International Journal of Pure and Applied Mathematics
Volume 76 No. 1 2012, 21-28
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

STABILITY ANALYSIS OF LINEAR SYSTEMS WITH TIME DELAYS

Kreangkri Ratchagit
Department of Mathematics
Maejo University
Chiangmai, 50290, THAILAND

Abstract: This paper addresses exponential stability problem for a class of linear systems with time delay. By constructing a suitable augmented Lyapunov-Krasovskii functional combined with Leibniz-Newton's formula, new sufficient conditions for the exponential stability of the systems are first established in terms of LMIs.

AMS Subject Classification: 15A09, 52A10, 93D05
Key Words: exponential stability, time delay, Lyapunov function, linear matrix inequalities

1. Introduction

Stability analysis of linear systems with time delays $\dot{x}(t) = Ax(t) + Dx(t-h)$ is fundamental to many practical problems and has received considarable attention [1, 2, 3, 4, 5, 6]. In delay stability criteria, the main concerns is to enlarge the feasible region of stability criteria in given time-delay. By constracting a

Received: May 13, 2011

suitable argumented Lyapunov functionalsand utilizing free weigtht matrices, some less conservative conditions for asymptotic stability are derived in [7, 8, 9, 10] for systems with time delay. This paper gives the improved results for the exponential stability of systems with time delay. By constructing argumented Lyapunov functionals combined with LMI technique, we propose new criteria for the exponential stability of the system. The delay stability conditions are formulated in terms of LMIs, being thus solvable by utilizing Matlab's LMI Control Toolbox available in the literature to date. The approach allows us to apply in exponential stability of linear systems with time delay.

The paper is organized as follows: Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Exponential stability conditions of the system are presented in Section 3.

2. Preliminaries

The following notations will be used in this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes the $n-$dimensional space with the scalar product $\langle .,. \rangle$ and the vector norm $\| \ . \ \|$; $M^{n\times r}$ denotes the space of all matrices of $(n \times r)-$dimensions; A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min/\max}(A) = \min/\max\{\mathrm{Re}\lambda; \lambda \in \lambda(A)\}$; $x_t := \{x(t+s) : s \in [-h, 0]\}, \|x_t\| = \sup_{s\in[-h,0]} \| \ x(t+s) \ \|$; $C([0,t], R^n)$ denotes the set of all R^n-valued continuous functions on $[0,t]$; Matrix A is called semi-positive definite $(A \geq 0)$ if $\langle Ax, x\rangle \geq 0$, for all $x \in R^n; A$ is positive definite $(A > 0)$ if $\langle Ax, x\rangle > 0$ for all $x \neq 0; A > B$ means $A - B > 0$. $*$ denotes the symmetric term in a matrix.

Consider linear neutral system

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}x(t-\tau) + \bar{C}\dot{x}(t-\tau), \quad t > 0 \tag{2.1}$$

where $x(t) \in R^n$ is the state vector of the system, $\bar{A}, \bar{B}, \bar{C} \in R^{n\times n}$ are constant matrices with $|\bar{C}|$ ¡ 1, $\tau \geq 0$ denotes the time delay. We assume the initial condition $x(t+\theta) = \phi(\theta), \forall\theta \in [-\tau, 0]$, where ϕ is a continuously differentiable function on $-[\tau, 0]$.

Let $z(t) = e^{\alpha t}x(t)$ we have

$$\dot{z}(t) = Az(t) + Bz(t-\tau) + C\dot{z}(t-\tau), \tag{2.2}$$

where $A = (\bar{A} + \alpha I), B = (\bar{B} - \alpha\bar{C})e^{\alpha\tau}, C = \bar{C}e^{\alpha\tau}$.

Definition 2.1. Given $\alpha > 0$. The zero solution of system (2.1) is α-exponentially stable if there exist a positive number $N > 0$ such that every solution $x(t,\phi)$ satisfies the following condition:

$$\| x(t,\phi) \| \leq Ne^{-\alpha t} \| \phi \|, \quad \forall t \in R^+.$$

We end this section with the following technical well-known propositions, which will be used in the proof of the main results.

