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ON SEMIGROUPS OF REGULAR HYPERSUBSTITUTIONS

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Abstract: It is known that the set of all hypersubstitutions of a given type forms a semigroup (see [4] and [1]). Semigroup's properties of hypersubstitutions have been studied by many authors. In this paper, we characterize Green's relations of every subsemigroups of the semigroup of regular hypersubstitutions. Moreover, we give a partial solution concerning $\mathcal{G}$-subsemigroups of this semigroup.

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1. Introduction

Let S be a semigroup. An element a of S is said to be *von Neumann regular* if $a = axa$ for some $x \in S$; and S is called a *von Neumann regular semigroup* if every element of S is von Neumann regular. Green's relations on S, $\mathcal{R}^S$, $\mathcal{L}^S$, $\mathcal{H}^S$, $\mathcal{D}^S$, and $\mathcal{J}^S$, are defined by, for $a, b \in S$,

(i) $a\mathcal{L}^S b$ if and only if there exist $x, y \in S^1$ such that $a = xb$ and $b = ya$,

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(ii) $a\mathcal{R}^Sb$ if and only if there exist $x, y \in S^1$ such that $a = bx$ and $b = ay$,

(iii) $a\mathcal{J}^Sb$ if and only if there exist $x, y, u, v \in S^1$ such that $a = xby$ and $b = uav$,

$\mathcal{H}^S := \mathcal{R}^S \cap \mathcal{L}^S$ and $\mathcal{D}^S := \mathcal{R}^S \circ \mathcal{L}^S$. For $a \in S$, let L_a^S denote the $\mathcal{L}^S$-class containing a. For R_a^S, H_a^S, D_a^S and J_a^S, we define analogously. For a subsemigroup T of S, Green's relations on T are denoted by $\mathcal{R}^T$, $\mathcal{L}^T$, $\mathcal{H}^T$, $\mathcal{D}^T$ and $\mathcal{J}^T$. It is routine matter to check that

$$\mathcal{L}^T \subseteq \mathcal{L}^S \cap (T \times T),\ \mathcal{R}^T \subseteq \mathcal{R}^S \cap (T \times T),\ \mathcal{H}^T \subseteq \mathcal{H}^S \cap (T \times T),$$
$$\mathcal{D}^T \subseteq \mathcal{D}^S \cap (T \times T),\ \mathcal{J}^T \subseteq \mathcal{J}^S \cap (T \times T).$$

It is known in general that Green's relations on any subsemigroup of a given semigroup S need not be the restrictions of the corresponding relations on S. Consider the additive semigroups of integers and natural numbers, if $S = \mathbb{Z}$, then $\mathcal{L}^S = \mathcal{R}^S = \mathcal{H}^S = \mathcal{D}^S = \mathcal{J}^S = S \times S$; and if $T = \mathbb{N}$, then $\mathcal{L}^T = \mathcal{R}^T = \mathcal{H}^T = \mathcal{D}^T = \mathcal{J}^T = \{(n, n) : n \in \mathbb{N}\}$. However, if T is a von Neumann regular subsemigroup of S, then the following holds (see [5]).

Theorem 1. *Let S be a semigroup. If T is a von Neumann regular subsemigroup of S, then*

$$\mathcal{R}^T = \mathcal{R}^S \cap (T \times T), \mathcal{L}^T = \mathcal{L}^S \cap (T \times T), \mathcal{H}^T = \mathcal{H}^S \cap (T \times T).$$

Definition 2. A subsemigroup T of a semigroup S is called a $\mathcal{G}$-semigroup if

$$\mathcal{L}^T = \mathcal{L}^S \cap (T \times T),\ \mathcal{R}^T = \mathcal{R}^S \cap (T \times T),\ \mathcal{H}^T = \mathcal{H}^S \cap (T \times T),$$
$$\mathcal{D}^T = \mathcal{D}^S \cap (T \times T),\ \mathcal{J}^T = \mathcal{J}^S \cap (T \times T).$$

Let $X = \{1, 2, 3, \ldots, n\}$ and let

$$\begin{aligned} P_n &:= \{\alpha : A \to X : A \subseteq X, \alpha \text{ is a mapping}\} \\ T_n &:= \{\alpha : X \to X : \alpha \in P_n\}. \end{aligned}$$

It is well-known that P_n and T_n are von Neumann regular semigroups under the usual composition of functions. The next result describes Green's relations on P_n (see [5]),

(i) $\alpha\mathcal{L}\beta$ if and only if $im\ \alpha = im\ \beta$,

(ii) $\alpha\mathcal{R}\beta$ if and only if $ker\ \alpha = ker\ \beta$,

(iii) $\alpha\mathcal{D}\beta$ if and only if $|im\ \alpha| = |im\ \beta|$.

