International Journal of Pure and Applied Mathematics
Volume 76 No. 1 2012, 15-19
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

WEIGHTED COMPOSITION OPERATORS ON FUNCTION SPACES

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Abstract: In this paper we investigate some properties of weighted composition operators acting on analytic function spaces.

AMS Subject Classification: 47B33, 47B38
Key Words: weighted composition operator, spectral set, Caratheodory region

1. Introduction

Let U be the open unit disk in the complex plain. The Hilbert space $\mathcal{H}$ under consideration satisfy the following axioms:

Axiom 1. $\mathcal{H}$ is a subspace of the space of all analytic functions on U.

Axiom 2. $z\mathcal{H} \subset \mathcal{H}$ and $1 \in \mathcal{H}$.

Axiom 3. For each $\lambda \in U$, the linear functional of evaluation at λ, e_λ, is bounded on $\mathcal{H}$.

A complex-valued function φ on U for which $\varphi f \in \mathcal{H}$, for every $f \in \mathcal{H}$, is called a multiplier of $\mathcal{H}$ and the multiplier φ on $\mathcal{H}$ determines a multiplication operator M_φ on $\mathcal{H}$ by $M_\varphi f = \varphi f$, $f \in \mathcal{H}$. The set of all multipliers of $\mathcal{H}$ is denoted by $\mathcal{M}(\mathcal{H})$. Each multiplier is a bounded analytic function on U. In fact $\|\varphi\|_U \leq \|M_\varphi\|$ where

$$\|\varphi\|_U = sup\{|\varphi(z)| : z \in U\}.$$

A good source on this topic is [6].

Received: April 21, 2011

We will use $\sigma(A)$ to denote respectively the spectrum of A. For a region Ω in the complex plane we use $H(\Omega)$ and $H^{\infty}(\Omega)$ respectively for analytic functions on Ω and bounded analytic functions on Ω. Also by $H(\bar{\Omega})$ we mean the set of all analytic functions in a neighborhood of Ω.

Some sources on Hilbert spaces of analytic functions are [1]-[11].

2. Main Result

Recall that an open connected subset Ω of the plane is called a Caratheodory region if its boundary equals the boundary of the unbounded component of $\mathbb{C}\backslash\Omega$. For example the open unit disk U is a Caratheodory region. It is easy to see that Ω is a Caratheodory region if and only if Ω is the interior of the polynomially convex hull of $\bar{\Omega}$. In this case the Farrell-Rubel-Shields theorem holds (see [3], Theorem 5.1, p. 151). If f is a bounded analytic function on a Caratheodory region Ω, then there is a sequence $\{p_n\}_n$ of polynomials which converges to f pointwise boundedly, i.e., $\|p_n\|_{\Omega} \leq c$ for a constant c and $p_n(z) \longrightarrow f(z)$ for all $z \in \Omega$.

Note that a compact set K in $\mathbb{C}$ is called a spectral set for a bounded operator T on a Banach space X if $\sigma(T) \subset K$ and $\|f(T)\| \leq \|f\|_K$ for every rational function f with poles off K.

From now on let $\mathcal{H}$ be a Hilbert space of analytic functions on the open unit disk U satisfying the axioms 1, 2, 3.

A few comments are in order. Note that by axiom 2, polynomials are in $\mathcal{H}$. By axiom 3, point evaluations are continuous, so the Riesz representation theorem imply that for each λ in U there is a unique function $k_\lambda \in \mathcal{H}$ such that $e_\lambda(f) = f(\lambda) =< f, k_\lambda >$, $f \in \mathcal{H}$. The function k_λ is called the reproducing kernel for the point λ. Also by axiom 3, if $\lambda \in U$ then $ran(M_z - \lambda) \subset ker e_\lambda$.

Studying the fixed points of weighted composition operators entails a study of the iterate behavior of holomorphic self maps. The holomorphic self maps of the open unit disc U are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in U. The maps of that are not elliptic are called of non-elliptic type. The iterate of a non-elliptic map can be characterized by the Denjoy-Wolff Iteration Theorem (see [2], [5]). By ψ_n we denote the nth iterate of ψ and by $\psi'(w)$ we denote the angular derivative of ψ at $w \in \partial U$. Note that if $w \in U$, then $\psi'(w)$ has the natural meaning of derivative.

Theorem 2.1. (Denjoy-Wolff Iteration Theorem) *Suppose ψ is a holomorphic self-map of U that is not an elliptic automorphism:*

(i) If ψ has a fixed point $w \in U$, then $\psi_n \to w$ uniformly on compact subsets of U, and $|\psi'(w)| < 1$.

(ii) If ψ has no fixed point in U, then there is a point $w \in \partial U$ such that $\psi_n \to w$ uniformly on compact subsets of U, and the angular derivative of ψ exists at w, with $0 < \psi'(w) \leq 1$.

We call the unique attracting point w, the Denjoy-Wolff point of ψ.

