MAXWELL’S EQUATIONS IN A CURVED SPACE TIME

K. Ghosh

Department of Physics
St. Xavier’s College
30 Mother Teresa Sarani, Kolkata, 700016, INDIA

Abstract: In this article we will discuss a few mathematical aspects of classical electrodynamics with particular reference to the Robertson-Walker-Friedmann cosmological model with closed spatial geometry. We will first give an alternate derivation of the Gauss’ law for the electric field of a point charge in the flat space-time not using the Gauss’ divergence theorem. We will thereafter derive an exact expression for the electrostatic energy. The expression for the electrostatic energy obtained in this article agrees with the standard expression when the sources are not point particles. However we will show that the electrostatic self-energy of a point charge may be zero. We will also consider the consistency of the Maxwell’s equations for the electromagnetic potentials in the Robertson-Walker-Friedmann universe with closed spatial geometry. We find that the contravariant electromagnetic fields are invariant under the gauge transformations and so also are the inhomogeneous Maxwell’s equations when they are expressed in terms of the contravariant electromagnetic fields. However the inhomogeneous Maxwell’s equations are not invariant under the gauge transformations when they are expressed in terms of the electromagnetic potentials. This may also remain valid in an arbitrary curved space-time. We will give a naive interpretation of this aspect related with the formulation of the differential geometry of the curved spaces.

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1. Introduction

In this article we will discuss a few mathematical aspects of classical electrodynamics.

In Section 2 we will discuss an alternate derivation of the Poisson’s equation of electrostatics for the static sources [1]. It may not be possible to apply the Gauss’ divergence theorem to a vector field when the vector field is singular [1]. We will demonstrate in this section that using the spherical polar coordinates, we can derive an expression analogous to the integral version of the divergence theorem for the electrostatic field of a point charge through explicit integrations over the field coordinates. We do not have to use the Gauss’ divergence theorem directly. We also discuss an improved version of the proof of the no-work law of electrostatics. We have discussed these issues in detail in [2]. We will derive an exact expression for the electrostatic energy. The expression for the electrostatic energy obtained in this article agrees with the standard expressions [1,3] when the sources are not point particles. However we will show that the electrostatic self-energy of a point charge may be zero. This is related with the derivation of the electrostatic energy, expressed only in terms of the electric field, from the expression of the electrostatic energy expressed in terms of the potential and is discussed in the text. The vanishing expression for the self-energy of a point charge is in contrast to the standard expressions [1,3] where the electrostatic self-energy of a point charge is infinite. The applicability of the Gauss’ integral law to singular fields is important for these discussions. The zero self energy is significant for the point particle model of the elementary particles. We have a regular value for the electrostatic self-energy of a point particle even within the framework of classical electrodynamics. We note that the zero self-energies of the point particles give us a way to experimentally study the point particle model of the elementary particles. The electrostatic self-energies of the elementary particles are finite and non-zero when they are considered to be of finite volume. Hence we can compare the two models with reference to a relativistic theory. This is also significant in the context of the quantum theories.

Section 3 is related with the consistency of the Maxwell’s equations in an arbitrary curved space-time. We will consider the closed universe Robertson-Walker cosmological models. The contravariant or the covariant electromagnetic fields are invariant under the gauge transformations and so also are the inhomogeneous Maxwell’s equations written in terms of the electromagnetic fields, eqns(19). However it is easy to find that in the comoving coordinate system, the inhomogeneous Maxwell’s equation for the azimuthal component of
the electromagnetic potentials, eqn(20), is not invariant under the gauge transformations. We will give a naive interpretation of this result. This is significant in the context of the differential geometry of the curved manifolds.

