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AN OVERVIEW OF SEPARATION AXIOMS IN RECENT RESEARCH

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Abstract: The aim of this paper is to exhibit the recent research on separation axioms, T_S-space, pairwise T_S-space, semi star generalized W-$T_{\frac{1}{2}}$ space, pairwise semi star generalized W-$T_{\frac{1}{2}}$ spaces and pairwise complemented spaces and its properties.

AMS Subject Classification: 54E55
Key Words: T_S-space, pairwise T_S-space, semi star generalized W-$T_{\frac{1}{2}}$ space, pairwise semi star generalized W-$T_{\frac{1}{2}}$ spaces

1. Introduction

Separation axioms are one among the most common, important and interesting concepts in Topology. They can be used to define more restricted classes of topological spaces. The separation axioms of topological spaces are usually denoted with the letter "T" after the German "Trennung" which means separation. Most of the weak separation axioms are defined in terms of generalized closed sets and their definitions are deceptively simple. However, the structure and the properties of those spaces are not always that easy to comprehend.

The separation axioms that were studied together in this way were the axioms for Hausdorff spaces, regular spaces and normal spaces. Separation

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axioms and closed sets in topological spaces have been very useful in the study of certain objects in digital topology [1, 2]. Khalimsky, Kopperman and Meyer[3] proved that the digital line is a typical example of $T_{\frac{1}{2}}$-spaces. There were many definitions offered, some of which assumed to be separation axioms before the current general definition of a topological space.

For example, the definition given by Felix Hausdorff in 1914 is equivalent to the modern definition plus the Hausdorff separation axiom. The first step of generalized closed sets was done by Levine in 1970 [5] and he initiated the notion of $T_{\frac{1}{2}}$-spaces in unital topology which is properly placed between T_0-space and T_1-spaces by defining $T_{\frac{1}{2}}$-space in which every generalized closed set is closed. After the works of Levine on semi open sets, several mathematicians turned their attention to the generalizations of various concepts of topology by considering semi open sets instead of open sets. When open sets are replaced by semi open sets, new results were obtained. Consequently, many separation axioms has been defined and studied.

Meanwhile, in 1963, Kelly [20] introduced the concept of bitopological spaces by using quasi metric space as a natural structure. This spaces is a richer in structure than that of topological spaces and it is much of use in the study of generalizations of topological notions and implications in bitopological situation. Then he initiated the study of separation properties for bitopological spaces and introduced the terms pairwise Hausdorff, pairwise regular, pairwise completely regular and pairwise normal spaces. After this, researchers are turned their interest to extend the concepts of topological spaces into bitopological settings. In recent years many separation axioms and generalizations of closed sets have been developed by various authors.

Thus, the study of topological invariants is the prime objective of the Topology. Keeping this in mind several authors invented new separation axioms. The present paper is not an exception to this trend. The main focus of the present paper is to focus the new separation axioms in both unital and bitopological spaces and study their properties.

In the year 2000, the concepts of semi star generalized open and semi star generalized closed sets were introduced and studied by K. Chandrasekhara Rao and K. Joseph [28] in topological spaces. It was observed that every closed set is a semi star generalized closed set. Further K. Chandrasekhara Rao extended the concepts to bitopological spaces [29] in 2005. No separation axiom has been defined so far by using the concept of semi star generalized closed sets in topological and bitopological spaces. So the recent research focuses its attention on the definition of a new class of separation axioms using the above mentioned sets in topological and bitopological spaces.

After the work of Hdeib on w-closed sets in 1992 [16], gw-closed sets, rgw-closed sets were introduced and discussed by K.Y. Al-Zoubi [18] and Ahmad Al Omari [19] in 2005 and 2007 respectively. They also provided separation axioms, namely, gw-$T_{\frac{1}{2}}$-spaces and rgw-$T_{\frac{1}{2}}$-spaces using gw-closed set and rgw-closed set. It is identified that no generalization of closed sets using w-closed set and s^*g-closed set has been made so far. Therefore by defining new closed set using s^*g-closed set and w-closed set another class of separation axioms is propounded in unital and bitopological spaces.

