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ESTIMATES OF DIVIDED DIFFERENCES OF REAL-VALUED FUNCTIONS DEFINED WITH A NOISE

Sergei P. Sidorov[1 §], Vladimir Balash[2]

[1]Department of Mechanics and Mathematics
Saratov State University
83, Astrakhanskaya, Saratov 410012, RUSSIAN FEDERATION
[2]School of Information Systems, Computing and Mathematics
Brunel University
London, UNITED KINGDOM

Abstract: Suppose that point evaluations of a real-valued function f at points $x_0, \dots, x_k$ are known with a noise error that continuously uniformly distributed on [-1,1]. In this paper we find the lower bound of the "worst"–case error (as well as the mean error) of absolute value of k-th divided difference of the function f. The case of the Gaussian-type error of point evaluations of functions is examined as well.

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1. Introduction

Let $k \in \mathbb{N}$. Let X_k denote the set of all sequence of real numbers $x = (x_i)_0^k$ with $x_i < x_{i+1}$.

Denote Y_k the set of all sequences of real numbers $y = (y_i)_0^k$, such that $\|y\|_\infty \le 1$, where $\|y\|_\infty := \max_i |y_i|$.

Given $x = (x_i)_0^k \in X_k$, $y = (y_i)_0^k \in Y_k$, let $F(x, y)$ denotes the set of all real-valued functions defined on $\mathbb{R}$, such that $f(x_i) = y_i$, $0 \le i \le k$.

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§Correspondence author

Denote by $[z_0, z_1, \ldots, z_p]f$ the p-th order divided difference of the function f at knots $z_0 < z_1 < \ldots < z_p$. We refer the reader to [2] for basic properties of divided differences.

Given $x = (x_i)_0^k \in X_k$ and $y = (y_i)_0^k \in Y_k$, denote

$$\lambda_k(x, y) := \inf_{f \in F(x,y)} \sup_{\tilde{x}=(\tilde{x}_i)_0^k \in X_k} | [\tilde{x}_0, \ldots, \tilde{x}_k]f |$$

The main goal of this paper is to estimate the values

$$\lambda_k(x) := \sup_{y \in Y_k} \lambda_k(x, y), \tag{1}$$

$$\mu_k(x) := \mathbb{E}_y \left(\lambda_k(x, y)\right), \tag{2}$$

where $\mathbb{E}$ is the expectation of $\lambda_k(x, y)_p$ taken over $y = (y_i)_0^k$, where y_i's are independently uniformly distributed on [-1,1].

Note that if point evaluations of a real-valued function f at points $x_0, \ldots, x_k$ are known with an error that continuously uniformly distributed on [-1,1], then

1. $\lambda_k(x)$ is the lower bound of the "worst"–case error of k-th divided difference of the function f at points $\tilde{x}_0, \ldots, \tilde{x}_k$, whenever f is.
2. $\mu_k(x)$ is the lower bound of the mean error of absolute value of k-th divided difference of the function f at points $\tilde{x}_0, \ldots, \tilde{x}_k$, whenever f is.

The paper is organized as follows. Section 2 evaluates the quantity $\lambda_k(x)$ and estimates this value in the case when the data points $x_0, \ldots, x_k$ are uniformly spaced. Also we examine the problem of optimal choices of the data points $-1 \le x_0 < \ldots < x_k \le 1$ to minimize the "worst"–case error of k-th divided difference of the function f, and we will show that optimal knots are Chebyshev points. In Section 3 one can find the estimates of the value $\mu_k(x)$. Finally, we consider estimates of the mean error of absolute value of k-th divided difference of a function f when the error of point evaluations of the function is normally distributed. It is interesting that optimal points that minimizing this error are not coincide with Chebyshev points.

