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LMI APPROACH TO STABILITY OF DISCRETE TIME-DELAY SYSTEMS

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Abstract: This paper is concerned with asymptotic stability of switched discrete time-delay systems. The system to be considered is subject to interval time-varying delays, which allows the delay to be a fast time-varying function and the lower bound is not restricted to zero. Based on the discrete Lyapunov functional, a switching rule for the asymptotic stability for the system is designed via linear matrix inequalities. Numerical example is included to illustrate the effectiveness of the result.

AMS Subject Classification: 37C75, 93C30, 93D20
Key Words: switching design, discrete system, asymptotic stability, Lyapunov function, linear matrix inequality

1. Introduction

The stability analysis and synthesis problem of switched systems is one of the fundamental and challenging research topics, and various approaches has been obtained so far. For arbitrary switching law, a common Lyapunov function gives

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stability [1-3]. On the other hand, time-delay phenomena are very common in practical systems. A switched system with time-delay individual subsystems is called a switched time-delay system; in particular, when the subsystems are linear, it is then called a switched time-delay linear system. During the last decades, the stability analysis of switched linear continuous/discrete time-delay systems has attracted a lot of attention [4, 5, 6, 7]. The main approach for stability analysis relies on the use of Lyapunov-Krasovskii functionals and linear matrix inequlity (LMI) approach for constructing a common Lyapunov function [8, 9, 10]. Although many important results have been obtained for switched linear continuous-time systems, there are few results concerning the stability of switched linear discrete systems with time-varying delays. It was shown in [5, 7, 11] that when all subsystems are asymptotically stable, the switching system is asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems has been studied in [10], but the result was limited to constant delays. In [11], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the average dwell time scheme.

This paper studies asymptotic stability problem for switched linear discrete systems with interval time-varying delays. Specifically, our goal is to develop a constructive way to design switching rule to asymptotically stabilize the system. By using improved Lyapunov-Krasovskii functionals combined with LMIs technique, we propose new criteria for the asymptotic stability of the system. Compared to the existing results, our result has its own advantages. First, the time delay is assumed to be a time-varying function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, the delay function is bounded but not restricted to zero. Second, the approach allows us to design the switching rule for stbility in terms of of LMIs, which can be solvable by utilizing Matlab's LMI Control Toolbox available in the literature to date.

The paper is organized as follows: Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Switching rule for the asymptotic stability is presented in Section 3. Numerical example of the result is given in Section 4.

2. Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional space with the scalar product of two vectors $\langle x, y\rangle$ or $x^T y$; $R^{n\times r}$ denotes the space of all matrices of $(n \times r)-$ dimension. A^T denotes the transpose of A; a matrix A is symmetric if $A = A^T$.

Matrix A is semi-positive definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A \geq B$ means $A - B \geq 0$. $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min}(A) = \min\{Re\lambda : \lambda \in \lambda(A)\}$.

Consider a discrete systems with interval time-varying delay of the form

$$\begin{aligned} x(k+1) &= A_\gamma x(k) + B_\gamma x(k - d(k)), \quad k = 0, 1, 2, ... \\ x(k) &= v_k, \quad k = -d_2, -d_2 + 1, ..., 0, \end{aligned} \tag{2.1}$$

where $x(k) \in R^n$ is the state, $\gamma(.) : R^n \to \mathcal{N} := \{1, 2, \ldots, N\}$ is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system. Moreover, $\gamma(x(k)) = i$ implies that the system realization is chosen as the i^{th} system, $i = 1, 2, ..., N$. It is seen that the system (2.1) can be viewed as an autonomous switched system in which the effective subsystem changes when the state $x(k)$ hits predefined boundaries. $A_i, B_i, i = 1, 2, ..., N$ are given constant matrices. The time-varying function $d(k)$ satisfies the following condition:

$$0 < d_1 \leq d(k) \leq d_2, \quad \forall k = 0, 1, 2, ...$$

Remark 2.1. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Definition 2.1. The switched system (2.1) is asymptotically stable if there exists a switching function $\gamma(.)$ such that the zero solution of the system is asymptotically stable.

Definition 2.2. The system of matrices $\{J_i\}, i = 1, 2, \ldots, N$, is said to be strictly complete if for every $x \in R^n \backslash \{0\}$ there is $i \in \{1, 2, \ldots, N\}$ such that $x^T J_i x < 0$.