2. T_S-Spaces

In this section we established the new separation axiom, which defines a new topological space, called T_S-space [(1).Chandrasekhara Rao, K. and Narasimhan, D., T_S-Spaces, Proc. Nat. Acad. Sci., 77(2A) (2007), 363-366, (2).Chandrasekhara Rao, and Narasimhan, D., Characterizations of T_S-Spaces, International Mathematical Forum, 4(21) (2008), 1033-1037.] with the help of s^*g-closed sets. Also some important topological properties namely, closed hereditary property and productive property are discussed for T_S-spaces. A necessary and sufficient condition for a topological space to be a T_S-space is also obtained. Further some relations between T_S-space and the already known topological spaces are established.

Definition 2.1. A topological space (X, τ) is called T_S-space if every s^*g-closed set is closed in X.

Example 2.2. Let $X = \{a, b, c\}, \tau = \{\phi, X, \{a\}\}$. Then s^*g -closed sets in X are $\phi, X, \{b, c\}$, which are closed in X. Hence (X, τ) is T_S-space.

Now, we begins with the topological properties (topological invariants).

Theorem 2.3. (1) Let (X, τ) be a T_S-space. If Y is a closed subspace of X, then (Y, τ_Y) is a T_S-space. (i.e) T_S-space is closed hereditary.

(2) Let I be an index set. Let $\{X_i, i \in I\}$ be a T_S-space. Then their product $X = \prod X_i$ is also a T_S-space. (i.e) T_S-space is productive.

(3) The image of a T_S-space under a continuous, open bijective map $f \colon X \to Y$ is a T_S-space, where Y is another topological space.

A necessary and sufficient condition for a topological space X to be a T_S-space is now characterized through the singleton sets.

Theorem 2.4. A topological space (X, τ) is T_S-space if and only if the singleton $\{x\}$ is either open or semi closed.

K. Chandrasekhara Rao and K. Joseph proved that s^*g-closed sets are closed under finite union and s^*g-open sets are closed under finite intersection. They proposed two problems regarding the intersection of two s^*g-closed sets and the union of two s^*g-open sets. By way of answering these problems, the following interesting results are obtained.

In a T_S-space,

(a) the intersection of two s^*g-closed sets is s^*g-closed,

(b) the union of two s^*g-open sets is s^*g-open.

Now, we shall try to relate it to the concepts that have been previously introduced. Now, it is convenient to establish the relation between the topological spaces with the existing separation axioms and the relatively new T_S-spaces.

(a) Every $T_{\frac{1}{2}}$-space is a T_S-space.

(b) Every T_b-space is a T_S-space.

(c) Every ${}_{\alpha}T_b$-space is a T_S-space.

(d) Every door space is a T_S-space.

The converses of the assertions of the above theorem are not true in general. That can be verified from the previous example. In a topological space every s^*g-closed set is gs-closed and sg-closed but not reversible in general. The following theorem shows that the reverse is true in T_b-spaces.

(a) Every gs closed set in a T_b-space is s^*g-closed.

(b) Every sg-closed set in a T_b -space is s^*g-closed.

(c) Every subset of a complemented T_b-space is s^*g -closed.

(d) Every subset of a complemented $T_{\frac{1}{2}}$-space is s^*g closed.

Theorem 2.5. Every αg-closed set in a ${}_{\alpha}T_b$-space is s^*g-closed.

From the above theorem, we observe that, Every subset of a complemented αT_b-space is s^*g-closed.

Theorem 2.6. (a) If (X, τ) is both T_p^*-space and ${}^*T_p$-space then X is a T_S-space.

(b) If (X,τ) is both T_c-space and $T^*_{\frac{1}{2}}$-space then X is a T_S-space.

Theorem 2.7. If X is complemented T_S-space, then X is a $T_{\frac{1}{2}}$-space.

Theorem 2.8. Let X be a complemented space. Then
1. Every δg-closed set is s^*g-closed in X.
2. For a subset A of X is δg-closed, then A is $\delta\widehat{g}$-closed.

In any topological space (X,τ), every $\delta\widehat{g}$-closed set is s^*g-closed. Then we obtain the following.

Theorem 2.9. (1) Every s^*g-closed set is closed in a T_S-space.

(2) Every $\delta\widehat{g}$-closed set is closed in a T_S-space.

(3) If (X,τ) is both T_S-space and ${}_g\widehat{T}_{\delta g}$ -space then X is a $T_{\frac{1}{2}}$-space.