2. Estimates of $\lambda_k(x)$

Theorem 2.1. *Let $x = (x_i)_0^k \in X_k$. Then $\lambda_k(x) = \sum_{p=0}^{k} A_p$, where*

$$A_p = A_p(x_0, \ldots, x_k) = \left(\prod_{0 \le m \le k;\ m \ne p} |x_m - x_p|\right)^{-1},\ p = 0, \ldots, k.$$

Proof. Denote by $L_p f(\zeta; z_0, z_1, \dots, z_p)$ the evaluation at point ζ of Newton interpolation polynomial of function f at knots $z_0, z_1, \dots, z_p$:

$$L_p f(\zeta; z_0, z_1, \dots, z_p) = \sum_{j=0}^{p} [z_0, \dots, z_j] f \cdot \prod_{i=0}^{j-1} (\zeta - z_i), \quad \zeta - z_{-1} := 1$$

1) First we will find the upper estimation of the quantity $\lambda_k(x)$.

Given $y = (y_i)_0^k \in Y_k$, let $h \in C(\mathbb{R})$ be such that $h(x_i) = y_i$, $0 \le i \le k$. Define a function $g_y \in C(\mathbb{R})$ by

$$g_y(\cdot) = L_k h(\cdot; x_0, \dots, x_k).$$

Then

$$\lambda_k(x) = \sup_{y \in Y} \inf_{f \in F(x,y)} \sup_{\tilde{x} = (\tilde{x}_i)_0^k \in X_k} [\tilde{x}_0, \dots, \tilde{x}_k] f \le \sup_{y \in Y} \sup_{\tilde{x} = (\tilde{x}_i)_0^k \in X_k} [\tilde{x}_0, \dots, \tilde{x}_k] g_y.$$

We have

$$\begin{aligned} [\tilde{x}_0, \dots, \tilde{x}_k] g_y &= [\tilde{x}_0, \dots, \tilde{x}_k] L_k h(\cdot; x_0, \dots, x_k) = \\ &= [\tilde{x}_0, \dots \tilde{x}_k] \left(\sum_{j=0}^{k} [x_0, \dots, x_p] g_y \cdot \prod_{l=0}^{j-1} (\cdot - x_l) \right) = \\ &= [x_0, \dots, x_k] g_y \cdot [\tilde{x}_0, \dots, \tilde{x}_k] \left(\prod_{l=0}^{k-1} (\cdot - x_l) \right) = [x_0, \dots, x_k] g_y. \end{aligned}$$

Denote $e_i(t) = t^i$, $i = 0, 1, \dots$. We have by definition of divided difference

$$[x_0, \dots, x_k] g_y = \frac{\begin{vmatrix} e_0(x_0) & e_0(x_1) & \dots & e_0(x_k) \\ \dots & \dots & \dots & \dots \\ e_{k-1}(x_0) & e_{k-1}(x_1) & \dots & e_{k-1}(x_k) \\ y_0 & y_1 & \dots & y_k \end{vmatrix}}{\det(e_i(x_p))_{i=0,\dots,k}^{p=0,\dots,k}}.$$

Laplace expansion along the last row of the determinant yields

$$[x_0, \dots, x_{k-1}] g_y \le \sup_{y \in Y} \frac{\left| \sum_{p=0}^{k} y_p (-1)^p \det(e_i(x_l))_{i=0,\dots,k-1}^{l=0,\dots,k,\ l \ne p} \right|}{\det(e_i(x_l))_{i=0,\dots,k}^{l=0,\dots,k}}. \tag{3}$$

Note that all determinants in the sum are Vandermonde determinants and therefore are positive. Thus, the supremum is achieved at $y_p = (-1)^p$, $p = 0,\dots,k$. Then for all $\tilde{x} \in X_k$

$$\sup_{y\in Y_k} [\tilde{x}_0,\dots,\tilde{x}_k]g_y \le$$
$$\le \frac{\sum_{p=0}^{k}\prod_{0\le l<m\le k;\ l,m\ne p}(x_m-x_l)}{\prod_{0\le l<m\le k}(x_m-x_l)} = \sum_{p=0}^{k}\frac{1}{\prod_{0\le m\le k;\ m\ne p}|x_m-x_p|}. \quad (4)$$