(iv) $\mathcal{J} = \mathcal{D}$

Using Theorem 1, we obtain

$$\mathcal{R}^{T_n} = \mathcal{R}^{P_n} \cap (T_n \times T_n),\ \mathcal{L}^{T_n} = \mathcal{L}^{P_n} \cap (T_n \times T_n),$$

$$\mathcal{H}^{T_n} = \mathcal{H}^{P_n} \cap (T_n \times T_n).$$

Problem. Characterize all $\mathcal{G}$-subsemigroup of P_n and T_n.

In [6], I. Levi characterized a certain class of $\mathcal{G}$-semigroups as a partial solution of the above problem. In this paper, we are interested in the semigroup of regular hypersubstitutions defined by Plonka. Let us recall some basic definitions.

A type is defined as a mapping $\tau = \{(f_i, n_i) : i \in I\}$ from a sequence $(f_i)_{i\in I}$ of operation symbols to $\mathbb{N}$. For each $i \in I$, $\tau(f_i) := n_i$ is called the arity of f_i and f_i is called an n_i-ary operation symbol.

Let $\tau = \{(f_i, n_i) : i \in I\}$ be a type. Let $X := \{x_1, x_2, x_3, \ldots\}$ be a countably infinite alphabet of variables such that the sequence of the operation symbols $(f_i)_{i\in I}$ is disjoint with X, and let $X_n := \{x_1, x_2, x_3, \ldots, x_n\}$ be an n-element alphabet where $n \in \mathbb{N}$. Here f_i is n_i-ary for a natural number $n_i \geq 1$. An n-ary $(n \geq 1)$ term of type τ is inductively defined as follows:

(i) every variable $x_i \in X_n$ is an n-ary term,

(ii) if $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

Let $W_\tau(X_n)$ be the smallest set containing $x_1, \ldots, x_n$ and being closed under finite application of (ii). The set of all terms of type τ over the alphabet X is defined by

$$W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n).$$

For a term $t \in W_\tau(X)$, the set of all variables occurring in t will be denoted by $var(t)$; and the number of all operation symbols occurring in t will be denoted by $op(t)$.

Any mapping $\sigma : \{f_i : i \in I\} \to W_\tau(X)$ is called a *hypersubstitution* of type τ if $\sigma(f_i)$ is an n_i-ary term of type τ for every $i \in I$. Any hypersubstitution σ of type τ can be uniquely extended to a map $\hat{\sigma}$ on $W_\tau(X)$ as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$,

(ii) $\hat{\sigma}[t] {:=} \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \ldots, t_{n_i})$.

By using such extension, the binary operation $\circ_h$ can be defined on the set $Hyp(\tau)$ of all hypersubstitutions of type τ. Let $\sigma_1, \sigma_2 \in Hyp(\tau)$. We define

$$(\sigma_1 \circ_h \sigma_2)(f_i) := \hat{\sigma}_1[\sigma_2(f_i)].$$

The set $Hyp(\tau)$ is closed under this associative composition operation and so forms a semigroup. In fact, $Hyp(\tau)$ is a monoid. Indeed, the identity hypersubstitution σ_{id} mapping every f_i to $f_i(x_1, \ldots, x_{n_i})$, acts as an identity element. A hypersubstitution σ of type τ is said to be *regular* if $var(\sigma(f_i)) = \{x_1, \ldots, x_{n_i}\}$ for all $i \in I$, and let $Reg(\tau)$ denote the set of all regular hypersubstitutions of type τ. It was proved in [4] that $Reg(\tau)$ forms a submonoid of $Hyp(\tau)$.

Throughout this paper, we restrict ourselves to the semigroup $Reg(f, n)$ of type $\{(f, n)\}$ for $n \in \mathbb{N} \setminus \{1\}$. Let $\sigma_t \in Reg(f, n)$ denote a hypersubstitution mapping f to t. The following lemma is useful for the study.