Theorem 2.10. (a) Every ${}_\alpha T_c$-space X is a T_S-space if X is $T^*_{\frac{1}{2}}$-space.

(b) Every $T_{\alpha gs}$-space X is a T_S-space.

(c) If (X,τ) is a T_d-space and $T_{\frac{1}{2}}$-space then X is a T_S-space.

(d) If (X,τ) is both ${}^\Omega T_{\frac{1}{2}}$-space and $T^\Omega_{\frac{1}{2}}$-space then X is a T_S-space.

(e) If (X,τ) is both T_c-space and $T^\Omega_{\frac{1}{2}}$-space. Then X is a T_S-space.

(f) Suppose X is a ${}^\sharp T_b$-space and $T^\sharp_b$-space. Then X is a T_S-space.

(g) Suppose X is a ${}^\sharp T_c$-space and $T^\sharp_{\frac{1}{2}}$-space. Then X is a T_S-space.

(h) Suppose X is a ${}_{gs}T^\sharp_c$-space and ${}_{gs}T^\sharp_{\frac{1}{2}}$-space. Then X is a T_S-space.

(i) Suppose X is a ${}^\sharp T_{\frac{1}{2}}$-space and $T^\sharp_{\frac{1}{2}}$-space. Then X is a T_S-space.

3. Pairwise T_S-Spaces

As a continuation of the previous section, the concept of pairwise T_S-spaces [(1).Chandrasekhara Rao, K. and Narasimhan, D., Pairwise T_S-Spaces, Thai Journal of Mathematics, 6(1) (2008), 1-8, (2).Chandrasekhara Rao, K and Narasimhan, D., Characterizations of pairwise complemented spaces, Antarctica J. Math., No.3(1),(2006),17-27.] is extended to bitopological spaces and its basic properties are discussed. In this section another new separation axiom is defined in bitopological spaces in which $\tau_1\tau_2$-s^*g closed sets and τ_2-closed sets are coincide and $\tau_2\tau_1$-s^*g closed sets and τ_1-closed sets are coincide.

Definition 3.1. A bitopological space (X, τ_1, τ_2) is called a pairwise T_S-space if every $\tau_1\tau_2$-s^*g closed set is τ_2- closed in X and every $\tau_2\tau_1$-s^*g closed set is τ_1- closed in X.

Example 3.2. Let $X = \{a, b, c\}$, $\tau_1 =\{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{a\}, \{a, c\}\}$. Then (X, τ_1, τ_2) is a pairwise T_S-space.

Theorem 3.3. Let (X, τ_1, τ_2) be a pairwise T_S-space.
(a) If Y is a τ_2-closed subspace of X, then $(Y, \tau_{1/Y}, \tau_{2/Y})$ is a $\tau_1\tau_2$-T_S-space and
(b)If Y is a τ_1-closed subspace of X, then $(Y, \tau_{1/Y}, \tau_{2/Y})$ is a $\tau_2\tau_1$-T_S-space

In the next theorem we establish that a pairwise T_S-space is productive.

Theorem 3.4. Let I be a index set. Let $\{X_i, i \in I)$ be a pairwise T_S-space. Then their product $X =\prod X_i$ is a pairwise T_S-space.

Theorem 3.5. The inverse image of a $\tau_1\tau_2$-s^*g closed set under a pairwise continuous, pairwise open bijective map f: $X \to Y$ is $\tau_1\tau_2$-s^*g closed, where X and Y are bitopological spaces.

Theorem 3.6. The image of a pairwise T_S-space under a pairwise continuous, pairwise open bijective map $f : X \to Y$ is a pairwise T_S-space, where X and Y are bitopological spaces.

Theorem 3.7. In a pairwise T_S-space,
(a) the intersection of two $\tau_1\tau_2$-s^*g closed sets is $\tau_1\tau_2$-s^*g closed,
(b) the union of two $\tau_1\tau_2$-s^*g open sets is $\tau_1\tau_2$-s^*g open,
(c) every pairwise $T_{\frac{1}{2}}$-space is a pairwise T_S-space,
(d) every pairwise T_b-space is a pairwise T_S-space,
(e) every pairwise ${}_{\alpha}T_b$-space is a pairwise T_S-space,
(f) every pairwise door space is a pairwise T_S-space.