It is easy to see that the system $\{J_i\}$ is strictly complete if and only if

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

where

$$\alpha_i = \{x \in R^n : \quad x^T J_i x < 0\}, i = 1, 2, ..., N.$$

Proposition 2.1. (see [12]) *The system* $\{J_i\}, i = 1, 2, \dots, N,$ *is strictly complete if there exist* $\delta_i \geq 0, i = 1, 2, \dots, N, \sum_{i=1}^{N} \delta_i > 0$ *such that*

$$\sum_{i=1}^{N} \delta_i J_i < 0.$$

If $N = 2$ *then the above condition is also necessary for the strict completeness.*

3. Main Results

Let us set

$$\mathcal{W}_i = \begin{pmatrix} Q - P & A_i^T S - R^T & R^T B_i \\ S^T A_i - R & P - S - S^T & S^T B_i \\ B_i^T R & B_i^T S & -Q \end{pmatrix}.$$

$$\begin{aligned} J_i(R, Q) = (d_2 - d_1)Q + R^T A_i + A_i^T R, \quad \lambda_1 = \lambda_{\min}(P). \\ \alpha_i = &\{x \in R^n : \quad x^T J_i(R, Q)x < 0\}, \ i = 1, 2, ..., N, \\ \bar{\alpha}_1 = &\alpha_1, \quad \bar{\alpha}_i = \alpha_i \setminus \bigcup_{j=1}^{i-1} \bar{\alpha}_j, \quad i = 2, 3, \dots, N. \end{aligned} \tag{3.1}$$

The main result of this paper is summarized in the following theorem.

Theorem 3.1. *The switched system (2.1) is asymptotically stable if there exist symmetric positive definite matrices* $P > 0, Q > 0$ *and matrices* R, S *satisfying the following conditions:*

(i) $\exists \delta_i \geq 0, i = 1, 2, \dots, N, \sum_{i=1}^{N} \delta_i > 0 : \sum_{i=1}^{N} \delta_i J_i(R, Q) < 0.$

(ii) $\mathcal{W}_i(P, Q, R, S) < 0, \quad i = 1, 2, ..., N.$

The switching rule is chosen as $\gamma(x(k)) = i$, *whenever* $x(k) \in \bar{\alpha}_i$.

Proof. Consider the following Lyapunov-Krasovskii functional for any ith system (2.1)

$$V(k) = V_1(k) + V_2(k) + V_3(k),$$

where

$$V_1(k) = x^T(k)Px(k), \quad V_2(k) = \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i),$$

$$V_3(k) = \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j+1}^{k-1} x^T(l)Qx(l),$$

We can verify that

$$\lambda_1\|x(k)\|^2 \leq V(k). \tag{3.2}$$

Let us set $\xi(k) = [x(k)\, x(k+1)\, x(k-d(k))]^T$, and

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} P & 0 & 0 \\ R & S & 0 \\ P & R & S \end{pmatrix}.$$

Then, the difference of $V_1(k)$ along the solution of the system is given by

$$\begin{aligned} \Delta V_1(k) &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\ &= \xi^T(k)H\xi(k) - 2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.3}$$

because of

$$\xi^T(k)H\xi(k) = x(k+1)Px(k+1).$$

Using the expression of system (2.1)

$$0 = x(k+1) - A_i x(k) - B_i x(k-d(k)),$$

we have

$$-2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ x(k+1) - A_i x(k) - B_i x(k-d(k)) \\ 0 \end{pmatrix} \xi(k)$$

$$= -\xi^T(k)G^T \begin{pmatrix} 0.5I & 0 & 0 \\ -A_i & I & -B_i \\ 0 & 0 & 0 \end{pmatrix} \xi(k) - \xi^T(k) \begin{pmatrix} 0.5I & -A_i^T & 0 \\ 0 & I & 0 \\ 0 & -B_i^T & 0 \end{pmatrix} G\xi(k).$$

Therefore, from (3.2) it follows that

$$\Delta V_1(k) = \xi^T(k)W_i\xi(k), \tag{3.4}$$

where

$$W_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & 0 \end{pmatrix} - G^T \begin{pmatrix} 0.5I & 0 & 0 \\ -A_i & I & -B_i \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0.5I & -A_i^T & 0 \\ 0 & I & 0 \\ 0 & -B_i^T & 0 \end{pmatrix} G.$$