2. Discussions on Poisson’s Equation

In this section we will consider an alternate derivation of the Poisson’s equation for the electric field of a static source. We first note that we can not apply the integral version of the divergence theorem if the vector field is divergent at any point inside the region of interest. The expression \( E_x(\Delta x) = E_x(r = 0) + \Delta x \left[ \frac{dE_x}{dx} \right]_{r=0} \) is not well-defined for a point charge at the origin. In one dimension we can use the relation: \( \int_{x_1}^{x_2} (df/dx)dx = f(x_2) - f(x_1) \) only if \( f(x) \) is well-behaved for all \( x \) between and including the limits. We should also note that \( \left( \frac{x^2}{r^2} \right) \) is not well-defined at \( r = 0 \). In the following section we will demonstrate that in the spherical polar coordinates an explicit volume integration of the divergence of the electric field of a point charge gives us the total charge when the charge is at the origin. Also for a point charge at the origin the volume integral of the divergence of the electric field is vanishing when the volume of integration does not include the origin. This together with the fact that the total flux of the electric field of a point charge at the origin over an \( S^2 \) centred at the origin is given by \( \frac{2}{\epsilon_0} \) lead us to a relation for the electric field of a point charge analogous to the integral version of the divergence theorem. We can generalize this relation for a point charge not at the origin and have the Poisson’s equation for the electrostatic field.

We first consider a point charge at a radius vector \( \vec{a} \) on the \( Z \)-axis. In the following discussions we will ultimately consider the point charge to be at the origin. We use the inverse square law in the spherical polar coordinates for the electric field of a point charge at the position vector \( \vec{a} \). The electric field is azimuthally symmetric with the corresponding component being zero. We integrate the divergence of the electric field over a sphere centred at the origin. The radius vector of the point charge is \( \vec{a} \) and the radius of the sphere is greater than \( |\vec{a}| \).

We first consider the integral of the radial derivative term:

\[
4\pi\epsilon_0 I_r = \int dv \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{r^2 \vec{R} \cdot \hat{r}}{R^3} \right) \right].
\]  

Here \( \vec{R} = \vec{r} - \vec{a} \) and the radius of the volume of integration is greater than \( |\vec{a}| \).
We considered $\vec{a}$ to be on the Z-axis. The measure in the integral contains a $(\sin \theta)$ term in the spherical polar coordinate system. Hence the complete integrand is divergent of degree two in the triple integration. Thus the integral is well-behaved and the Riemann sum remains independent of our choice of the point of evaluation of the integrand. In particular we can evaluate the integrand in the Riemann-integral infinitesimally away from the position of the point charge and use the rule of the total integrals to the above radial integral to have,

$$4\pi \epsilon_0 I_r = \int d\Omega \frac{r^3 q}{[r^2 + a^2 - 2r a \cos \theta]^{3/2}} \tag{2}$$

$$-\int d\Omega \frac{r^2 q(\vec{a} \cdot \vec{r})}{[r^2 + a^2 - 2r a \cos \theta]^{3/2}}$$

In the limit $a = 0$ the above expression gives the result $I_r = \frac{q}{\epsilon_0}$.

We now consider the integral of the polar angular derivative term. This integral is also a total integral in the polar angle $\theta$. The upper limit of the integral gives vanishing contributions. The integral can take finite value only from the lower limit $\theta = 0$ part. This term and hence the value of the corresponding integral is given by the following expression:

$$4\pi \epsilon_0 I_\theta = -[2\pi \int_0^{r'} r' dr \sin(\delta \theta) (\vec{R} \cdot \hat{\theta} \vec{R}^3)] \tag{3}$$

Here $r' > a$. In the limit $\delta \theta \to 0$ and $a \to 0$ the above integral vanishes. There is no azimuthal component of the electric field when the point charge is on the polar axis.

Hence we have $\int \vec{\nabla}_r \vec{E} \cdot dv = \frac{q}{\epsilon_0}$ when the point charge is at the origin.

When the point charge is at the origin $4\pi \epsilon_0 \int \vec{\nabla}_r \vec{E} \cdot dv$ is vanishing if the integrating volume does not include the origin. If we consider any annular region centred at the point charge at the origin, the volume integral of $(\vec{\nabla}_r \vec{E})$ is given by the following expression:

$$\int qd\Omega [(\frac{r_2}{r_2})^3 - (\frac{r_1}{r_1})^3] = 0. \tag{4}$$

It is easy to show that the volume integral of $(\vec{\nabla}_r \vec{E})$ is vanishing for any arbitrary volume not including the point charge at the origin.

Thus we have the well-known Poisson’s equation for a point charge at the origin,

$$\vec{\nabla}_r (\frac{\vec{r}}{r^2}) = 4\pi \delta^3 (\vec{r}). \tag{5}$$
The above discussions together with the fact that the total flux of the electric field over a closed surface is $\oint \mathbf{E} \cdot d\mathbf{S}$ allows us to write the corresponding integral version of Gauss’s divergence theorem for the electric field of a point charge at the origin.

