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INCIDENCE COLORINGS OF THE POWERS OF CYCLES

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Abstract: The incidence chromatic number of a graph G, denoted by $\chi_i(G)$, is the smallest positive integer of colors such that G has an incidence coloring. We determine for all n except $2k^2 - 3k + 1$ cases for each $k \geq 3$ that if n is divisible by $2k+1$, then $\chi_i(C_n^k) = 2k+1$, otherwise $\chi_i(C_n^k) = 2k+2$. Moreover, we show that if n is divisible by 5, then $\chi_i(C_n^2) = 5$. Otherwise $\chi_i(C_n^2) = 6$.

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Key Words: incidence coloring, powers of cycles

1. Introduction

Let G be a graph and let $V(G)$, $E(G)$, and $\Delta(G)$ denote a vertex set, an edge set and the maximum degree of G, respectively. For vertices u and v, we write uv for an edge joining u and v. Throughout this paper, all graphs are finite, undirected, and simple. For any vertex v in $V(G)$, we let $N_G(v)$ be the set of all neighbors of v in G. The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is equal to $|N_G(v)|$. The *order* of a graph G is the cardinality $|V(G)|$ and the *size* of a graph G is the cardinality $|E(G)|$.

For vertices u and v in $V(G)$, the *distance between u and v*, denoted by $dist_G(u,v)$, is the length of the shortest path jointing them. The *square* of a

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graph G, denoted by G^2, is defined such that $V(G^2) = V(G)$ and two vertices u and v are adjacent in G^2 if and only if $dist_G(u,v) \leq 2$. If 2 is replaced by k, we call the obtained graph the *k-th power* of G.

In 1993, R. A. Brualdi and J. Q. Massey [2] introduced the concept of incidence coloring. Let

$$I(G) = \{(v,e) : v \in V(G), e \in E(G), v \text{ is incidenct with } e\}$$

be the set of incidences of a graph G. We say that two incidences (v,e) and (w,f) are adjacent provided one of the following holds:

(*i*) $v = w$;
(*ii*) $e = f$;
(*iii*) the edge $vw = e$ or $vw = f$.

The configurations associated with (*i*)–(*iii*) are pictured in Figure 1.

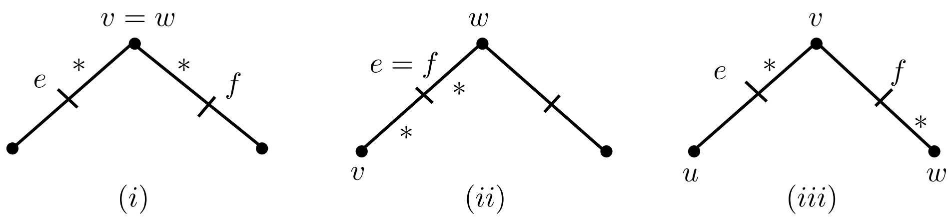

Figure 1: Cases of two incidences being adjacent.

An *incidence coloring* of a graph G is a mapping $\lambda : I(G) \to C$, where C is a color-set, such that adjacent incidences of G are assigned distinct colors, and λ is a *k-incidence coloring* if $|C| = k$. The *incidence chromatic number* of G, denoted by $\chi_i(G)$, is the minimum cardinality of C for which $\lambda : I(G) \to C$ is an incidence coloring. For a vertex v, we use $I(v)$ to denote the set of incidences of the form (v, vw) and use $A(v)$ to denote the set of incidences of the form (w, wv) respectively. Obviously, for each edge xy of G, there are two incidences with respect to xy which are (x, xy) and (y, yx). For an incidence (x, xy), the edge xy is the edge with respect to the incidence (x, xy). From the definition of incidence coloring, the simple lower bound of the incidence chromatic number of any graph G with at least one edge is $\Delta(G) + 1$, that is $\chi_i(G) \geq \Delta(G) + 1$.

R. A. Brualdi and J. Q. Massey [2] proved that $\chi_i(G) \leq 2\Delta(G)$ for every graph G and they posed the incidence coloring conjecture (ICC), which states that for every graph G, $\chi_i(G) \leq \Delta(G) + 2$. In 1997, B. Guiduli [6] showed that the concept of incidence coloring is a special case of directed star arboricity, introduced by I. Algor and N. Alon [1] and pointed out that the ICC false by using