Proposition 2.1. *Assume that* $a(\gamma) \in R^{n_{\mathsf{x}}}$ *and* $b(\gamma) \in R^{n_{\mathsf{y}}}$ *and* $\gamma \in \Omega$. *Then, for any symmetric positive definite matrix* $X \in R^{n_{\mathsf{x}} \times n_{\mathsf{x}}}$ *and any matrix* $M \in R^{n_{\mathsf{y}} \times n_{\mathsf{y}}}$, *the following holds:*

$$-2\int_{\Omega} b^T(\gamma)a(\gamma)d\gamma \leq \int_{\Omega} \begin{bmatrix} a(\gamma) \\ b(\gamma) \end{bmatrix}^T \begin{bmatrix} X & XM \\ M^TX & (XM+I)^TX^{-1}(XM+I) \end{bmatrix} \begin{bmatrix} a(\gamma) \\ b(\gamma) \end{bmatrix} d\gamma$$

Proposition 2.2. (Cauchy Inequality) *For any symmetric positive definite marix* $N \in M^{n\times n}$ *and* $a, b \in R^n$ *we have*

$$\pm a^Tb \leq a^TNa + b^TN^{-1}b.$$

Proposition 2.3. (see [11]) *For any symmetric positive definite matrix* $M \in M^{n\times n}$, *scalar* $\gamma > 0$ *and vector function* $\omega : [0,\gamma] \to R^n$ *such that the integrations concerned are well defined, the following inequality holds*

$$\Big(\int_0^{\gamma} \omega(s)\,ds\Big)^T M \Big(\int_0^{\gamma} \omega(s)\,ds\Big) \leq \gamma\Big(\int_0^{\gamma} \omega^T(s)M\omega(s)\,ds\Big).$$

Proposition 2.4. (see [12]) *Let* E, H *and* F *be any constant matrices of appropriate dimensions and* $F^TF \leq I$. *For any* $\epsilon > 0$, *we have*

$$EFH + H^TF^TE^T \leq \epsilon EE^T + \epsilon^{-1}H^TH.$$

Proposition 2.5. (Schur Complement Lemma, see [13]) *Given constant matrices* X, Y, Z *with appropriate dimensions satisfying* $X = X^T, Y = Y^T > 0$. *Then* $X + Z^TY^{-1}Z < 0$ *if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

3. Main Results

Theorem 3.1. *Consider the neutral system (2.1) with decay rate α, delay time $\tau > 0$. The system (2.1) is exponentially stable with decay rate α if there exist symmetric and positive-definite matrices P, Q_1, Q_2, V, and W such that the following inequality holds:*

$$\Pi = \begin{bmatrix} \Pi_{11} & -WB & PC & A^TB^TV & A^TQ_2 & \tau(W+P) \\ * & -Q_1 & 0 & B^TB^TV & B^TQ_2 & 0 \\ * & * & -Q_2 & C^TB^TV & C^TQ_2 & 0 \\ * & * & * & -V & 0 & 0 \\ * & * & * & * & -Q_2 & 0 \\ * & * & * & * & * & -V \end{bmatrix} < 0 \tag{3.1}$$

where $\Pi_{11}=P(A+B)+(A+B)^TP+WB+B^TW+Q_1$.