2) Let $h \in C(\mathbb{R})$ be such that $h(x_i) = (-1)^{k+i}$, $i = 0,\dots,k$. To find the lower bound of $\lambda_k(x)$ let us define $g \in C(\mathbb{R})$ by $g(\zeta) = L_k h(\zeta; x_0,\dots,x_k)$. Let $y^* = (y_i^*)_0^k \in Y_k$ be with $y_i^* = L_k h(x_i; x_0,\dots,x_k)$, $i = 0,\dots,k$. By definition

$$\lambda_k(x) := \sup_{y\in Y_k}\ \inf_{f\in F(x,y)}\ \sup_{\tilde{x}=(\tilde{x}_i)_0^k\in X_k} |\ [\tilde{x}_0,\dots,\tilde{x}_k]f\ |.$$

Therefore we can get a lower bound of $\lambda_k(x)$ by choosing $y = y^*$ and $\tilde{x}_i = x_i$, $0 \le i \le k$:

$$\lambda_k(x) \ge \inf_{f\in F(x,y^*)} [x_0,\dots,x_k]f = [x_0,\dots,x_k]g =$$
$$= \frac{\sum_{p=0}^{k}\det(e_i(x_l))_{i=0,\dots,k-1}^{l=0,\dots,k,\ l\ne p}}{\det(e_i(x_l))_{i=0,\dots,k}^{l=0,\dots,k}} = \sum_{p=0}^{k}\frac{1}{\prod_{0\le m\le k;\ m\ne p}|x_m-x_p|}. \quad \square$$

In particular, as it follows from Theorem 2.1 that if $k = 1$ then $\lambda_1(x) = 2/(x_1 - x_0)$; if $k = 2$, then $\lambda_2(x) = 2/(x_1 - x_0)(x_2 - x_1)$.

The proof of Theorem 2.1 shows that the "worst" y is $y = (y_i)_0^k$ with $y_i = (-1)^i$, and the optimal function f is the algebraic polynomial of degree k satisfying the interpolation conditions $f(x_i) = (-1)^i$, $0 \le i \le k$.

It can be proved analogously that given $x = (x_i)_0^k \in X_k$ and $0 \le s \le k-1$

$$\sup_{y\in Y_k}\ \inf_{f\in F(x,y)}\ \sup_{\zeta\in(x_s,x_{s+1})} |\ [x_0,\dots,x_s,\zeta,x_{s+1},\dots,x_{k-1}]f\ | = \sum_{p=0}^{k} A_p,$$

where points $x_0,\dots,\zeta,\dots,x_{k-1}$ are arranged in ascending order.

Consider the special case in which the data points $x_0,\dots,x_k$ are uniformly spaced.

Corollary 2.2. *Let $\Delta > 0$. Denote $x_\Delta = (x_i)_0^k$ with $x_{i+1} - x_i = \Delta$ for all $0 \le i \le k-1$. Then*

$$\lambda_k(x_\Delta) = \frac{2^k}{k!}\Delta^{-k}.$$

Proof. If $x_{i+1} - x_i = \Delta$ for all $0 \le i \le k-1$ then

$$\sum_{p=0}^{k} \frac{1}{\prod_{0\le m\le k;\ m\ne p} |x_m - x_p|} =$$

$$= \frac{1}{\Delta^k} \sum_{p=0}^{k} \frac{1}{\prod_{0\le m\le k;\ m\ne p} |m-p|} = \frac{1}{k!\Delta^k} \sum_{p=0}^{k} \binom{k}{p} = \frac{2^k}{k!}\Delta^{-k}. \quad \square$$

The next proposition follows from Corollary 2.2.