Lemma 3. *If $\sigma_s, \sigma_t \in Reg(f, n)$, then $op(\hat{\sigma}_s[\sigma_t(f)]) \geq op(\sigma_t(f))$. Moreover, if $op(s) > 1$, then $op(\hat{\sigma}_s[\sigma_t(f)]) > op(\sigma_t(f))$.*

Proof. The result can be proved by induction on the complexity of the term s. □

Hereafter, let S_n be the set of all bijections on $\{1, 2, 3, \ldots, n\}$, let σ_α when $\alpha \in S_n$ denote the hypersubstitution $\sigma_{f(x_{\alpha(1)}, \ldots, x_{\alpha(n)})}$ and let

$$PH_n := \{\sigma_\alpha : \alpha \in S_n\}.$$

Clearly, PH_n is a finite subgroup of $Reg(f, n)$. Therefore, for every subsemigroup S of $Reg(f, n)$, if $PH_n \cap S$ is non-empty, then it is a subgroup of S. The next proposition shows that the set of all von Neumann regular elements of $Reg(f, n)$ is PH_n.

Proposition 4. *A hypersubstitution $\sigma_t \in Reg(f, n)$ is von Neumann regular if and only if $\sigma_t \in PH_n$. Consequently, $Reg(f, n)$ is not a von Neumann regular semigroup.*

Proof. ($\Leftarrow$). Obvious because PH_n is a group.

($\Rightarrow$). Assume that σ_t is von Neumann regular and $\sigma_t \notin PH_n$. Then $\sigma_t = \sigma_t \sigma_{t'} \sigma_t$ for some $\sigma_{t'} \in Reg(f, n)$. By Lemma 3,

$$op(t) = op(\sigma_t(f)) = op(\hat{\sigma}_t[\hat{\sigma}_{t'}[\sigma_t(f)]]) = op(\hat{\sigma}_t[\hat{\sigma}_{t'}[t]]) > op(\hat{\sigma}_{t'}[t]) \geq op(t).$$

This is a contradiction. □

2. Main Results

To prove our main theorem, we prepare the following propositions.

Proposition 5. *Let S be a subsemigroup of $Reg(f,n)$ and $\sigma_t, \sigma_{t'} \in S$. Then $\sigma_t \mathcal{R}^S \sigma_{t'}$ if and only if $\sigma_t = \sigma_{t'}\sigma_\alpha$ for some $\sigma_\alpha \in PH_n \cap S^1$.*

Proof. Assume at first that $\sigma_t = \sigma_{t'}\sigma_\alpha$ for some $\sigma_\alpha \in PH_n \cap S^1$. Then $\sigma_t\sigma_{\alpha^{-1}}{=}\sigma_{t'}$. Since $\alpha \in PH_n \cap S^1$, $\alpha^{-1} \in PH_n \cap S^1$. Therefore, $\sigma_t \mathcal{R}^S \sigma_{t'}$.

Conversely, assume that $\sigma_t \mathcal{R}^S \sigma_{t'}$. Then there exist $\sigma_a, \sigma_b \in S^1$ such that $\sigma_t = \sigma_{t'}\sigma_a$ and $\sigma_{t'} = \sigma_t\sigma_b$. Since $a \notin X_n$, let $a = f(a_1, \ldots, a_n)$. By Lemma 3, $op(t) = op(\sigma_t(f)) = op(\hat{\sigma}_{t'}[\sigma_a(f)]){=}\, op(\hat{\sigma}_{t'}[f(a_1, \ldots, a_n)]){=}\, op(t'(\hat{\sigma}_{t'}[a_1], \ldots, \hat{\sigma}_{t'}[a_n])) \geq op(t')$. Similarly, since $\sigma_{t'}{=}\sigma_t\sigma_b$, $op(t') \geq op(t)$. Hence $op(t) = op(t')$. Consequently, $op(a) = 1$ and $op(b) = 1$.