However the converses of the assertions of the above theorem are not true as can be seen from the following example:

Example 3.8. In Example 3.2, (X, τ_1, τ_2) is a pairwise T_S-space but not a pairwise $T_{\frac{1}{2}}$-space, pairwise T_b-space, pairwise ${}_{\alpha}T_b$-space or a pairwise door space.

Theorem 3.9. If (X, τ_1, τ_2) is both pairwise T_p^*-space and pairwise ${}^*T_p$-space then X is a pairwise T_S-space.

The following theorem yields the necessary and sufficient condition for a bitopological space X to be a pairwise T_S-space.

Theorem 3.10. A bitopological space (X, τ_1, τ_2) is a pairwise T_S-space if and only if the singleton $\{x\}$ is either τ_i-open or τ_j-semi closed, $i, j = 1, 2$ and $i \neq j$.

Theorem 3.11. If a bitopological space (X, τ_1, τ_2) is pairwise T_c-space and (X, τ_i) is $T^*_{\frac{1}{2}}$-space, $i = 1, 2$, then X is a pairwise T_S-space.

Theorem 3.12. In any bitopological space (X, τ_1, τ_2), every $\tau_i\tau_j$-$\delta\widehat{g}$ closed set is $\tau_i\tau_j$-s^*g closed, $i, j = 1, 2$ and $i \neq j$.

Since every $\tau_i\tau_j$-s^*g closed set is τ_j-closed in a pairwise T_S-space, every $\tau_i\tau_j$-$\delta\widehat{g}$ closed set is τ_j-closed in a pairwise T_S-space, $i, j = 1, 2$ and $i \neq j$.

Theorem 3.13. (a) Every pairwise ${}_g\widehat{T}_{\delta g}$ -space (X, τ_1, τ_2) is pairwise $T_{\frac{1}{2}}$-space if it is pairwise T_S-space.

(b) Every pairwise ${}_\alpha T_c$-space X is a pairwise T_S-space if (X, τ_i) is $T^*_{\frac{1}{2}}$-spaces, $i = 1, 2$.

(c) Every pairwise $T_{\alpha gs}$-space X is a pairwise T_S-space.

(d) If (X, τ_1, τ_2) is a pairwise T_d-space and pairwise $T_{\frac{1}{2}}$-space then X is a pairwise T_S-space.

(e) If (X, τ_1, τ_2) is both pairwise ${}^\Omega T_{\frac{1}{2}}$ -space and pairwise $T^\Omega_{\frac{1}{2}}$-space then X is a pairwise T_S-space.

(f) If (X, τ_1, τ_2) is both pairwise T_c-space and pairwise $T^\Omega_{\frac{1}{2}}$-space, then X is a pairwise T_S-space.

4. Semi Star Generalized w-$T_{\frac{1}{2}}$-Spaces

Now, we introduce the concept of semi star generalized w-closed sets [Chandrasekhara Rao, K. and Narasimhan, D., Semi star generalized w-closed sets, South East Asian j. Math. & Math. Sc., Vol.8 No.1 (2009), 31-38.] in topological spaces and we prove that an arbitrary union of s^*gw-closed sets is s^*gw-closed set and an arbitrary intersection of s^*gw-open sets is s^*gw-open. Also a necessary and sufficient condition for a set A of X to be a s^*gw-closed set is obtained. Further a new separation axiom which defines semi star generalized

w-$T_{\frac{1}{2}}$-space, is introduced with the help of the above mentioned closed sets and different types of s^*gw-continuity [Chandrasekhara Rao, K. and Narasimhan, D, Semi Star Generalized w-continuity in topological spaces, Journal of Advanced Research in Pure Mathematics, Vol.1 (1), (2009) 23-28.] are studied.

Definition 4.1. A set A of a topological space (X, τ) is called semi star generalized w-closed (s^*gw-closed) if $cl_w(A) \subseteq U$ whenever$A \subseteq U$ and U is semi open in X.

Example 4.2. Let X be the set of all real numbers $\Re$, $\tau = \{\phi, \Re, \Re - Q\}$, where Q is the set of all rational numbers. Then Q is s^*gw-closed.