The difference of $V_2(k)$ is given by

$$\begin{aligned}\Delta V_2(k) &= \sum_{i=k+1-d(k+1)}^{k} x^T(i)Qx(i) - \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i)\\ &= \sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Qx(i) + x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k))\\ &\quad + \sum_{i=k+1-d_1}^{k-1} x^T(i)Qx(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Qx(i).\end{aligned} \tag{3.5}$$

Since $d(k) \geq d_1$ we have

$$\sum_{i=k+1-d_1}^{k-1} x^T(i)Qx(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Qx(i) \leq 0,$$

and hence from (3.5) we have

$$\Delta V_2(k) \leq \sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Qx(i) + x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)). \tag{3.6}$$

The difference of $V_3(k)$ is given by

$$\begin{aligned}\Delta V_3(k) &= \sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j}^{k} x^T(l)Qx(l) - \sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j+1}^{k-1} x^T(l)Qx(l)\\ &= \sum_{j=-d_2+2}^{-d_1+1}[\sum_{l=k+j}^{k-1} x^T(l)Qx(l) + x^T(k)Q(\xi)x(k)\\ &\quad - \sum_{l=k+j}^{k-1} x^T(l)Qx(l) - x^T(k+j-1)Qx(k+j-1)]\\ &= \sum_{j=-d_2+2}^{-d_1+1}[x^T(k)Qx(k) - x^T(k+j-1)Qx(k+j-1)]\\ &= (d_2-d_1)x^T(k)Qx(k) - \sum_{j=k+1-d_2}^{k-d_1} x^T(j)Qx(j).\end{aligned} \tag{3.7}$$

Since $d(k) \leq d_2$, and

$$\sum_{i=k=1-d(k+1)}^{k-d_1} x^T(i)Qx(i) - \sum_{i=k+1-d_2}^{k-d_1} x^T(i)Qx(i) \leq 0,$$

we obtain from (3.6) and (3.7) that

$$\Delta V_2(k)+\Delta V_3(k) \leq (d_2-d_1+1)x^T(k)Qx(k)-x^T(k-d(k))Qx(k-d(k)). \quad (3.8)$$

Therefore, combining the inequalities (3.4), (3.8) gives

$$\Delta V(k) \leq x^T(k)J_i(R,Q)x(k) + \xi^T(k)W_i\xi(k), \quad (3.9)$$

where

$$W_i = \begin{pmatrix} Q-P & A_i^TS-R^T & R^TB_i \\ S^TA_i-R & P-S-S^T & S^TB_i \\ B_i^TR & B_i^TS & -Q \end{pmatrix}.$$

Therefore, we finally obtain from (3.9) and the condition (ii) that

$$\Delta V(k) < x^T(k)J_i(R,Q)x(k), \quad \forall i = 1,2,....,N, k = 0,1,2,....$$

We now apply the condition (i) and Proposition 2.1., the system $J_i(R,Q)$ is strictly complete, and the sets α_i and $\bar{\alpha}_i$ by (3.1) are well defined such that

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

$$\bigcup_{i=1}^{N} \bar{\alpha}_i = R^n \backslash \{0\}, \quad \bar{\alpha}_i \cap \bar{\alpha}_j = \emptyset, i \neq j.$$

Therefore, for any $x(k) \in R^n$, $k = 1,2,...$, there exists $i \in \{1,2,\ldots,N\}$ such that $x(k) \in \bar{\alpha}_i$. By choosing switching rule as $\gamma(x(k)) = i$ whenever $x(k) \in \bar{\alpha}_i$, from the condition (3.9) we have

$$\Delta V(k) \leq x^T(k)J_i(R,Q)x(k) < 0, \quad k = 1,2,...,$$

which, combining the condition (3.2) and the Lyapunov stability theorem [12], concludes the proof of the theorem.

Remark 3.1. Note that theresult sproposed in [4,5,6] for switching systems to be asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems studied in [9] was limited to constant delays. In [10], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the averaged well time scheme.