Setting $a = f(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ when α is a mapping on $\{1, 2, 3, \ldots, n\}$. Since $t = \hat{\sigma}_{t'}[a]$ and $var(t) = X_n$, α is surjective. And, then α is bijective. Hence $\sigma_\alpha \in PH_n \cap S^1$. This completes the proof. □

Proposition 6. *Let S be a subsemigroup of $Reg(f,n)$ and let $\sigma_t, \sigma_{t'} \in S$. Then $\sigma_t \mathcal{L}^S \sigma_{t'}$ if and only if $\sigma_t = \sigma_\alpha\sigma_{t'}$ for some $\sigma_\alpha \in PH_n \cap S^1$.*

Proof. Assume at first that $\sigma_t = \sigma_\alpha\sigma_{t'}$ for some $\sigma_\alpha \in PH_n \cap S^1$. Then $\sigma_{\alpha^{-1}}\sigma_{t'}{=}\sigma_{t'}$. Since $\alpha^{-1} \in PH_n \cap S^1$, $\sigma_t \mathcal{L}^S \sigma_{t'}$.

Conversely, assume that $\sigma_t \mathcal{L}^S \sigma_{t'}$. Then there exist $\sigma_a, \sigma_b \in S^1$ such that $\sigma_t = \sigma_a\sigma_{t'}$ and $\sigma_{t'} = \sigma_b\sigma_t$. Thus $op(t) = op(\sigma_t(f)){=}op(\hat{\sigma}_a[\sigma_{t'}(f)]) {=}op(\hat{\sigma}_a[t']) \geq op(t')$. Similarly, $op(t') \geq op(t)$ because $\sigma_{t'}{=}\sigma_b\sigma_t$. Hence $op(t) = op(t')$. Consequently, $op(a) = 1$ and $op(b) = 1$. We conclude $\sigma_a{=}\sigma_{f(x_{\alpha(1)},\ldots,\alpha(n))}$ for some α which is bijective. □

Proposition 7. *Let S be a subsemigroup of $Reg(f,n)$ and let $\sigma_t, \sigma_{t'} \in Reg(f,n)$. Then $\sigma_t \mathcal{J}^S \sigma_{t'}$ if and only if $\sigma_t = \sigma_\beta\sigma_{t'}\sigma_\alpha$ for some $\sigma_\beta, \sigma_\alpha \in PH_n \cap S^1$.*

Proof. Assume at first that $\sigma_t = \sigma_\beta\sigma_{t'}\sigma_\alpha$ for some $\sigma_\beta, \sigma_\alpha \in PH_n \cap S^1$. Then $\sigma_{\beta^{-1}}\sigma_{t'}\sigma_{\alpha^{-1}}{=}\sigma_{t'}$. Since $PH_n \cap S^1$ is a group, $\sigma_t \mathcal{J}^S \sigma_{t'}$.

Conversely, assume that $\sigma_t \mathcal{J}^S \sigma_{t'}$. Then there exist σ_a, σ_b, σ_c, $\sigma_d \in S^1$ such that $\sigma_t = \sigma_a\sigma_{t'}\sigma_b$ and $\sigma_{t'} = \sigma_c\sigma_t\sigma_d$. Set $b = f(b_1, \ldots, b_n)$. Since $\sigma_a \in Reg(f,n)$, $op(t) = op(\sigma_t(f)) = op(\hat{\sigma}_a[\hat{\sigma}_{t'}[\sigma_b(f)]]) \geq op(\hat{\sigma}_{t'}[b]) = op(\hat{\sigma}_{t'}\ [f(b_1, \ldots, b_n)]) = op(t'(\hat{\sigma}_{t'}\ [b_1], \ldots, \hat{\sigma}_{t'}[b_n])) > op(t')$. Similarly, $op(t') \geq op(t)$ because $\sigma_{t'}{=}\sigma_c\sigma_t\sigma_d$ and $\sigma_d \in Reg(f,n)$. Hence $op(t) = op(t')$. Consequently, $op(a) = 1$ and $op(b) = 1$.

Setting $a = f(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ and $b = f(x_{\beta(1)}, \ldots, x_{\beta(n)})$ with α, β are mappings on $\{1, \ldots, n\}$. Since $\sigma_t = \sigma_a\sigma_{t'}\sigma_b$ and $var(t) = X_n, \alpha$ and β are surjective. So, α and β are bijective. □

Proposition 8. *Let S be a subsemigroup of $Reg(f,n)$. Then $\mathcal{D}^S = \mathcal{J}^S$.*

Proof. It is enough to show that $\mathcal{D}^S \supseteq \mathcal{J}^S$. Let $\sigma_t, \sigma_{t'} \in Reg(f,n)$ be such that $\sigma_t \mathcal{J}^S \sigma_{t'}$. By Proposition 7, there exist $\sigma_\alpha, \sigma_\beta \in PH_n \cap S^1$ such that $\sigma_t{=}\sigma_\beta\sigma_{t'}\sigma_\alpha$. It follows that $\sigma_{\alpha^{-1}}\sigma_{t'} = \sigma_t\sigma_\beta$. Since $\sigma_t \mathcal{R}^S \sigma_t\sigma_\beta$ and $\sigma_{\alpha^{-1}}\sigma_{t'} \mathcal{L}^S \sigma_{t'}$, $\sigma_t \mathcal{D}^S \sigma_{t'}$. □

If A and B are subsets of a semigroup S, then we write AB to mean $\{ab : a \in A, b \in B\}$.

