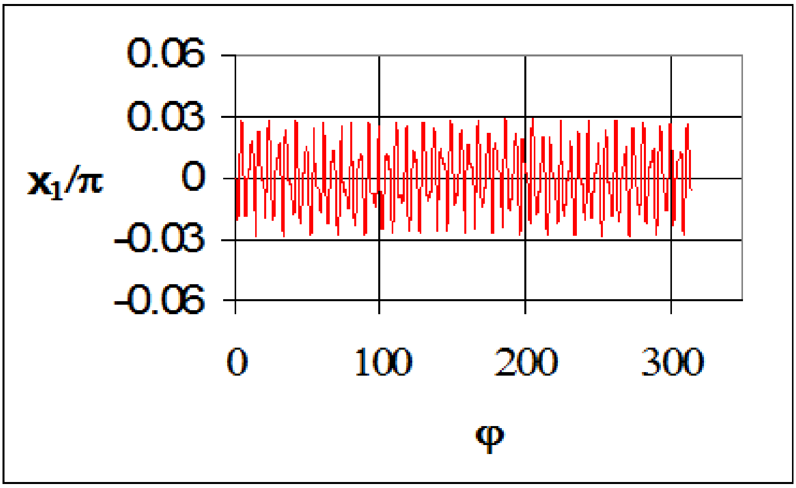

Figure 4: Anomaly-history of non-linear response

presented here.

References

[1] G.V. Groves, Motion of a satellite in the earth's gravitational field, *Proc. R. Soc. Lon. Ser.-A. Mathematical and Physical Sciences*, **254** (1960), 48-65.

[2] K.B. Bhatnagar, M. Mehra, The motion of a geosynchronous satellite – I, *Ind. J. of Pure Appl. Math.*, **17**, No. 12 (1986), 1438-1452.

[3] V.A. Brumberg, On relativistic equations of motion of an Earth satellite, *Celest. Mech. Dyn. Astr.*, **88** (2004), 209-225.

[4] S. Schweighart, R. Sedwick, High fidelity linearized J2 model for satellite formation flight, *J. Guid. Contr. Dyn.*, **25**, No. 6 (2002), 1073-1080.

[5] D.A. Vallado, *Fundamentals of Astrodynamics and Applications*, Microcosm Press, Segundo, CA, and Kluwer Academic Press, Boston, MA (2001), 1-130.

[6] O. Montenbruck, E. Gill, *Satellite Orbits, Models, Methods, Applications*, Springer, New York (2000), 14-113.

[7] M. Inarrea, V. Lanchares, Chaotic pitch motion of an asymmetric non-rigid spacecraft with viscous drag in circular orbit, *Int. J. Nonlinear Mech.*, **41** (2006), 86-100.

[8] K.D. Kumar, Satellite attitude stabilization using fluid rings, *Acta Mech.*, **208** (2009), 117-131.

[9] E. Osei-Frimpong, A.Y. Aidoo, Analysis of nonlinear orbiting earth-satellite pitch attitude libration equations, *Int. Elect. J. of Pure and Appl. Math.*, **1**, No. 1 (2010), 61-74.

[10] J. Hale Francis, *Introduction Space Flight*, Prentice Hall, Englewood Cliffs, NJ (1994).

[11] E. Osei-Frimpong, A.Y. Aidoo, A.K. Amos, Analytical solution to a highly nonlinear earth-satellite pitch attitude libration, *Int. J. Pure and Appl. Math.*, **58**, No. 4 (2010), 425-434.

[12] A.M. Si Mohammed, M. Benyettou, Y. Bentoutou, A. Boudjemai, Y. Hashida, M.N. Sweeting, Three-axis active control system for gravity gradient stabilized microsatellite, *Acta Astronaut.*, **64** (2009), 796-809.

[13] A.K. Misra, Dynamics and control of tethered satellite systems, *Acta Astronaut.*, **63** (2008), 1169-1177.

[14] K.D. Kumar, M.J. Tahk, H.C. Bang, Satellite attitude stabilization radiation pressure using solar radiation and magnetotorquer, *Control Eng. Pract.*, **17**, No. 2 (2009), 267-279.

[15] T.R. Patel, K.D. Kumar, K. Behdinan, Variable structure control for satellite attitude stabilization in elliptic orbits using solar radiation pressure, *Acta Astronaut.*, **64** (2009), 359-773.

[16] D. Viennot, Geometric phases in adiabatic Floquet theory, Abelian gerbes and Cheon's anholonomy, *J. Phys A-Math. Theor.*, **42**, No. 39 (2009), 1-22.

[17] C. Göçek, Stability analysis of periodically switched linear systems using Floquet Theory, *Math. Probl. Eng.*, **1** (2004), 1-10.

[18] C. Lee, D. Han, S. Hong, Modal analysis of periodically time-varying linear rotor systems using Floquet theory, In: *7-th IFToMM-Conf. Rotor Dyn.*, Vienna, Austria (2006), 1-9.

[19] A.K. Amos, On spacecraft tumbling, *J. Chin. Soc. Mech. Eng.*, **23** (2000), 41-47.