Proof. Consider a Lyapunov functional candidate:

$$V(z_t) = V_1(z_t) + V_2(z_t) + V_3(z_t),$$

where

$$V_1(z_t) = z^T(t)Pz(t),$$
$$V_2(z_t) = \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{z}^T(s)B^TXB\dot{z}(s)dsd\beta + \int_{t-\tau}^{t} z^T(s)Q_1z(s)ds, \text{ and}$$
$$V_3(z_t) = \int_{t-\tau}^{t} \dot{z}^T(s)Q_2\dot{z}(s)ds$$

Apply Newton Leibnitz formula, $z(t) - z(t-\tau) = \int_{t-\tau}^{t} \dot{z}(\sigma)d\sigma$, equation (2.2) can be rewritten as:

$$\dot{z}(t) = (A+B)z(t) - B\int_{t-\tau}^{t} \dot{z}(s)ds + C\dot{z}(t-\tau) \tag{3.2}$$

We then find the time derivative of $V(z_t)$ along the solution of equation (3.2). We have

$$\begin{aligned} \dot{V}_1(z_t) &= 2z^T(t)P\dot{z}(t) \\ &= 2z^T(t)P\Big\{(A+B)z(t) - B\int_{t-\tau}^{t} \dot{z}(s)ds + C\dot{z}(t-\tau)\Big\} \end{aligned}$$

$$= 2z^T(t)P(A+B)z(t) - 2z^T(t)PB\int_{t-\tau}^{t}\dot{z}(s)ds \qquad (3.3)$$
$$+ 2z^T(t)PC\dot{z}(t-\tau).$$

Utilize Proposition 2.1. with the choice of $a(\gamma) = B\dot{z}(s), b(\gamma) = Pz(t)$, we have

$$-2\int_{t-\tau}^{t} z^T(t)PB\dot{z}(s)ds$$
$$\leq \int_{t-\tau}^{t}\begin{bmatrix}B\dot{z}(s)\\ z(t)P\end{bmatrix}^T\begin{bmatrix}X & XM\\ M^TX & (XM+I)^TX^{-1}(XM+I)\end{bmatrix}\begin{bmatrix}B\dot{z}(s)\\ z(t)P\end{bmatrix}ds$$
$$= \int_{t-\tau}^{t}\dot{z}^T(s)B^TXB\dot{z}(s)ds + 2z^T(t)PM^TXB\int_{t-\tau}^{t}\dot{z}(s)ds$$
$$+ \tau z^T(t)P(M^TX+I)X^{-1}(XM+I)Pz(t) \quad (3.4)$$

Substitute equation (3.4) into equation (3.3) and choosing $W = PM^TX$ and$V = \bar{\tau}X$, we obtain

$$\begin{aligned}\dot{V}_1(z_t) \leq\ & 2z^T(t)P(A+B)z(t) + \int_{t-\tau}^{t}\dot{z}^T(s)B^TXB\dot{z}(s)ds\\
& + 2z^T(t)PM^TXB\int_{t-\tau}^{t}\dot{z}(s)ds\\
& + \tau z^T(t)P(M^TX+I)X^{-1}(XM+I)Pz(t) + 2z^T(t)PC\dot{z}(t-\tau)\\
=\ & 2z^T(t)P(A+B)z(t) + 2z^T(t)WB\big[z(t) - z(t-\tau)\big]\\
& + \int_{t-\tau}^{t}\dot{z}^T(s)B^TXB\dot{z}(s)ds\\
& + \tau z^T(t)(W+P)X^{-1}(W^T+P)z(t) + 2z^T(t)PC\dot{z}(t-\tau)\\
=\ & 2z^T(t)P(A+B)z(t) + 2z^T(t)WBz(t)\\
& - 2z^T(t)WBz(t-\tau) + \int_{t-\tau}^{t}\dot{z}^T(s)B^TXB\dot{z}(s)ds\\
& + z^T(t)(W+P)\bar{\tau}^2V^{-1}(W+P)^Tz(t) + 2z^T(t)PC\dot{z}(t-\tau), \quad (3.5)\end{aligned}$$

$$\begin{aligned}\dot{V}_2(z_t) =\ & \tau\dot{z}^T(t)B^TXB\dot{z}(t) - \int_{t-\tau}^{t}\dot{z}^T(s)B^TXB\dot{z}(s)ds + z^T(t)Q_1z(t)\\
& - z^T(t-\tau)Q_1z(t-\tau)\end{aligned}$$