Theorem 4.3. *Let (X, τ) be a topological space and $A \subseteq X$. Then the following are true.*

(a) If A is w-closed, then A is s^*gw-closed.,

(b) If A is semi open and s^*gw-closed, then A is w-closed.

(b) If A is s^*gw-closed, then A is gw-closed.

Since, $cl_w(A) \subseteq cl(A)$, we have the following theorem:

Theorem 4.4. *Every s^*g-closed set is s^*gw-closed and every closed set is s^*gw-closed.*

Now we find the condition under which the superset of a s^*gw-closed set to be a s^*gw-closed.

Theorem 4.5. *If A is s^*gw-closed in X and $A \subseteq B \subseteq cl_w(A)$, then B is s^*gw-closed.*

Next we observe that s^*gw-closed sets are closed under finite union.

Theorem 4.6. *If A and B are s^*gw-closed sets then so is $A \cup B$.*

We note that in any topological space, the arbitrary union of $cl(A_i), i \in I$ is contained in closure of arbitrary union of subsets A_i. But the equality holds when the collection $\{A_i, i \in I\}$ is locally finite. Hence we discuss the following.

Theorem 4.7. *The arbitrary union of s^*gw-closed sets $A_i, i \in I$ in a topological space (X, τ) is s^*gw-closed if the family $\{A_i, i \in I\}$ is locally finite.*

Theorem 4.8. *Let $B \subseteq A \subseteq X$ where A is open and s^*gw-closed in X. Then B is s^*gw-closed relative to A if and only if B is s^*gw-closed relative to X.*

Now we establish a necessary and sufficient condition for a set A of X to be s^*gw-closed set.

Theorem 4.9. *A set A is s^*gw-closed in X if and only if $cl_w(A) - A$ contains no nonempty semi closed set.*

Corollary 4.10. *Let A be s^*gw-closed. Then A is w-closed if and only if $cl_w(A) - A$ is semi closed.*

Theorem 4.11. *If A is s^*gw-closed and $A \subseteq B \subseteq cl_w(A)$ then $cl_w(B) - B$ contains no nonempty semi closed set.*

Definition 4.12. A set A of a topological space (X, τ) is called semi star generalized w-open (s^*gw-open) if and only if A^C is s^*gw-closed.

Example 4.13. In example 4.2, $\Re - Q$ is s^*gw-open.

Theorem 4.14. *A set A is s^*gw-open if and only if $F \subseteq int_w(A)$ whenever F is semi closed and $F \subseteq A$.*

The finite union of two s^*gw-open sets is not s^*gw-open in general. But it is so when both sets are separated. Two sets A and B are said to be separated if $A \cap cl(B) = \phi = cl(A) \cap B$. Then we have, if A and B are separated s^*gw-open sets, then $A \cup B$ is s^*gw-open set. When we think about intersection we obtain, s^*gw-open sets are closed under finite intersection, that is if A and B are s^*gw-open sets, then so is $A \cap B$.

The arbitrary intersection of s^*gw-open sets is s^*gw-open set when the indexed family of s^*gw-open sets is locally finite.

Theorem 4.15. *The arbitrary intersection of s^*gw-open sets $A_i, i \in I$ in a topological space (X, τ) is s^*gw-open if the family $\{A_i^C, i \in I\}$ is locally finite.*

Now we find the condition for the subset of s^*gw-open set to be s^*gw-open.

Theorem 4.16. *If A is s^*gw-open in X and $int_w(A) \subseteq B \subseteq A$, then B is s^*gw-open.*

Theorem 4.17. *A set A is s^*gw-closed in X if and only if $cl_w(A) - A$ is s^*gw-open.*

Theorem 4.18. *If A set A is s^*gw-open in a topological space (X, τ), then $G = X$ whenever G is semi open and $int_w(A) \cup A^C \subseteq G$.*

Theorem 4.19. *The intersection of a s^*gw-open set and w-open set is s^*gw-open.*

The next theorem deals with the product of two s^*gw-open sets in topological spaces.

Theorem 4.20. *If $A \times B$ is s^*gw-open subset of $(X \times Y, \tau \times \sigma)$, then A*

is s^*gw-open subset in (X,τ) and B is s^*gw-open subset in (Y,σ).

