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SEMI-HEREDITARILY HYPERCYCLICITY FOR A TUPLE OF OPERATORS

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Abstract: In this paper we characterize the equivalent conditions for a tuple of commutative bounded linear operators, satisfying the hypercyclicity criterion.

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1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let $\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}, ..., {T_n}^{k_n} : k_i \in \mathbb{Z}_+, \ \ i = 1, ..., n\}$ be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, by $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}$, i.e. $\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}\}$. We say that $\mathcal{T}_d^{(k)}$ is hypercyclic provided there exist $x_1, ..., x_k \in X$ such that $\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\}$ is dense in the k copies of X, $X \oplus ... \oplus X$.

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For simplicity we state and prove our results for a pair that is a tuple with $n = 2$, and the general case follows by a similar method. Note that if T_1, T_2 are commutative bounded linear operators on a Banach space X, and $\{m_j\}$, $\{n_j\}$ are two sequences of natural numbers, then we say $\{{T_1}^{m_j}{T_2}^{n_j} : j \geq 0\}$ is hypercyclic if there exists $x \in X$ such that $\{{T_1}^{m_j}{T_2}^{n_j}x : j \geq 0\}$ is dense in X.

Definition 1.2. A pair (T_1, T_2) is called topologically mixing if for any given open sets U and V, there exist two positive integers M and N such that $T_1^m T_2^n(U) \cap V \neq \emptyset$ for all $m \geq M$ and $n \geq N$.

Definition 1.3. We say that a pair $\mathcal{T} = (T_1, T_2)$ is hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences $(\{m_k\}, \{n_k\})$ of integers provided for all pair of subsequences $(\{m_{k_j}\}, \{n_{k_j}\})$ of $(\{m_k\}, \{n_k\})$, the sequence $\{{T_1}^{m_{k_j}}{T_2}^{n_{k_j}} : j \geq 1\}$ is hypercyclic. We say that a pair $\mathcal{T}$ is hereditarily hypercyclic, if it is hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences.

Definition 1.4. We say that a pair $\mathcal{T} = (T_1, T_2)$ is semi-hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences $(\{m_k\}, \{n_r\})$ of integers provided for all pair of subsequences $(\{m_{k_i}\}, \{n_{r_j}\})$ of $(\{m_k\}, \{n_r\})$, the sequence $\{{T_1}^{m_{k_i}}{T_2}^{n_{r_j}} : i, j \geq 1\}$ is hypercyclic. We say that a pair $\mathcal{T}$ is semi-hereditarily hypercyclic, if it is semi-hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences.

Clearly, if a pair $\mathcal{T} = (T_1, T_2)$ is hereditarily hypercyclic, then it is semi-hereditarily hypercyclic.

Definition 1.5. An strictly increasing sequence of positive integers $\{n_k\}$ is said to be syndetic if $\sup_n\{n_{k+1} - n_k\} < \infty$.

The formulation of the hypercyclicty criterion in the next section was given by N. S. Feldman ([4]). Here, we want to extend some properties of hypercyclic operators to a pair of commuting operators, and although the techniques work for any n-tuple of operators but for simplicity we prove our results only for the case $n = 2$. For some other topics we refer to [1–16].

2. Main Results

In this section we characterize the equivalent conditions for a pair of operators, satisfying the hypercyclicity criterion.

Theorem 2.1. (The Hypercyclicity Criterion for a Tuple) *Suppose X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ is a pair of* *continuous linear mappings on X. If there exist two dense subsets Y and Z in X, and a pair of strictly increasing sequences $\{m_j\}$ and $\{n_j\}$ such that:*

1. $T_1^{m_j}T_2^{n_j} \to 0$ on Y as $j \to \infty$,

2. There exists a sequence of function $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_j}T_2^{n_j}S_j z \to z$, then $\mathcal{T}$ is a hypercyclic tuple.

Theorem 2.2. (see [18]) *Let $\mathcal{T}$ be a pair of operators T_1 and T_2 on the an infinite dimensional Banach space X. Also, let T_i^* has no eigenvalues for $i = 1, 2$. Then the followings are equivalent:*

(i) $\mathcal{T}_d^{(2)}$ is hypercyclic.

(ii) for every nonempty open subsets U, V of X, there exists a pair of integers (m, n) such that $T_1^m T_2^n(U) \cap V \neq \emptyset$ and $T_1^{m+1}T_2^{n+1}(U) \cap V \neq \emptyset$.