Corollary 2.3. *Let $\Delta > 0$. Denote $z_\Delta = (z_i)_{i\in\mathbb{Z}}$ with $z_{i+1} - z_i = \Delta$ for all $i \in \mathbb{Z}$. Denote $A := \{a = (a_i)_{i\in\mathbb{Z}} : \ \|a\|_\infty \le 1\}$. Given $a = (a_i)_{i\in\mathbb{Z}} \in A$, denote $G(z_\Delta, a)$ the set of all real-valued function defined on $\mathbb{R}$ such that $f(z_i) = a_i$, $i \in \mathbb{Z}$. If $0 \le s \le k-1$ and $p \in \mathbb{Z}$, then*

$$\sup_{a\in A} \inf_{f\in G(z_\Delta,a)} \sup_{\zeta\in(z_{p+s},z_{p+s+1})} [z_p, \dots, \zeta, \dots, z_{p+k-1}]f = \frac{2^k}{k!}\Delta^{-k}. \tag{5}$$

Note that the problem (5) is similar to the task of finding the infimum of a norm of a fixed order derivative in a class of smooth interpolants, which satisfies given interpolation conditions. The last task is the well-known Favard's problem [4] (see also [10], [1], [3] for comments). It is worth noting the papers [8], [9], that also study the problems of extremal interpolation. In the Corollary 2.3 we relax the condition on smoothness of interpolant and consider the problem of finding the infimum of the value of a fixed order divided difference in a class of real-valued interpolants, which satisfy given interpolation conditions.

One can consider the problem of the optimal choice of knots taken from [-1,1]:

$$\lambda_k = \inf_{x\subset[-1,1],\ x\in X_k} \lambda_k(x), \tag{6}$$

where infimum is taken over all k-partitions x of interval [-1,1], $x = (-1 \le x_0 < \dots < x_k \le 1)$.

Corollary 2.4. *Let λ_k be defined as above. Then $\lambda_k = 2^{k-1}$.*

Proof. It follows from the proof of Theorem 2.1 that

$$\lambda_k := \inf_{x\subset[-1,1]} \sup_{y\in Y} \inf_{f\in F(x,y)} \sup_{\tilde{x}=(\tilde{x}_i)_0^k\in X_k} [\tilde{x}_0, \dots, \tilde{x}_k]f = \inf_{x\subset[-1,1]} [x_0, \dots, x_k]g_{y^*}, \tag{7}$$

where g_{y^*} is the algebraic polynomial of degree k satisfying the interpolation conditions $g_{y^*}(x_i) = (-1)^{k+i}$, $0 \le i \le k$. It follows from continuity of g_{y^*} that there are points $a_1, \dots, a_k$ such that $x_{i-1} < a_i < x_i$, $1 \le i \le k$, and

$$g_{y^*}(x) = c \prod_{i=1}^{k} (x - a_i).$$

Then the problem (7) is equivalent to the problem of finding the infimum of the absolute value of leading coefficient c of such g_{y^*}, where infimum is taken over all $-1 \le a_0 < \ldots < a_n \le 1$. It is known [6] that this infumum is equal to 2^{k-1}, a_i are Chebyshev nodes and $g_{y^*}(x) = \cos(k \arccos x)$ is optimal. □

The next proposition shows that if f is defined with an noise error then it can not be approximated well by Lagrange interpolatory polynomials.