We can derive a corresponding expression for the divergence of the electric field of a point charge not at the origin. We follow the above derivation with $\mathbf{r}$ and $\mathbf{a}$ replaced by $\mathbf{R} = (\mathbf{r} - \mathbf{r}')$ and $\mathbf{a}' = (\mathbf{r}' - \mathbf{r})$ respectively and consider the explicit volume integrals over a sphere centred at $\mathbf{r}'$. Here $\mathbf{r}'$ is the position of the point charge.

We consider the inverse square law to find the flux of the electric field and we have the Poisson’s equation for the electrostatic field of a point charge not at the origin,

$$\nabla \cdot \left( \frac{\mathbf{E}}{R^2} \right) = 4\pi \delta^3(\mathbf{R}).$$

We note that we can put $\mathbf{a} = 0$ in equations (2) and (3) instead of taking the limits. A different derivation is given in [1] which may not be well-defined if we put a corresponding parameter to be vanishing instead of taking the limit. We can use the above procedures given in this section to apply the Gauss’ integral law to any field of the form $(\frac{R}{r^n})$ with $n < 3$. In this context we note that an expression of the form $(\frac{m^m}{r^n})$ with $n \geq m$ is not well-defined at $r = 0$ although the limit $r \to 0$ may exist.

We now consider the no-work law for the electric field. For a point charge source the electric field is singular at the point charge and we may not be able to apply the Stoke’s theorem. We can consider an annular region surrounding the point charge with the inner boundary being an infinitesimally small circle centred at the point charge and apply Stoke’s theorem to the electric field in this region. The work done for the inner boundary vanishes out of spherical symmetry and we have the no work law for the outer closed line.

We conclude our discussions on Maxwell’s equations for steady sources with a few discussions on the derivation of the curl of $\mathbf{B}$ law. In the derivation of $(\nabla \times \mathbf{B})$ law we have a ‘boundary’ term [3]:

$$I_B = \int \nabla_{\mathbf{r}'}.\left[ \frac{\mathbf{J}(\mathbf{r}')}{R} \right] dv'.$$  (7)

We can not use Gauss’ divergence theorem directly as $\frac{\mathbf{J}(\mathbf{r}')}{R}$ is divergent at $\mathbf{R} = 0$. We break the integral into two parts:

$$I_B = \int_{\delta v} \nabla_{\mathbf{r}'}.\left[ \frac{\mathbf{J}(\mathbf{r}')}{R} \right] dv' + \int_{V_{-\delta v}} \nabla_{\mathbf{r}'}.\left[ \frac{\mathbf{J}(\mathbf{r}')}{R} \right] dv'.$$  (8)
Here $\delta v$ is an arbitrarily small sphere surrounding $\vec{r}$. For a non-singular current distribution the first term vanishes in the limit $R \to 0$. In the second term $\frac{\vec{J}(\vec{r})}{R}$ is well-behaved everywhere within $V - \delta v$. We can use the divergence theorem to transform the volume integral into two boundary surface integrals. For the inner boundary $\int \frac{\vec{J}(\vec{r})}{R}.d\vec{s}_1$ is vanishing for a regular current distribution as the radius of $\delta v$ become arbitrarily small. For the outer boundary $\vec{J}.d\vec{s}_2$ is zero as the current should be tangential at the boundary of the source. However this derivation is valid when the current density is non-singular and the derivatives of $\vec{J}$ are sufficiently well-behaved within the source apart from the boundary.

We will now derive an exact expression for the electrostatic energy. The expression for the electrostatic energy obtained in this section agrees with the standard expressions [1,3] when the sources are not point charges. However we show that the electrostatic self-energy of a point charge is vanishing instead of being infinite. This is in contrast to the standard expressions [1,3] where the electrostatic self-energy of a point charge is infinite.