A space (X,τ) is a generalized w-$T_{\frac{1}{2}}$ (simply, gw-$T_{\frac{1}{2}}$) if every gw-closed set in (X,τ) is w-closed. A space X is a regular generalized w-$T_{\frac{1}{2}}$ (simply, rgw-$T_{\frac{1}{2}}$) if every rgw -closed set in (X,τ) is w-closed. In continuation, the notion of new topological space, called s^*gw-$T_{\frac{1}{2}}$-space is introduced in unital topology.

Definition 4.21. A space (X,τ) is a semi star generalized w-$T_{\frac{1}{2}}$ (simply, s^*gw-$T_{\frac{1}{2}}$) if every s^*gw-closed set in (X,τ) is w-closed.

Theorem 4.22. *For a space (X,τ), the following are equivalent.*

(a) X is a s^*gw-$T_{\frac{1}{2}}$,

(b) Every singleton is either semi closed or w-open.

Since, every s^*gw-closed set is rgw-closed, gw-closed, we have the following:

Theorem 4.23. *Every rgw-$T_{\frac{1}{2}}$ space is s^*gw-$T_{\frac{1}{2}}$ space and every gw-$T_{\frac{1}{2}}$ space is s^*gw-$T_{\frac{1}{2}}$ space.*

In view of s^*gw-continuity,the following are discussed.

Definition 4.24. A map $f : X \to Y$ is called

(a) s^*gw-continuous if the inverse image of w-closed set in Y is s^*gw-closed in X.

(b) s^*gw-irresolute if the inverse image of s^*gw-closed set in Y is s^*gw-closed in X.

Concerning the composition of functions, the composition of two s^*gw-continuous functions is not s^*gw-continuous in general and we have the following results.

Theorem 4.25. *(a) The composition of two s^*gw-irresolute functions is s^*gw-irresolute, or equivalently, If f, g are s^*gw-irresolute, then gof is also s^*gw-irresolute.*

(b) If f is s^*gw-irresolute and g is s^*gw-continuous, then gof is also s^*gw-continuous.

Theorem 4.26. *a) Every s^*gw-continuous function is gw-continuous,*

b) Every s^*gw-continuous function is rgw-continuous.

Since every w-closed set is s^*gw-closed, we have the following theorem:

Theorem 4.27. *Every s^*gw-irresolute map is s^*gw-continuous map.*

The next theorem shows that s^*gw-$T_{\frac{1}{2}}$ spaces are preserved under s^*gw-irresolute map provided it is also a pre w-closed map.

Theorem 4.28. *Let $f : X \to Y$ be onto s^*gw-irresolute and pre w -closed map. If X is s^*gw-$T_{\frac{1}{2}}$ then Y is also s^*gw-$T_{\frac{1}{2}}$.*

Now, we define a new class of closed map in topological spaces.

Definition 4.29. A map $f : X \to Y$ is called s^*gw-closed if image of a closed set in X is s^*gw-closed in Y.

Theorem 4.30. *a) Every s^*gw-closed function is gw-closed,*

b) Every s^*gw-closed function is rgw-closed.

Note that, A subset $A \subseteq X$ is said to be w-c-closed provided that there is a proper subset B for which $A=cl_w(B)$.
A map $f : X \to Y$ is said to be
(a) gw-c-closed if $f(A)$ is gw-closed in Y for every w-c-closed subset $A \subseteq X$ and
(b) rgw-c-closed if $f(A)$ is rgw-closed in Y for every w-c-closed subset $A \subseteq X$.
Now, it is intended to define another type of closed map, as given below.

Definition 4.31. A map $f : X \to Y$ is called

(a) s^*gw-c-closed if $f(A)$ is s^*gw-closed in Y for every w-c-closed subset $A \subseteq X$.

(b) irresolute map if inverse image of semi closed in Y is semi closed in X.

Theorem 4.32. *Let $f : X \to Y$ be irresolute map and s^*gw-c-closed. Then $f(A)$ is s^*gw-closed in Y for every s^*gw-closed subset A of X.*

Theorem 4.33. *Let $f : X \to Y$ be semi open preserving and w-irresolute map, if B is s^*gw-closed in Y, then $f^{-1}(B)$ is s^*gw-closed in X.*

Theorem 4.34. *Let $f : X \to Y$ be irresolute map and s^*gw-closed and A is s^*g-closed in X. Then $f(A)$ is s^*gw-closed.*

Theorem 4.35. *Let $f : X \to Y$ be irresolute map and pre w-closed and A is s^*gw-closed in X. Then $f(A)$ is s^*gw-closed.*

Definition 4.36. A map $f : X \to Y$ is said to be w-contra-S-map if for every semi open subset V of Y, $f^{-1}(V)$ is w-closed.