Paley graphs of order p with $p \equiv 1 (\mathrm{mod}\ 4)$. Following the analysis in [1], they showed that $\chi_i(G) \geq \Delta(G)+\Omega(\log \Delta(G))$, where $\Omega = (1/8)-o(1)$. According to a tight upper bound for directed star arboricity, they gave an upper bound for the incidence chromatic number, namely $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$. R. A. Brualdi and J. Q. Massey [2] determined the incidence chromatic numbers of trees, complete graphs and complete bipartite graphs which was corrected some parts of the proof by W. C. Shiu and P. K. Sun [9]. In 1998, D. L. Chen, X. K. Liu and S. D. Wang [3] determined the incidence chromatic numbers of paths, cycles, fans, wheels, adding-edge wheels and complete multipartite graphs. In 2000, X. D. Chen, D. L. Chen, and S. D. Wang [4] determined the incidence chromatic number of $P_n \times P_m$ and generalized complete graphs. In 2002, S. D. Wang, D. L. Chen, and S. C. Pang [10] determined the incidence chromatic number of Halin graphs and outerplanar graphs with $\Delta(G) \leq 4$. In the same year, W. C. Shiu, P. C. B. Lam, and D. L. Chen [8] determined the incidence chromatic number of some cubic graphs. In 2004, M. H. Dolama, E. Sopena, and X. Zhu [5] determined the incidence chromatic number of K_4-minor free graphs and also gave the upper bound for k-degenerated graphs and planar graphs. In 2008, D. Li and M. Liu [7] determined the incidence chromatic number of the k-th power of paths, the squre of trees, which are $\min\{n, 2k+1\}$ and $\Delta(T^2)+1$ respectively, where n is the order of a graph G. And they also gave an upper bound for the incidence chromatic number of the square of a Halin graph.

In this paper, we determine for all n except $2k^2-3k+1$ cases for each $k \geq 3$ that if n is divisible by $2k+1$, then $\chi_i(C_n^k) = 2k+1$, otherwise $\chi_i(C_n^k) = 2k+2$. Moreover, we show that if n is divisible by 5, then $\chi_i(C_n^2) = 5$. Otherwise $\chi_i(C_n^2) = 6$.

2. Power of Cycles

Let C_n be a cycle with n vertices and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ in the usual arrangement. It is known that if n is divisible by 3, then $\chi_i(C_n) = 3$, otherwise $\chi_i(C_n) = 4$ (see [3]). If $2k+1 \geq n$, then $C_n^k = K_n$. R. A. Brualdi and J. Q. Massey [2] showed that $\chi_i(K_n) = n$. Thus $\chi_i(C_n^k) = n$ if $2k+1 \geq n$. So we consider only the case $n > 2k+1$. From now on, we let $n = (2k+1)t + r$ where r and t are integers such that $t \geq 1$ and $0 \leq r \leq 2k$ for C_n^k.

Observation 1. *Let G be a Δ-regular graph. If G has a $(\Delta+1)$-incidence coloring λ, then $\lambda(A(v))$ is a singleton for each vertex v.*

Proof. Note that $\lambda(v, vu)$ and $\lambda(v, vw)$ must be distinct for each $u \neq w$. Thus $\lambda(I(v))$ is a set of cardinality Δ. Observe that $\lambda(A(v))$ must be disjoint from $\lambda(I(v))$. But λ is a $(\Delta+1)$-incidence coloring, hence $\lambda(A(v))$ is a singleton. □

Observation 2. *Let λ be an incidence labeling of G^k such that $\lambda(A(v))$ is a singleton for each vertex v. The incidence labeling λ is an incidence coloring of G^k if and only if $dist_G(u, v) \geq 2k + 1$ for each distinct vertices u and v in which $\lambda(A(u)) = \lambda(A(v))$.*

Proof. Let λ be an incidence labeling of G^k such that $\lambda(A(v))$ is a singleton for each vertex v.

Necessity. Assume that λ is an incidence coloring of G^k. Consider distinct vertices u and v in which $\lambda(A(u)) = \lambda(A(v))$. Suppose $dist_G(u, v) \leq 2k$. Then uv is an edge or there is a vertex w such that $dist_G(u, w) \leq k$ and $dist_G(v, w) \leq k$ in G. Thus a pair of (u, uv) and (v, vu) or a pair of (u, uw) and (v, vw) in G^k have the same color. This contradiction completes the proof.

Sufficiency. Suppose $\lambda(A(u)) = \lambda(A(v))$ implies $dist_G(u, v) \geq 2k + 1$. Note that if the distinct incidences (u, e) and (v, f) are adjacent in G^k, then $u = v$ or $uv \in E((G^k)^2) = E(G^{2k})$ i.e., $dist_G(u, v) \leq 2k$. So $\lambda(u, e) \neq \lambda(v, f)$ by assumption. Thus λ is an incidence coloring of G^k. □

Lemma 3. *In a graph C_n^k, $\chi_i(C_n^k) = 2k + 1$ if and only if n is divisible by $2k + 1$.*

Proof. Necessity. Let $n = (2k + 1)t + r$ such that $t \geq 1$ and $1 \leq r \leq 2k$. Suppose there is a $(2k+1)$-incidence coloring λ. By Observation 1, $\lambda(A(v_i))$ is a singleton for $1 \leq i \leq n$. By Observation 2, $\lambda(A(v_1))$, $\lambda(A(v_2)), \ldots, \lambda(A(v_{2k+1}))$ are distinct. Again, $\lambda(A(v_2))$, $\ldots$, $\lambda(A(v_{2k+1}))$, $\lambda(A(v_{2k+2}))$ are distinct. Thus $\lambda(A(v_1)) = \lambda(A(v_{2k+2}))$. Similarly, we have $\lambda(A(v_i)) = \lambda(A(v_{(2k+1)j+i})$ for each i and j. Then $\lambda(A(v_n)) = r$. But $dist_G(v_r, v_n) < 2k + 1$. By Observation 2, λ is not a $(2k + 1)$-incidence coloring, a contradiction.