Corollary 2.5. *Let $x \in X_k$ be such that $x \subset [-1, 1]$. Then*

$$\sup_{y \in Y_k} \inf_{f \in F(x,y)} \sup_{\zeta \in [-1,1]} |f(\zeta) - Lf(\zeta; x_0, \dots, x_k)| \ge 1,$$

Proof. It is known that when interpolating at point ζ a given continuous function f by Lagrange polynomial of degree n at the nodes $x_0, \dots, x_k$ we get the error

$$f(\zeta) - Lf(\zeta; x_0, \dots, x_k) = [x_0, \dots, x_k, \zeta]f \cdot \prod_{i=0}^{k} (\zeta - x_i).$$

Then

$$\sup_{y \in Y_k} \inf_{f \in F(x,y)} \sup_{\zeta \in [-1,1]} |f(\zeta) - Lf(\zeta; x_0, \dots, x_{k-1})| \ge$$

$$\ge \lambda_{k+1}(x) \inf_{-1 \le x_0 < \ldots < x_k \le 1} \sup_{\zeta} \left| \prod_{i=0}^{k} (\zeta - x_i) \right|,$$

It follows from Corollary 2.4 that for every partition $x = (-1 \le x_0 < \ldots < x_{k+1} \le 1)$ we have $\lambda_{k+1}(x) \ge 2^k$. It is well-known that

$$\inf_{-1 \le x_0 < \ldots < x_k \le 1} \sup_{\zeta} \left| \prod_{i=0}^{k} (\zeta - x_i) \right| = \frac{1}{2^k}.$$

□

3. Estimates of $\mu_k(x)$

Lemma 3.1. *If $x = (x_i)_0^k \in X_k$ then*

$$\mu_k(x) = \frac{1}{2^{k+1}\prod_{i=0}^k A_i} \int_{-A_k}^{A_k} \cdots \int_{-A_0}^{A_0} |\sum_{p=0}^k \zeta_p| d\zeta_0 \dots d\zeta_k,$$

where A_i, $0 \le i \le k$, are defined in Theorem 2.1.

Proof. It follows from $\lambda_k(x,y) = |\ [x_0,\dots,x_k]g_y\ |$, for every $y = (y_i)_0^k \in Y_k$, where g_y is the algebraic polynomial of degree k satisfying the interpolation conditions $g_y(x_i) = y_i$, $0 \le i \le k$, that

$$\mu_k(x) := \mathbb{E}_y\left(\lambda_k(x,y)\right) = \mathbb{E}_y\left(|\ [\tilde{x}_0,\dots,\tilde{x}_k]g_y\ |\right),$$

where where $\mathbb{E}$ is the expectation taken over $y = (y_i)_0^k$, where y_i's are independently uniformly distributed on [-1,1]. We have

$$\mathbb{E}_y\left(|\ [x_0,\dots,x_k]g_y\ |\right) =$$

$$= \frac{1}{2^{k+1}}\int_{-1}^1 \cdots \int_{-1}^1 \frac{|\sum_{p=0}^k y_p(-1)^p \det(e_i(x_l))_{i=0,\dots,k-1}^{l=0,\dots,k,\ l\neq p}|}{\det(e_i(x_l))_{i=0,\dots,k}^{l=0,\dots,k}} dy_0\dots dy_k =$$

$$= \frac{1}{2^{k+1}}\int_{-1}^1 \cdots \int_{-1}^1 \left|\sum_{p=0}^k \frac{y_p(-1)^p}{\prod_{0\le m\le k;\ m\neq p}|x_m - x_p|}\right| dy_0\dots dy_k \quad (8)$$

Then

$$\mathbb{E}_y\left(|\ [x_0,\dots,x_k]g_y\ |\right) = \frac{1}{2^{k+1}} \int_{-1}^1 \cdots \int_{-1}^1 \left|\sum_{p=0}^k (-1)^p A_p y_p\right| dy_0\dots dy_k =$$

$$= \frac{1}{2^{k+1}\prod_{i=0}^k A_i} \int_{-A_k}^{A_k} \cdots \int_{-A_0}^{A_0} |\sum_{p=0}^k \zeta_p| d\zeta_0 \dots d\zeta_k. \quad \square$$