Theorem 4.37. *Let $f : X \to Y$ be w-contra-S-map and s^*gw-c-closed. Then $f(A)$ is s^*gw-closed in Y for every subset A of X.*

Theorem 4.38. *If $f : X \to Y$ is s^*gw-continuous {resp. s^*gw-irresolute} and X is s^*gw-$T_{\frac{1}{2}}$ then f is w-continuous {w-irresolute}.*

Theorem 4.39. *A function $f : X \to Y$ is bijective, semi open preserving and s^*gw-continuous map. Then f is s^*gw-irresolute map.*

Theorem 4.40. *Let $f : X \to Y$ is s^*gw-closed if and only if for each subset B of Y and for each w-open set U containing $f^{-1}(B)$, there is an s^*gw-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

5. Pairwise Semi Star Generalized W-$T_{\frac{1}{2}}$-Spaces

In this section we first extend the concept of semi star generalized w-closed sets in bitopological spaces as a continuation of the previous section.

(Chandrasekhara Rao, K. and Narasimhan, D., Semi star generalized w-closed sets in Bitopological Spaces, Int. J. Contemp. Math. Sciences, Vol 4(12) (2009), 587-595.)

Definition 5.1. A set A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-semi star generalized w - closed ($\tau_1\tau_2$ -s^*gw closed) if τ_2-$cl_w(A) \subseteq U$ whenever$A \subseteq U$ and U is τ_1-semi open in X.

Example 5.2. Let X be the set of all real numbers $\Re$, $\tau_1 = \{\phi, \Re, \Re - Q\}$, τ_2=$\{\phi, \Re, Q\}$,where Q is the set of all rational numbers. Then $\Re - Q$ is $\tau_1\tau_2$-s^*gw closed.

Theorem 5.3. *Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. Then the following are true.*

(a) If A is τ_2-w closed, then A is $\tau_1\tau_2$-s^*gw closed.

(b) If A is τ_1-semi open and τ_2-s^*gw closed, then A is τ_2-w closed.

(b) If A is τ_2-s^*gw-closed, then A is gw-closed.

Since, τ_2-$cl_w(A) \subseteq \tau_2$-$cl(A)$, we have the following theorem.

Theorem 5.4. *Every $\tau_1\tau_2$-s^*g closed set is $\tau_1\tau_2$-s^*gw closed and every τ_2-closed set is $\tau_1\tau_2$-s^*gw closed.*

Theorem 5.5. *If A is $\tau_1\tau_2$-s^*gw closed in X and $A \subseteq B \subseteq \tau_2$-$cl_w(A)$, then B is $\tau_1\tau_2$-s^*gw closed.*

If A and B are $\tau_1\tau_2$-s^*gw closed sets then so is $A \cup B$. The arbitrary union of $\tau_1\tau_2$-s^*gw closed sets $A_i, i \in I$ in a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$-s^*gw closed if the family $\{A_i, i \in I\}$ is locally finite in (X, τ_2).

Theorem 5.6. *Let $B \subseteq A \subseteq X$ where A is τ_1-open and $\tau_1\tau_2$-s^*gw closed in X. Then B is $\tau_1\tau_2$-s^*gw closed relative to A if and only if B is $\tau_1\tau_2$-s^*gw closed relative to X.*

From the above theorem we have, if A is $\tau_1\tau_2$-s^*gw closed, τ_1-semi open in X and F is τ_2-w closed in X, then $A \cap F$ is τ_2-w closed in X.

Theorem 5.7. *If a set A is $\tau_1\tau_2$-s^*gw closed in X then τ_2-$cl_w(A) - A$ contains no nonempty τ_1-semi closed set.*

Hence we prove, let A be $\tau_1\tau_2$-s^*gw closed. Then A is τ_2-w-closed if and only if τ_2-$cl_w(A) - A$ is τ_1-semi closed.