$$
\begin{aligned}
&= \big[Az(t)+Bz(t-\tau)+C\dot z(t-\tau)\big]^T\tau B^TXB\big[Az(t)+Bz(t-\tau)\\
&\quad + C\dot z(t-\tau)\big] - \int_{t-\tau}^{t}\dot z^T(s)B^TXB\dot z(s)ds + z^T(t)Q_1z(t)\\
&\quad - z^T(t-\tau)Q_1z(t-\tau) \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
\dot V_3(z_t) =&\ \dot z^T(t)Q_2\dot z(t) - \dot z^T(t-\tau)Q_2\dot z(t-\tau)\\
=&\ \big[Az(t)+Bz(t-\tau)+C\dot z(t-\tau)\big]^TQ_2[Az(t)+Bz(t-\tau)\\
&+ C\dot z(t-\tau)\big] - \dot z^T(t-\tau)Q_2\dot z(t-\tau) \qquad (3.7)
\end{aligned}
$$

Combining equations (3.5)-(3.7), we have

$$
\begin{aligned}
\dot V(z_t) \le&\ 2z^T(t)P(A+B)z(t) + 2z^T(t)WBz(t) - 2z^T(t)WBz(t-\tau)\\
&+ z^T(t)(W+P)\bar\tau^2V^{-1}(W+P)^Tz(t) + 2z^T(t)PC\dot z(t-\tau)\\
&+ z^T(t)A^TB^TVBAz(t) + z^T(t)A^TB^TVBBz(t-\tau)\\
&+ z^T(t)A^TB^TVBC\dot z(t-\tau) + z^T(t-\tau)B^TB^TVBAz(t)\\
&+ z^T(t-\tau)B^TB^TVBBz(t-\tau) + z^T(t-\tau)B^TB^TVBC\dot z(t-\tau)\\
&+ \dot z(t-\tau)C^TB^TVBAz(t) + \dot z^T(t-\tau)C^TB^TVBBz(t-\tau)\\
&+ \dot z^T(t-\tau)C^TB^TVBC\dot z(t-\tau) + z^T(t)Q_1z(t) - z^T(t-\tau)Q_1z(t-\tau)\\
&+ z^T(t)A^TQ_2Az(t) + z^T(t)A^TQ_2Bz(t-\tau) + z^T(t)A^TQ_2C\dot z(t-\tau)\\
&+ z^T(t-\tau)B^TQ_2Az(t) + z^T(t-\tau)B^TQ_2Bz(t-\tau)\\
&+ z^T(t-\tau)B^TQ_2C\dot z(t-\tau) + \dot z^T(t-\tau)C^TQ_2Az(t)\\
&+ \dot z^T(t-\tau)C^TQ_2Bz(t-\tau) + \dot z^T(t-\tau)C^TQ_2C\dot z(t-\tau)\\
&- \dot z^T(t-\tau)Q_2\dot z(t-\tau)\\
=&\ z^T(t)\big[P(A+B) + (A+B)^TP + \bar\tau^2(W+P)V^{-1}(W+P)^T + Q_1\\
&+ WB + B^TW + A^TB^TVBA + A^TQ_2A\big]z(t)\\
&+ 2z^T(t-\tau)\big[-WB + A^TB^TVBB + A^TQ_2B\big]z(t)\\
&+ 2\dot z^T(t-\tau)\big[PC + A^TB^TVBC + A^TQ_2C\big]z(t)\\
&+ z^T(t-\tau)\big[B^TB^TVBB - Q_1 + B^TQ_2B\big]z(t-\tau)\\
&+ 2\dot z^T(t-\tau)\big[B^TB^TVBC + B^TQ_2C\big]z(t-\tau)\\
&+ \dot z^T(t-\tau)\big[C^TB^TVBC + C^TQ_2C - Q_2\big]\dot z(t-\tau)\\
=&\ \xi^T\Pi^*\xi, \qquad (3.8)
\end{aligned}
$$