Corollary 3.2. *If $k = 1$ and $x = (x_0, x_1) \in X_1$ then*

$$\mu_1(x) = \frac{2}{3}\frac{1}{x_1 - x_0}.$$

Proof. Assuming $a_0 \ge a_1$ we have

$$\int_{-a_1}^{a_1}\int_{-a_0}^{a_0} |\zeta_0 + \zeta_1| d\zeta_0 d\zeta_1 = \int_{-a_1}^{a_1}\int_{-\zeta_1}^{a_0} (\zeta_0 + \zeta_1) d\zeta_0 d\zeta_1 = 2\left(a_0^2 a_1 + \frac{1}{3}a_1^3\right).$$

Thus, if $k = 1$ then $A_0 = A_1 = 1/(x_1 - x_0)$ and

$$\mathbb{E}_y\left(\mid [x_0, x_1]g_y \mid\right) = \frac{2}{3}\frac{1}{x_1 - x_0}. \qquad \square$$

Corollary 3.3. *If $k = 2$ and $x = (x_0, x_1, x_2) \in X_2$ then we have*

$$\mu_2(x) = \frac{1}{3(x_1 - x_0)(x_2 - x_1)} - \frac{1}{6(x_2 - x_0)^2}.$$

Proof. Suppose $a_0 = a_1 + a_2$ then

$$\int_{-a_2}^{a_2}\int_{-a_1}^{a_1}\int_{-a_0}^{a_0} |\zeta_0 + \zeta_1 + \zeta_2| d\zeta_0 d\zeta_1 d\zeta_2 =$$

$$= \int_{-a_2}^{a_2}\int_{-a_1}^{a_1}\int_{-(\zeta_1+\zeta_2)}^{a_0} (\zeta_0 + \zeta_1 + \zeta_2) d\zeta_0 d\zeta_1 d\zeta_2 =$$

$$= 4a_1a_2\left(a_0^2 + \frac{1}{3}a_1^2 + \frac{1}{3}a_2^2\right). \tag{9}$$

If $k = 2$ then

$$A_0 = \frac{1}{(x_1 - x_0)(x_2 - x_0)},\; A_1 = \frac{1}{(x_2 - x_1)(x_1 - x_0)},\; A_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)}.$$

Note that $A_1 = A_0 + A_1$. Using (9) it is easy to verify that

$$\mathbb{E}_y\left(\mid [x_0, x_1, x_2]g_y \mid\right) = \frac{1}{3(x_1 - x_0)(x_2 - x_1)} - \frac{1}{6(x_2 - x_0)^2}. \qquad \square$$

In the next proposition we find the lower and the upper estimations of $\mu_k(x)$.

Theorem 3.4. *Let $k \geq 3$ and $x = (x_i)_0^k \in X_k$. Then*

$$\frac{1}{\sqrt{3}}\left(\sum_{i=1}^{k} A_i^2\right)^{1/2} \leq \mu_k(x) \leq \frac{1}{\sqrt{3}}\left(\sum_{i=0}^{k} A_i^2\right)^{1/2}. \tag{10}$$

Proof. It follows from Lemma 3.1 and the CauchySchwarz inequality that

$$\mu_k(x) \leq \frac{1}{2^{k+1}\prod_{i=0}^{k} A_i}\left(I \cdot \int_{-A_k}^{A_k} \dots \int_{-A_0}^{A_0} (\sum_{p=0}^{k} \zeta_p)^2 d\zeta_0 \dots d\zeta_k\right)^{1/2}, \tag{11}$$

where

$$I := \int_{-A_k}^{A_k} \dots \int_{-A_0}^{A_0} d\zeta_0 \dots d\zeta_k = 2^{k+1} \prod_{p=0}^{k} A_p.$$