The fundamental expression for the electrostatic energy, expressed in terms of the electrostatic potential, is given by the following expression [1,3,4]:

$$\xi = \frac{1}{2} \int \rho(\vec{r}') V(\vec{r}') d\tau.$$  \hspace{1cm} (9)

The integral can be taken to be over the complete volume. Here $V(\vec{r}')$ is the potential at the point $\vec{r}'$ due to all the other source elements. We denote such quantities by $V_R(\vec{r}')$. However the charge density is given by the divergence of the electric field due to the complete source. We denote such quantities by the suffix $T$. Thus the energy is given by the following expression:

$$\xi = \frac{\epsilon_0}{2} \int (\nabla \cdot \vec{E}) V_R d\tau.$$  \hspace{1cm} (10)

For a non-singular source density both $V_R$ and $\vec{E}_R$ can be replaced by $V_T$ and $\vec{E}_T$ respectively. The corresponding fields and the potentials are non-singular and are vanishing in the limit the source volume element tends to zero. We can replace $V_R$ by $V_T$ in eqn.(10) and we can also apply the Gauss’integral law to the vector field $(V_T \vec{E}_T)$. We then have the conventional expression for the electrostatic energy expressed in terms of the electric field:

$$\xi = \frac{\epsilon_0}{2} \int (\vec{E}_T \cdot \vec{E}_T) d\tau.$$  \hspace{1cm} (11)

Here $\vec{E}_T$ is the complete field at the point $(x, y, z)$. 
However the situation is different when we have point charges. In these cases $V_R$ is finite at the point charges. On the other hand $V_T$ is divergent at the point charges and the value of the energy becomes different if we replace $V_R$ by $V_T$ in eqn(10). In fact replacing $V_R$ by $V_T$ in eqn(10) for point charges gives us the usual divergent value for the electrostatic self energy of point charges provided we can apply the Gauss’ integral law to the fields divergent as $\frac{1}{r^3}$.

We have mentioned at the beginning of this section that the application of the Gauss’ law to fields divergent as $\frac{1}{r^3}$ is a non-trivial issue.

In this section we evaluate the electrostatic energy from eqn(10) without replacing $V_R$ by $V_T$.

When we have an isolated point charge, $V_R$ is zero at the point charge but is finite for any other point. Let us consider the point charge to be at the origin. The product $(V_R\vec{E}_T)$ varies as $\frac{1}{r^3}$ but is zero at the point charge. As we have discussed earlier, we can not apply the divergence theorem to such an ill-behaved field. We note that $V_R$ is zero where the delta function representing the source density is non-vanishing. Hence in expression (9) we can delete an infinitesimal volume element surrounding the point charge from the region of volume integration. In the limit of zero volume this infinitesimal volume element surrounding the point charge will have zero contribution. The electrostatic energy is given by the following expression:

$$\xi' = \frac{1}{2} \int (\vec{\nabla} \cdot \vec{E}_T) V_R d\tau.$$  \hspace{1cm} (12)

Here $v$ is an infinitesimal volume centred at the point charge. We can now apply the divergence theorem to the vector field $(V_R\vec{E}_T)$. The integrand in the above expression, within the annular region of volume integration, is given by $(V_T\vec{E}_T)$. It is easy to find that $\xi'$ is vanishing. This is consistent with the value of the energy obtained directly from eqn.(10). We need not to consider any limiting internal structure of the point charges to explain the self-energy of the point charges. This is considered in [3] to explain the divergent expression for the electrostatic self-energy. We will discuss this issue again.

We now derive the expression for the electrostatic energy when we have more than one point charge. We consider the situation with two point charges. The electrostatic energy in terms of the potential is given by the following expression:

$$\xi = \frac{1}{2} \int (\vec{\nabla} \cdot \vec{E}_T) V_R d\tau.$$  \hspace{1cm} (13)

In this case we no longer have $V_R = 0$ at the point charges. Hence we can not delete the infinitesimally small volume elements containing the delta functions.
(the point charges) from our region of integration. We find that the integral over the delta functions give us the well-known expression for the energy of interaction. We can delete two infinitesimal volume elements surrounding the point charges from our region of volume integration provided we take into account this interaction energy. The expression for the electrostatic energy with two point charges is then given by the following expression:

\[
\xi = \frac{1}{2} \int \left( V - v_1 - v_2 \right) (\nabla \cdot \vec{E}) V_T d\tau + q_1 q_2 \frac{1}{4\pi \epsilon_0 R_{12}}.
\]  

(14)