Sufficiency. Let $\lambda(A(v_i)) = j$ where $i \equiv j \bmod (2k+1)$ and $j \in \{1, 2, \ldots, 2k+1\}$. By Observation 2, λ is a $(2k + 1)$-incidence coloring. □

Lemma 4. *If $1 \leq r \leq t$, then $\chi_i(C_n^k) = 2k + 2$.*

Proof. By Lemma 3, $\chi_i(C_n^k) \geq 2k + 2$. Define $\lambda(A(v_1))$, $\lambda(A(v_2))$, $\ldots$, $\lambda(A(v_{(2k+2)r}))$ to be $1, 2, \ldots, 2k + 2, \ldots, 1, 2, \ldots, 2k + 2$ and $\lambda(A(v_{(2k+2)r+1}))$, $\ldots$, $\lambda(A(v_n))$ to be $1, 2, \ldots, 2k + 1, \ldots, 1, 2, \ldots, 2k + 1$. By Observation 2, λ is a $(2k + 2)$-incidence coloring. □

For convenience, we write $A - (x, xy)$ for a set obtained by deleting (x, xy)

from A instead of $A - \{(x, xy)\}$.

Lemma 5. *If* $r = 2k$, *then* $\chi_i(C_n^k) = 2k + 2$.

Proof. By Lemma 3, $\chi_i(C_n^k) \geq 2k + 2$. Define $\lambda(A(v_{2k+1})), \ldots, \lambda(A(v_n))$ to be $1, 2, \ldots, 2k + 1, \ldots, 1, 2, \ldots, 2k + 1$. Define $\lambda(A(v_i) - (v_{k+i}, v_i v_{k+i})) = i$ for $i = 1, \ldots, 2k$. Other remaining incidences are assigned $2k + 2$. By Observation 2, this labeling has no conflict except possibly at incidences involving v_i for $i = 1, \ldots, 2k$. We can check for each i to verify that λ is a $(2k + 2)$-incidence coloring. □

Lemma 6. *If* $n = 5t + 3$, *then* $\chi_i(C_n^2) = 6$.

Proof. By Lemma 3, $\chi_i(C_n^2) \geq 6$. Define $\lambda(A(v_4)), \ldots, \lambda(A(v_n))$ to be $1, 2, \ldots, 5, \ldots, 1, 2, \ldots, 5$. Define $\lambda(v_{n-1}, v_{n-1}v_1) = \lambda(v_n, v_n v_1) = 1$, $\lambda(v_n, v_n v_2) = \lambda(v_1, v_1 v_2) = 2$, $\lambda(v_2, v_2 v_1) = \lambda(v_3, v_3 v_1) = 3$, $\lambda(v_3, v_3 v_2) = \lambda(v_2, v_2 v_4) = 4$, $\lambda(v_4, v_4 v_3) = \lambda(v_5, v_5 v_3) = 5$, and $\lambda(v_1, v_1 v_3) = \lambda(v_2, v_2 v_3) = 6$. By Observation 2, this labeling has no conflict except possibly at incidences involving v_i for $i = 1, 2$, or 3. We can check for each i to verify that λ is a 6-incidence coloring. □

Lemma 7. *In a graph* C_2^7, $\chi_i(C_7^2) = 6$.

Proof. Define $\lambda(A(v_1)) = 1, \lambda(A(v_2)) = \lambda(v_5, v_5 v_6) = 2$, $\lambda(A(v_4)) = \lambda(v_1, v_1 v_7) = 3$, $\lambda(A(v_5)) = 4$, $\lambda(v_1, v_1 v_3) = \lambda(v_4, v_4 v_3) = \lambda(A(v_7) - (v_1, v_1 v_7)) = 5$, and $\lambda(v_2, v_2 v_3) = \lambda(v_5, v_5 v_3) = \lambda(A(v_6) - (v_5, v_5 v_6)) = 6$.

It is straightforward to check that λ is a 6-incidence coloring. □

Theorem 8. *If* n *is divisible by* 5, *then* $\chi_i(C_n^2) = 5$. *Otherwise* $\chi_i(C_n^2) = 6$.

Proof. This result is obtained by Lemmas 3 – 7. □

Remark. By using Lemmas 3, 4, and 5, we can determine for all n except $2k^2 - 3k + 1$ cases for each $k \geq 3$ that if n is divisible by $2k + 1$, then $\chi_i(C_n^k) = 2k + 1$, otherwise $\chi_i(C_n^k) = 2k + 2$. For example, we can determine $\chi_i(C_n^3)$ for all n except $n = 9, 10, 11, 12, 17, 18, 19, 25, 26,$ and 33.

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