where $\xi = \left[z(t)z(t-\tau)\dot{z}(t-\tau)\right]^T$ and $\Pi^* = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ * & Z_{22} & Z_{23} \\ * & * & Z_{33} \end{bmatrix}$ with

$$\begin{aligned} Z_{11} &= P(A+B) + (A+B)^T P + \bar{\tau}^2 (W+P) V^{-1} (W+P)^T + Q_1 \\ &\quad + WB + B^T W + A^T B^T V B A + A^T Q_2 A \\ Z_{12} &= -WB + A^T B^T V B B + A^T Q_2 B \\ Z_{13} &= PC + A^T B^T V B C + A^T Q_2 C \\ Z_{22} &= -Q_1 + B^T B^T V B B + B^T Q_2 B \\ Z_{23} &= B^T B^T V B C + B^T Q_2 C \end{aligned}$$

$$Z_{33} = -Q_2 + C^T B^T V B C + C^T Q_2 C.$$

If $\Pi^* < 0$, we have $\dot{V}(z_t) < 0$. Using proposition 2.5., the resulting inequality, $\dot{V}(z_t) < 0$, is equivalent to the inequality (3.1). Therefore the system (2.1) is exponential stable. The proof is completed. □

4. Conclusion

In this paper, we have proposed new conditions for the exponential stability of linear systems with time delay. Based on the improved Lyapunov-Krasovskii functionals and linear matrix inequality technique, the conditions for the exponential stability of the systems have been established in terms of LMIs.

Acknowledgments

This work was supported by the Thai Research Fund Grant, the Higher Education Commission and Faculty of Science, Maejo University, Thailand.

References

[1] K. Ratchagit, Asymptotic stability of linear continuous time-varying systems with state delays in Hilbert spaces, *Journal of Computational Analysis and Applications*, **13**, No. 3 (2011), 554-564.

[2] K. Ratchagit, New delay-dependent conditions for the robust stability of linear polytopic discrete-time systems, *Journal of Computational Analysis and Applications*, **13**, No. 3 (2011), 463-469.

[3] M.C. de Oliveira, J.C. Geromel, Liu Hsu, LMI characterization of structural and robust stability: The discrete-time case, *Linear Algebra and its Applications*, **296** (1999), 27-38.

[4] V.N. Phat, P.T. Nam, Exponential stability and stabilization of uncertain linear time-varying systems using parameter dependent Lyapunov function, *Int. J. of Control*, **80** (2007), 1333-1341.

[5] V.N. Phat, P. Niamsup, Stability analysis for a class of functional differential equations and applications, *Nonlinear Analysis: Theory, Methods and Applications*, **71** (2009), 6265-6275.

[6] Y.J. Sun, Global stabilizability of uncertain systems with time-varying delays via dynamic observer-based output feedback, *Linear Algebra and its Applications*, **353** (2002), 91-105.

[7] O.M. Kwon, J.H. Park, Delay-range-dependent stabilization of uncertain dynamic systems with interval time-varying delays, *Applied Math. Conputation,* **208** (2009), 58-68.

[8] H. Shao, New delay-dependent stability criteria for systems with interval delay, *Automatica*, **45** (2009), 744-749.

[9] J. Sun, G.P. Liu, J. Chen, D. Rees, Improved delay-range-dependent stability criteria for linear systems with time-varying delays, *Automatica*, **46** (2010), 466-470.

[10] W. Zhang, X. Cai, Z. Han, Robust stability criteria for systems with interval time-varying delay and nonlinear perturbations, *J. Comput. Appl. Math.*, **234** (2010), 174-180.

[11] K. Gu, An integral inequality in the stability problem of time delay systems, In: *IEEE Control Systems Society and Proceedings of IEEE Conference on Decision and Control*, IEEE Publisher, New York (2000).

[12] Y. Wang, L. Xie, C.E. de Souza, Robust control of a class of uncertain nonlinear systems, *Syst. Control Lett.,* **19** (1992), 139-149.

[13] S. Boyd, L.El Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia (1994).

[14] J.K. Hale, S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York (1993).