Here \((v_1 \text{ and } v_2)\) are two infinitesimally small volume elements surrounding the point charges \(q_1\) and \(q_2\) respectively. One can show that the contribution from the volume integration is zero. We should be careful about the surface terms. Thus we find that the electrostatic self energy of a point charge is vanishing. We may consider a point charge to be the limiting situation of an infinitesimally small volume element whose total charge is always finite. One can then derive an expression for the total electrostatic energy from expression (11). If we neglect the internal structure of such a source and the corresponding electrostatic energy, we have an expression for the electrostatic energy which diverges inversely as the radius of the source tends to zero. This expression is similar to the conventional divergent expression of the electrostatic self-energy of a point charge. However we should note that the situation with a sphere whose radius tends to zero is different from a sphere whose radius is exactly zero. In the later situation the expression (11) is no longer valid. This is related with the applicability of the Gauss’ integral law to divergent fields. In this case the field \((V_T E_T)\) diverges as inverse cube of the distance from the point charges and we can not apply the Gauss’ integral law as is done to obtain eqn.(11). This is discussed below eqn.(11). The above discussions are significant when we consider the Lagrangian description of classical electrodynamics. We find that the applicability of the Gauss integral law to singular fields is a serious concern when we consider the expression for the energy-momentum stress tensor with point particle sources.

3. Consistency of The Maxwell’s Equations in Curved Spaces

In this section we will consider the consistency of the Maxwell’s equations in curved spaces. We will consider the closed universe Robertson-Walker cosmological model. We will briefly describe the corresponding situation with the spherical polar coordinates in the flat space-time. In the following discussions bold-face symbols will represent four dimensional space-time objects.
In the closed universe Robertson-Walker-Friedmann cosmological model the contravariant or the covariant electromagnetic fields are invariant under the gauge transformations and so also are the inhomogeneous Maxwell’s equations written in terms of the electromagnetic fields, eqns(19). However it is easy to find that in the comoving coordinate system, the inhomogeneous equation for the azimuthal component of the electromagnetic potentials, eqns(20), is not invariant under the gauge transformations. We will give a naive interpretation of this result. This is significant in the context of the differential geometry of the curved manifolds.

The space-time metric of the Robertson-Walker cosmological models, expressed in terms of the comoving observers, are given by the following expressions [5]:

$$ds^2 = -dt^2 + a^2(t)d\sigma^2.$$  \hspace{1cm} (15)

Here $t'$ is the proper time as measured by the comoving observers and $d\sigma^2$ represents the spatial metric. We consider the gauge invariance of the inhomogeneous Maxwell’s equations for the case of the closed spatial geometry. The space-time metric when the spatial geometry is closed is given by the following expression:

$$ds^2 = -dt^2 + a^2(t)[dr^2 + \sin^2 r(d\theta^2 + \sin^2 \theta d\phi^2)].$$ \hspace{1cm} (16)

Here $r'$ is an angle with the range $0 \leq r \leq \pi$. The other angles have the same range as that for the two dimensional sphere.

In this space-time the Ricci tensor has only diagonal components and are given by the following expressions:

$$R_{tt} = -\frac{3\ddot{a}}{a}. \hspace{1cm} (17)$$

and

$$R_{ii} = \frac{g_{ii}}{a^2}[a\ddot{a} + 2\dot{a}^2 + 2]. \hspace{1cm} (18)$$

Here $g_{ii}$ is the spacial metric component. In an arbitrary curved space-time the inhomogeneous Maxwell’s equations for the electromagnetic fields are given by the following expressions:

$$F^{\mu\nu} = 4\pi J^\mu. \hspace{1cm} (19)$$

Both the contravariant and the covariant electromagnetic fields are invariant under the gauge transformations in the space-time under consideration. Hence the above equations are invariant under the gauge transformations.
In an arbitrary curved space-time the inhomogeneous Maxwell’s equations for the electromagnetic four-potentials are given by the following equations [6]:

\[-(A^\alpha)_{;\mu}^\mu + (A^\mu)_{;\mu}^\alpha + R^\alpha_{\mu\nu} A^\mu = 4\pi J^\alpha.\]  

(20)