(iii) there exists a positive integer p such that for any nonempty open subsets U, V of X, there exists a pair of integers (m, n) such that $T_1^m T_2^n(U) \cap V \neq \emptyset$ and $T_1^{m+p}T_2^{n+p}(U) \cap V \neq \emptyset$.

Theorem 2.3. *Let $\mathcal{T}$ be a pair of operators T_1 and T_2 on the an infinite dimensional Banach space X. Also, let T_i^* has no eigenvalues for $i = 1, 2$. Then the followings are equivalent:*

(i) $\mathcal{T}_d^{(2)}$ is hypercyclic.

(ii) for every nonempty open subsets U, V of X and every neighborhood W of 0, there exist integers m and n such that$T_1^m T_2^n(U) \cap W \neq \emptyset$ and $T_1^m T_2^n(W) \cap V \neq \emptyset$.

Proof. (i) implies (ii): Let (U, V) be a pair of nonempty open subsets of X and W be a neighborhood of 0. Put $U_1 = U$, $V_1 = U_2 = W$ and $V_2 = V$. Since $\mathcal{T}_d^{(2)}$ is hypercyclic, there exists a tuple of nonnegative integers (m, n) such that $T_1^m T_2^n(U_1) \cap V_1 \neq \emptyset$ and $T_1^m T_2^n(U_2) \cap V_2 \neq \emptyset$. This proves (iii).

(ii) implies (i): In order to show that $\mathcal{T}_d^{(2)}$ is hypercyclic, we will show that the assertion (ii) in Theorem 2.2 holds. For this let (U, V) be any pair of nonempty open subsets of X. Also, let W be any neighborhood of 0. In assertion (ii) of the theorem substitute W by $W \cap T_1^{-1}T_2^{-1}(W)$. Then there exists a pair of integers (m, n) such that $T_1^m T_2^n(U) \cap (W \cap T_1^{-1}T_2^{-1}(W)) \neq \emptyset$ and $T_1^m T_2^n(W \cap T_1^{-1}T_2^{-1}(W)) \cap V \neq \emptyset$. Hence $T_1^i T_2^j(U) \cap W \neq \emptyset$ and $T_1^i T_2^j(W) \cap V \neq \emptyset$ for $(i, j) \in \{(m, n), (m+1, n+1)\}$. Now let u, v be arbitrary elements in U and V, respectively. Choose $k_0 \in \mathbb{N}$ such that $B(u, \frac{1}{k_0}) \subset U$ and $B(v, \frac{1}{k_0}) \subset V$. For $k \geq k_0$ put $U_k = B(u, \frac{1}{k})$, $V_k = B(v, \frac{1}{k})$ and $W_k = B(0, \frac{1}{k})$. Then there exists a pair of integers (m_k, n_k) such that $T_1^i T_2^j(U_k) \cap W_k \neq \emptyset$ and $T_1^i T_2^j(W_k) \cap V_k \neq \emptyset$ for $(i, j) \in \{(m_k, n_k), (m_k + 1, n_k + 1)\}$. So there exist $u_k, u'_k \in U_k$, $v_k, v'_k \in V_k$ and $w_k, w'_k \in W_k$ such that $u_k, u'_k \longrightarrow u$, $w_k, w'_k \longrightarrow 0$, $T_1^{m_k} T_2^{n_k} u_k \longrightarrow 0$, $T_1^{m_k+1} T_2^{n_k+1} w_k \longrightarrow v$, $T_1^{m_k+1} T_2^{n_k+1} u'_k \longrightarrow 0$, and $T_1^{m_k+1} T_2^{n_k+1} w'_k \longrightarrow v$. Hence, $u_k + w_k \longrightarrow u$ and so $T_1^{m_k} T_2^{n_k}(u_k + v_k) \longrightarrow v$. Also, $u'_k + w'_k \longrightarrow u$ and so $T_1^{m_k+1} T_2^{n_k+1}(u'_k + v'_k) \longrightarrow v$. Thus the sets $T_1^{m_k} T_2^{n_k}(U) \cap V$ and $T_1^{m_k+1} T_2^{n_k+1}(U) \cap V$ are both nonempty for k large enough and the proof is complete. □