Suppose that ψ is a holomorphic self map of U such that the composition operator C_ψ acts boundedly on $\mathcal{H}$, and φ belongs to $\mathcal{M}(\mathcal{H})$, the set of multipliers of $\mathcal{H}$. Then the weighted composition operator $C_{\varphi,\psi}$ acting on $\mathcal{H}$ is defined by $C_{\varphi,\psi} = M_\varphi C_\psi$.

Theorem 2.2. *Suppose that ψ is a holomorphic self map of U such that $||\psi||_U < 1$ and the composition operator C_ψ acts boundedly on $\mathcal{H}$. Also suppose that φ is a multiplier of $\mathcal{H}$ and w is the Denjoy-Wolff point of ψ. If $\sigma(M_z) = \bar{U}$ is a spectral set for the multiplication operator M_z and $\varphi(w) \neq 0$, then the operator $C_{\Phi,\psi}$ has a nonzero fixed point on $\mathcal{H}$ where $\Phi = \frac{\varphi}{\varphi(w)}$.*

Proof. Note that $w \in U$ since $||\psi||_U < 1$. Assume that $\varphi(w) \neq 0$. Choose δ with $|\psi'(w)| < \delta < 1$. Without loss of generality suppose that $w = 0$. So $|\psi(z)| < \delta|z|$ when z is sufficiency near to zero. If K is a compact subset of U, then by the Denjoy-Wolff Theorem, $\psi_n \to 0$ uniformly on K and $|\psi_{n+k}(z)| < \delta^k|\psi_n(z)|$ for sufficiently large n and every $k \in \mathbb{N}$, and $z \in K$. This implies that $\sum_{i=0}^{\infty} |\psi_i(z)|$ converges uniformly on compact subsets of U. Since φ is bounded, an application of Schwarz's Lemma shows that there exists a constant $M > 0$ such that $|\varphi(0) - \varphi(z)| < M|z|$ for every $z \in U$. But $\varphi(0) \neq 0$, thus

$$|1 - \frac{1}{\varphi(0)}\varphi(z)| < \frac{M}{|\varphi(0)|}|z| \quad (z \in U).$$

By substituting $\psi_i(z)$ instead of z in the above inequality, we get

$$|1 - \frac{1}{\varphi(0)}\varphi(\psi_i(z))| < \frac{M}{|\varphi(0)|}|\psi_i(z)|.$$

This implies that $\sum_{i=0}^{\infty} |1 - \frac{1}{\varphi(0)}\varphi(\psi_i(z))|$ and consequently $\prod_{i=0}^{\infty} \frac{1}{\varphi(0)}\varphi(\psi_i(z))$ converges uniformly on compact subsets of U. Set

$$g(z) = \prod_{i=0}^{\infty} \frac{1}{\varphi(0)}\varphi(\psi_i(z)).$$

Then g is a nonzero holomorphic function on U. Also, note that $\varphi \cdot g \circ \psi = \varphi(0)g$. Thus, generally there is a function $g \in H(U)$ such that

$$(**) \qquad C_{\varphi,\psi} g = \varphi(w) g.$$

Hence

$$\prod_{j=0}^{n-1} \varphi(\psi_j) g \circ \psi_n = \varphi(w)^n g,$$

so that

$$g = \varphi(w)^{-n} \prod_{j=0}^{n-1} \varphi(\psi_j) g \circ \psi_n.$$

Because $||\psi_n||_U < 1$, the function $g \circ \psi_n \in H^\infty(U)$; moreover, each factor of $\prod_{j=0}^{n-1} \varphi(\psi_j)$ belongs to $H^\infty(U)$ since φ belongs to $H^\infty(U)$. Now we show that $H^\infty(U) \subset \mathcal{M}(\mathcal{H})$. If $\varphi \in H^\infty(U)$, then there is a sequence p_n of polynomials such that $p_n \longrightarrow \varphi$ pointwise boundedly. Thus $p_n \longrightarrow \varphi$ pointwise on U and for a constant $c > 0$, $\|p_n\|_U \leq c$ for all n. Since $\sigma(M_z) = \bar{U}$ is a spectral set for the multiplication operator M_z, thus $\|M_{p_{\mathrm{n}}}\| \leq c$ for all natural numbers n. But the closed unit ball of $B(\mathcal{H})$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $A \in B(\mathcal{H})$, $M_{p_{\mathrm{n}}} \longrightarrow A$ in the weak operator topology. Using the fact that $M^*_{p_{\mathrm{n}}} \longrightarrow A^*$ in the weak operator topology and acting these operators on k_λ we obtain that $\overline{p_n(\lambda)} k_\lambda = M^*_{p_{\mathrm{n}}} k_\lambda \longrightarrow A^* k_\lambda$ weakly. Since $p_n(\lambda) \longrightarrow \varphi(\lambda)$ we see that $A^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda$. Because the closed linear span of $\{k_\lambda : \lambda \in U\}$ is dense in $\mathcal{H}$, we conclude that $A = M_\varphi$ and so φ is a multiplier. This implies that $g \in H^\infty(U) \subseteq \mathcal{H}$ and thus $(**)$ shows that the operator $C_{\Phi,\psi}$ has a nonzero fixed point on $\mathcal{H}$ as desired. □

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