Here the semicolons represent the appropriate covariant derivatives. The first differential operator in the left hand side of the above equation is given by the following expression:

\[(A^\alpha)_{;\mu}^\mu = g^{\mu\nu} [A^\alpha_{,\mu,\nu} + \Gamma^\alpha_{\rho\mu} A^\rho_{,\mu} + \Gamma^\alpha_{\rho\nu} A^\rho_{,\nu} + \Gamma^{\alpha\rho}_{\mu\nu} A^\lambda - \Gamma^{\alpha\rho}_{\nu\mu} A^\lambda_{,\rho} - \Gamma^{\alpha\rho}_{\nu\mu} \Gamma^\rho_{\lambda\rho} A^\lambda].\]  

(21)

The connection coefficients in the coordinate basis are given by the following expressions:

\[
\begin{align*}
\Gamma^t_{rr} &= \dot{a} a; \quad \Gamma^r_{rt} = \Gamma^r_{tr} = \frac{\dot{a}}{a}; \quad \Gamma^t_{\theta\theta} = \dot{\sin^2 r}; \quad \Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \frac{\dot{\theta}}{a}; \quad \Gamma^t_{\phi\phi} = \dot{\sin^2 r \sin^2 \theta}; \quad \Gamma^\phi_{t\phi} = \Gamma^\phi_{\phi t} = \frac{\dot{\phi}}{a}; \quad \Gamma^\phi_{\theta\theta} = \frac{5\cot^2 \theta - 4\cot^2 r - 1}{a^2 \sin^2 r \sin^2 \theta} - \frac{2 + 6\cot^2 r}{a^2 \sin^2 r \sin^2 \theta} \frac{(\partial T}{\partial \phi}) \frac{(\partial T}{\partial \phi)}, \end{align*}
\]

Here we have considered only that particular \(\frac{(\partial T}{\partial \phi})\) term which does not contain time derivatives of the scale factor \(a(t)\). The second differential operator in the left hand side of equation (20) do not contribute such terms.

This gives us non-trivial constraints on the gauge function so that the equations for \(A^\phi\) remain invariant under the gauge transformations. Similar situation is valid with \(A^\phi\) in the spherical polar coordinates in the flat space. The above aspect may also remain valid in an arbitrary curved space-time where we do not have a global Cartezian coordinate system.

We do not have the above problem in the Cartezian coordinates in the flat space-time because the corresponding partial differentiations commute and the unit vectors are fixed vectors. The above feature is related with non-trivial triple derivatives of a scalar field in a curved space-time. This may be related with tensorial character of the connection coefficients. We can naively say that
the rule for linear covariant derivatives in the curved geometry may not be carried for higher order covariant derivatives in general. We will later discuss this issue in detail. These are important for the analytical geometry of the curved spaces. The above discussions are also significant for electrodynamics in curved space-time and also for the corresponding quantum field theory [6,7,8]. We will discuss these issues later.

4. Conclusion

In this article we have discussed a few aspects of classical electrodynamics in a curved space-time.

In Section 2 we have given an alternate proof of the Poisson’s equation of electrostatics not using the Gauss’ divergence theorem. This shows us that we can extend the Gauss’ integral law to any vector field of the form \((\hat{\mathbf{R}})^n\) with \(n < 3\). We have derived an exact expression for the electrostatic energy. We found that the electrostatic self-energy of point charges may be zero. This is significant for the point particle model of the elementary particles. We have a regular value for the electrostatic self-energy of a point particle even within the framework of classical electrodynamics. This is significant in the context of the quantum theories. We have also shown that the electric field in an arbitrarily shaped conductor is not vanishing when the charge is only given by an ideal surface charge density at the surface of the conductor. We have briefly discussed an elementary explanation of the electrostatic behaviours of the conductors in terms of extended charge distributions confined near the surfaces of the conductors. These discussions are consistent with the corresponding quantum mechanical discussions [1].

In Section 3 we have shown that in the Robertson-Walker space-time the inhomogeneous Maxwell’s equations for the electromagnetic potentials are not invariant under the gauge transformations although the fields remain to be invariant under the gauge transformations. The above feature is related with non-trivial triple derivatives of a scalar field in a curved space-time. We can naively say that the rule for covariant linear derivatives in the curved geometry may not be carried for higher order derivatives in general. This is significant in the context of the differential geometry of the curved manifolds. We will later discuss this issue in detail.

References