Proposition 2.4. *Let $\mathcal{T}$ be a pair of operators T_1 and T_2 on the an infinite dimensional Banach space X. Then $\mathcal{T} = (T_1, T_2)$ is semi-hereditarily hypercyclic with respect to a pair of increasing sequences of nonnegative integers $(\{m_k\}, \{n_r\})$ if and only if for all given any two open sets U, V, there exists a pair of positive integers (M, N) such that $T_1^{m_k} T_2^{n_r}(U) \cap V \neq \emptyset$ for any $k > M$ and $r > N$.*

Proof. Let $\mathcal{T} = (T_1, T_2)$ be semi-hereditarily hypercyclic with respect to a pair of increasing sequences of nonnegative integers $(\{m_k\}, \{n_r\})$. Suppose that there exist some open sets U, V such that for all i, j, $T_1^{m_{k_i}} T_2^{n_{r_j}}(U) \cap V = \emptyset$ for some pair of subsequences $(\{m_{k_i}\}, \{n_{r_j}\})$ of $(\{m_k\}, \{n_r\})$. Since $\mathcal{T}$ is semi-hereditarily hypercyclic with respect to $(\{m_k\}, \{n_r\})$, thus $\{T_1^{m_{k_i}} T_2^{n_{r_j}} : i, j \geq 0\}$ is hypercyclic and so we get a contradiction.

Conversely, suppose that $\{m_{k_i}\}$ and $\{n_{r_j}\}$ are arbitrary subsequences of $\{m_k\}$ and $\{n_r\}$ respectively. Let U, V be open sets in X, so there exists a pair of positive integers (M, N) such that $T_1^{m_k} T_2^{n_r}(U) \cap V \neq \emptyset$ for any $k > M$ and $r > N$. This implies clearly that $T_1^{m_{k_i}} T_2^{n_{r_j}}(U) \cap V \neq \emptyset$ for any $i > M$ and $j > N$. Now the proof is complete. □

Theorem 2.5. *Let $\mathcal{T}$ be a pair of operators T_1 and T_2 on the an infinite dimensional Banach space X. Then $\mathcal{T} = (T_1, T_2)$ is semi-hereditarily hypercyclic with respect to a pair of increasing syndetic sequences of nonnegative integers if and only if $\mathcal{T}$ is topologically mixing.*

Proof. Suppose that $\mathcal{T} = (T_1, T_2)$ is semi-hereditarily hypercyclic with respect to a pair of increasing syndetic sequences of nonnegative integers $(\{m_k\}, \{n_r\})$. Let U and V be two nonempty open sets in X. We will show that there exist integers M, N such that $T_1^m T_2^n(U) \cap V \neq \emptyset$ for any $m > M$ and $n > N$. Put $M = sup\{m_{k+1} - m_k : k \geq 0\}$ and $N = sup\{n_{r+1} - n_r : r \geq 0\}$. For all $i = 0, ..., M$ and $j - 0, ..., N$ define $U_{ij} = U$ and $V_{ij} = T_1^{-i} T_2^{-j}(V)$. Thus there exist integers M_{ij} and N_{ij} such that for any $k > M_{ij}$ and $r > N_{ij}$ for all $i = 0, ..., M$ and $j = 0, ..., N$, we have $T_1^{m_k} T_2^{n_r}(U_{ij}) \cap V_{ij} \neq \emptyset$ that is also holds for any $k > m_{M_{ij}}$ and $r > n_{N_{ij}}$. Put $M_o = max\{m_{M_{ij}} : i = 0, 1, 2, ..., M \ j = 0, 1, 2, ..., N\}$ and $N_o = max\{n_{N_{ij}} : i = 0, 1, 2, ..., M \ j = 0, 1, 2, ..., N\}$. Let $M_o = m_{k_0}$ and $N_o = m_{r_0}$. Now if $m > M_0$ and $n > N_0$, there exist $k > k_0$, $r > r_0$, $0 \leq i \leq M$ and $0 \leq j \leq N$ such that $m = m_k + i$ and $n = n_r + j$. Thus we get

$$\begin{aligned} T_1^m T_2^n(U) \cap V &= T_1^{m_k+i} T_2^{n_r+j}(U_{ij}) \cap V = T_1^{m_k} T_2^{n_r}(U_{ij}) \cap T_1^{-i} T_2^{-j}(V) \\ &= T_1^{m_k} T_2^{n_r}(U_{ij}) \cap V_{ij} \neq \emptyset. \end{aligned}$$

Conversely if $\mathcal{T}$ is topologically mixing, then by Proposition 2.4, $\mathcal{T}$ is semi-hereditarily hypercyclic with respect to the pair of entire sequences and so the proof is complete. □

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