We have

$$\int_{-A_k}^{A_k} \dots \int_{-A_0}^{A_0} \big(\sum_{p=0}^{k} \zeta_p\big)^2 d\zeta_0 \dots d\zeta_k = 2^{k+1} =$$

$$= \int_{-A_k}^{A_k} \dots \int_{-A_0}^{A_0} \sum_{p=0}^{k} \zeta_p^2 d\zeta_0 \dots d\zeta_k = \frac{1}{3} 2^{k+1} \prod_{p=0}^{k} A_p \sum_{i=0}^{k} A_i^2, \quad (12)$$

then it follows from (11) that

$$\mu_k(x) \le \frac{1}{\sqrt{3}} \left(\sum_{i=0}^{k} A_i^2 \right)^{1/2}.$$

On the other hand,

$$\mu_k(x) \ge \frac{2}{2^{k+1} \prod_{i=0}^{k} A_i} \int_{-A_k}^{A_k} \dots \int_{-A_1}^{A_1} \int_{-(\sum_{i=1}^{k} \zeta_i)}^{A_0} \left(\sum_{i=0}^{k} \zeta_i \right) d\zeta_0 \dots d\zeta_k$$

$$= \frac{1}{2^{k+1} \prod_{i=0}^{k} A_i} \int_{-A_k}^{A_k} \dots \int_{-A_1}^{A_1} \left(A_0^2 + \left(\sum_{i=1}^{k} \zeta_i \right)^2 \right) d\zeta_1 \dots d\zeta_k$$

$$= \frac{1}{2^{k+1} \prod_{i=0}^{k} A_i} \Bigg(A_0^2 \int_{-A_k}^{A_k} \dots \int_{-A_1}^{A_1} d\zeta_1 \dots d\zeta_k$$

$$+ \int_{-A_k}^{A_k} \dots \int_{-A_1}^{A_1} \left(\sum_{i=1}^{k} \zeta_i^2 \right) d\zeta_1 \dots d\zeta_k \Bigg)$$

$$> \frac{1}{2} A_0 + \frac{1}{6A_0} \sum_{i=1}^{k} A_i^2. \quad (13)$$

Since the infimum of the last quantity in (13), considered as a function of A_0, is achieved at

$$A_0 = \frac{1}{\sqrt{3}} \left(\sum_{i=1}^{k} A_i^2 \right)^{1/2}$$

we get the lower bound in (10). □

Corollary 3.5. *Let $\Delta > 0$. Denote $x_\Delta = (x_i)_0^k$ with $x_{i+1} - x_i = \Delta$ for all $0 \le i \le k-1$. Then*

$$\frac{\Delta^{-k}}{k!\sqrt{3}} \left(\sum_{p=1}^{k} \binom{k}{p}^2 \right)^{1/2} \le \mu_k(x_\Delta) \le \frac{\Delta^{-k}}{k!\sqrt{3}} \left(\sum_{p=0}^{k} \binom{k}{p}^2 \right)^{1/2}.$$

4. The Case of Normally Distributed Errors

Let $k \in \mathbb{N}$. Let X_k denote the set of all sequence of real numbers $x = (x_i)_0^k$ with $x_i < x_{i+1}$.

Denote U_k the set of all sequences of real numbers $u = (u_i)_0^k$, such that u_i is a sequence of i.i.d. random variables such that $u_i \sim \mathcal{N}(0,1)$.

Given $x = (x_i)_0^k \in X_k$, $u = (u_i)_0^k \in U_k$, let $F(x,u)$ denotes the set of all real-valued function defined on $\mathbb{R}$, such that $f(x_i) = u_i$, $0 \le i \le k$.

Given $x = (x_i)_0^k \in X_k$, $u = (u_i)_0^k \in U_k$, denote

$$\lambda_k(x,u) := \inf_{f \in F(x,u)} \sup_{\tilde{x} = (\tilde{x}_i)_0^k \in X_k} \mid [\tilde{x}_0, \dots, \tilde{x}_k] f \mid$$

The main goal of this section is to estimate the value

$$\nu_k(x) := \mathbb{E}_u \left(\lambda_k(x,u) \right), \tag{14}$$

where $\mathbb{E}$ is the expectation of $\lambda_k(x,u)$ taken over $u = (u_i)_0^k$, where u_i's are identically independently normally distributed, $u_i \sim \mathcal{N}(0,1)$, $0 \le i \le k$.