4. Numerical Example

Example 4.1. Consider the switched discrete-time system (2.1), where $d_1 = 1, d_2 = 2$ and

$$(A_1, B_1) = \left(\begin{bmatrix} -0.1 & 0.01 \\ 0.02 & -0.2 \end{bmatrix}, \begin{bmatrix} -0.1 & 0.01 \\ 0.02 & -0.3 \end{bmatrix} \right),$$
$$(A_2, B_2) = \left(\begin{bmatrix} -0.11 & 0.02 \\ 0.01 & -0.2 \end{bmatrix}, \begin{bmatrix} -0.1 & 0.02 \\ 0.01 & -0.2 \end{bmatrix} \right).$$

By LMI toolbox of Matlab, we find that the conditions (i), (ii) of Theorem 3.1 are satisfied with $\delta_1 = 0.2, \delta_2 = 0.2$ and

$$P = \begin{bmatrix} 2.3616 & 0.0366 \\ 0.0366 & 1.7062 \end{bmatrix}, Q = \begin{bmatrix} 0.0957 & -0.0135 \\ -0.0135 & 0.1759 \end{bmatrix},$$

$$R = \begin{bmatrix} 1.7675 & 0.0180 \\ 0.0198 & 1.4436 \end{bmatrix}, S = \begin{bmatrix} 2.6855 & -0.0265 \\ -0.0219 & 2.4635 \end{bmatrix}.$$

In this case, we have

$$(J_1(P,Q), J_2(P,Q)) = \left(\begin{bmatrix} -0.1409 & 0.0112 \\ 0.0107 & -0.1880 \end{bmatrix}, \begin{bmatrix} -0.1767 & 0.0143 \\ 0.0137 & -0.1876 \end{bmatrix} \right).$$

Moreover, the sum

$$\delta_1 J_1(P,Q) + \delta_2 J_2(P,Q) = \begin{bmatrix} -0.0635 & 0.0051 \\ 0.0049 & -0.0751 \end{bmatrix}$$

is negative definite; i.e. the first entry in the first row and the first column $-0.0635 < 0$ is negative and the determinant of the matrix is positive. The sets α_1 and α_2 are given as

$$\alpha_1 = \{(x_1, x_2) : -0.1409x_1^2 + 0.0219x_1x_2 - 0.1880x_2^2 < 0\},$$
$$\alpha_2 = \{(x_1, x_2) : 0.1767x_1^2 - 0.0280x_1x_2 + 0.1876x_2^2 > 0\}.$$

Obviously, the union of these sets is equal to $R^2 \setminus \{0\}$. The switching regions are defined as

$$\overline{\alpha}_1 = \{(x_1, x_2) : -0.1409x_1^2 + 0.0219x_1x_2 - 0.1880x_2^2 < 0\},$$
$$\overline{\alpha}_2 = \alpha_2 \setminus \overline{\alpha}_1.$$

By Theorem 3.1 the system is asymptotically stable and the switching rule is chosen as $\gamma(x(k)) = i$ whenever $x(k) \in \bar{\alpha}_i$.

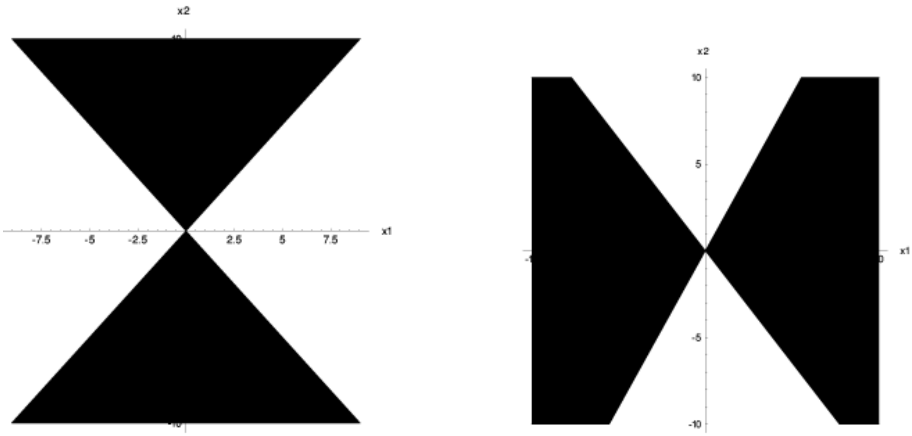

Figure 1: Region α_1 (left), and Region α_2 (right)

5. Conclusion

This paper has proposed a switching design for the asymptotic stability of switched linear discrete-time systems with interval time-varying delays. Based on the discrete Lyapunov functional, a switching rule for the asymptotic stability for the system is designed via linear matrix inequalities.

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