Theorem 9. *Let S be a subsemigroup of $Reg(f,n)$.*

(1) *A semigroup S is a $\mathcal{G}$-subsemigroup of $Reg(f,n)$ if and only if* (i), (ii) *and* (iii) *are fulfilled:*

   (i) $S(PH_n \backslash S) \subseteq (Reg(f,n) \backslash S)$,

   (ii) $(PH_n \backslash S)S \subseteq (Reg(f,n) \backslash S)$,

   (iii) $(PH_n \backslash S)S(PH_n \backslash S) \subseteq (Reg(f,n) \backslash S)$.

(2) *A subsemigroup of $Reg(f,n)$ containing PH_n is a $\mathcal{G}$-subsemigroup of $Reg(f,n)$.*

(3) *A subsemigroup of $Reg(f,n)$ contained in PH_n is a $\mathcal{G}$-subsemigroup of $Reg(f,n)$.*

Proof. (1) Assume at first that a semigroup S is a $\mathcal{G}-$subsemigroup of $Reg(f,n)$. We will show that condition (i) holds. Suppose not. Then there exist $\sigma_t \in S$ and $\sigma_\alpha \in PH_n \backslash S$ such that $\sigma_t\sigma_\alpha \in S$. Since $(\sigma_t, \sigma_t\sigma_\alpha) \in \mathcal{R}^{Reg(f,n)} \cap (S \times S) = \mathcal{R}^S$, by Proposition 5, $\sigma_\alpha \in S$. This is a contradiction. Therefore, (i) holds. Similarly, by using Proposition 6 and Proposition 7 we obtain $\mathcal{L}^{Reg(f,n)} \cap (S \times S) = \mathcal{L}^S$ and $\mathcal{J}^{Reg(f,n)} \cap (S \times S) = \mathcal{J}^S$, respectively. Thus (ii) and (iii) hold.

Conversely, assume that the conditions (i), (ii) and (iii) hold. It is enough to show that $\mathcal{R}^{Reg(f,n)} \cap (S \times S) \subseteq \mathcal{R}^S$. Let $(\sigma_t, \sigma_{t'}) \in \mathcal{R}^{Reg(f,n)} \cap (S \times S)$. By Proposition 5, $\sigma_{t'} = \sigma_t\sigma_\alpha$ for some $\sigma_\alpha \in PH_n$. Suppose that $\sigma_\alpha \notin S$. By assumption (i), $\sigma_{t'} = \sigma_t\sigma_\alpha \notin S$, which is a contradiction. Thus $\sigma_\alpha \in S$, therefore $(\sigma_t, \sigma_{t'}) \in \mathcal{R}^S$. Similarly, by using assumptions (ii) and (iii), we obtain $\mathcal{L}^{Reg(f,n)} \cap (S \times S) = \mathcal{L}^S$ and $\mathcal{J}^{Reg(f,n)} \cap (S \times S) = \mathcal{J}^S$, respectively. Hence S is a $\mathcal{G}$-subsemigroup of $Reg(f,n)$. Using Proposition 8 and (1), we obtain (2) and (3). □

Example. We consider a semigroup $Reg(f,3)$. Let $S := \langle \sigma_{(12)}, \sigma_t \rangle$ where $t = f(f(x_1,x_2,x_3), f(x_1,x_2,x_3), f(x_1,x_2,x_3))$. Clearly, S is a subsemigroup of $Reg(\{f,3\})$ and $S \cap PH_3 = \{\sigma_{(12)}, \sigma_{(1)}(=\sigma_{id})\}$. It is routine matter to check that the condition (i), (ii) and (iii) hold. Hence S is a $\mathcal{G}$-subsemigroup of $Reg(\{f,3\})$.

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