Note that when u_i's are normally distributed, the "worst"-case does not have sense since it is obvious that $\sup_{u \in U_k} \lambda_k(x,u) = \infty$.

Theorem 4.1. *If $x = (x_i)_0^k \in X_k$ then*

$$\nu_k(x) = \frac{2}{\sqrt{2\pi}} \left(\sum_{p=0}^{k} A_p^2 \right)^{1/2},$$

where A_p, $p = 0, \dots, k$, are defined in Theorem 2.1.

Proof. It follows from $\lambda_k(x,u) = \mid [x_0, \dots, x_k] g_u \mid$, for every $u = (u_i)_0^k \in U_k$, where g_u is the algebraic polynomial of degree k satisfying the interpolation conditions $g_u(x_i) = y_i$, $0 \le i \le k$, that

$$\nu_k(x) := \mathbb{E}_u \left(\lambda_k(x,u) \right) = \mathbb{E}_u \left(\mid [x_0, \dots, x_k] g_u \mid \right),$$

where where $\mathbb{E}$ is the expectation taken over $u = (u_i)_0^k$, where u_i's are identically independently standard normally distributed. We have

$$[x_0, \dots, x_k]g_u = \sum_{p=0}^{k} (-1)^p A_p u_p.$$

It is well-known [7] that any linear combination of independent normal variables is normal, with mean equals to the linear combination of its means, and the variance equals to the linear combination of its variances. Thus, $v := [\tilde{x}_0, \dots, \tilde{x}_k]g_u$ is normally distributed with zero mean and variance

$$\sigma^2 = \sum_{p=0}^{k} A_p^2.$$

It is well-known [5] that if v is a random variable distributed normally with mean μ and variance σ^2, $v \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\mathbb{E}(v|v > \alpha) = \mu + \sigma M(c),$$

where

$$M(c) = \frac{\phi(c)}{1 - \Phi(c)}$$

is called the inverse Mills ratio, $c = \frac{\alpha - \mu}{\sigma}$, ϕ denotes the standard normal density function, and Φ is the standard normal cumulative distribution function. Then it follows that

$$\nu_k(x, \tilde{x}) = 2\mathbb{E}(v|v > 0) = 2\sigma \frac{\phi(0)}{1 - \Phi(0)} = \frac{2}{\sqrt{2\pi}} \left(\sum_{p=0}^{k} A_p^2 \right)^{1/2}. \qquad \square$$

Corollary 4.2. *Let $\Delta > 0$. Denote $x_\Delta = (x_i)_0^k$ with $x_{i+1} - x_i = \Delta$ for all $0 \le i \le k - 1$. Then*

$$\nu_k(x_\Delta) = \frac{2\Delta^{-k}}{k!\sqrt{2\pi}} \left(\sum_{p=0}^{k} \binom{k}{p}^2 \right)^{1/2}.$$

One can consider the problem of the optimal choice of knots in [-1,1] to minimize $\nu_k(x)$. It leads us to the problem of minimizing the function

$$f(x_0, \dots, x_k) = \sum_{p=0}^{k} A_p^2(x_0, \dots, x_k),$$

over all $-1 \le x_0 < x_1 < \ldots < x_k \le 1$, where $A_p(x_0, \ldots, x_k)$, $p = 0, \ldots, k$, are defined in Theorem 2.1. It is worth noting that if $k \ge 2$ then the optimal knots of this problem do not coincide with Chebyshev points that are optimal for the problem of minimizing the "worst"-case error $\lambda_k(x)$, i.e. of minimizing the function

$$h(x_0, \ldots, x_k) = \sum_{p=0}^{k} A_p(x_0, \ldots, x_k)$$

over all $-1 \le x_0 < x_1 < \ldots < x_k \le 1$.

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