Theorem 5.8. *If A is $\tau_1\tau_2$-s^*gw closed and $A \subseteq B \subseteq \tau_2$-$cl_w(A)$ then τ_2-$cl_w(B) - B$ contains no nonempty τ_1-semi closed set.*

Definition 5.9. A set A of a topological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-semi star generalized w open ($\tau_1\tau_2$-s^*gw open) if and only if A^C is $\tau_1\tau_2$-s^*gw closed.

Example 5.10. In example 5.2, Q is $\tau_1\tau_2$-s^*gw open.

Theorem 5.11. *A set A is $\tau_1\tau_2$-s^*gw open if and only if $F \subseteq \tau_2$-$int_w(A)$ whenever F is τ_1 -semi closed and $F \subseteq A$.*

If A and B are separated $\tau_1\tau_2$-s^*gw open sets then $A \cup B$ is a $\tau_1\tau_2$-s^*gw open set.

If A and B are $\tau_1\tau_2$-s^*gw open sets then so is $A \cap B$.

Theorem 5.12. *The arbitrary intersection of $\tau_1\tau_2$-s^*gw open sets $A_i, i \in I$ in a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$-s^*gw open if the family $\{A_i^C, i \in I\}$ is locally finite in (X, τ_2).*

Now we look for the condition under which the subset of $\tau_1\tau_2$-s^*gw open set to be $\tau_1\tau_2$-s^*gw open.

Theorem 5.13. *If A is $\tau_1\tau_2$-s^*gw open in X and τ_2-$int_w(A) \subseteq B \subseteq A$, then B is $\tau_1\tau_2$-s^*gw open.*

Theorem 5.14. *If a set A is $\tau_1\tau_2$-s^*gw closed in X, then τ_2-$cl_w(A) - A$ is $\tau_1\tau_2$-s^*gw open.*

Theorem 5.15. *If A set A is $\tau_1\tau_2$-s^*gw open in a bitopological space (X, τ_1, τ_2), then $G = X$ whenever G is τ_1-semi open and τ_2-$int_w(A) \cup A^C \subseteq G$.*

Theorem 5.16. *The intersection of a $\tau_1\tau_2$-s^*gw open set and τ_1-w open set is always $\tau_1\tau_2$-s^*gw open.*

Theorem 5.17. *If $A\times B$ is $\tau_1\tau_2$-s^*gw open subset of $(X\times Y, \tau_1\times\sigma_1, \tau_2\times\sigma_2)$, then A is $\tau_1\tau_2$-s^*gw open subset in (X, τ_1, τ_2) and B is $\sigma_1\sigma_2$-s^*gw open subset in (Y, σ_1, σ_2).*

Definition 5.18. A space (X, τ_1, τ_2) is a pairwise regular generalized w-$T_{\frac{1}{2}}$ (simply, pairwise rgw-$T_{\frac{1}{2}}$) if every $\tau_1\tau_2$-rgw closed set in (X, τ_1, τ_2) is τ_2-w closed and $\tau_2\tau_1$-rgw closed set in (X, τ_1, τ_2) is τ_1-w closed.

From the previous section, a space (X, τ) is a semi star generalized w-$T_{\frac{1}{2}}$ (simply, s^*gw-$T_{\frac{1}{2}}$) if every s^*gw-closed set is w-closed. In this sequel we define the following.

Definition 5.19. A space (X, τ_1, τ_2) is a pairwise semi star generalized w-$T_{\frac{1}{2}}$ (simply, pairwise s^*gw-$T_{\frac{1}{2}}$) if every $\tau_1\tau_2$-s^*gw closed set in (X, τ_1, τ_2) is τ_2-w closed and $\tau_2\tau_1$-s^*gw closed set in (X, τ_1, τ_2) is τ_1-w closed.

Theorem 5.20. *For a space (X, τ_1, τ_2), the following are equivalent.*

(a) X is pairwise s^*gw-$T_{\frac{1}{2}}$,

(b) Every singleton is either τ_i-semi closed or τ_j-w open, $i \neq j$.

Since, every $\tau_1\tau_2$-s^*gwclosed set is $\tau_1\tau_2$-rgw closed, we have the following

Theorem 5.21. *Every pairwise rgw-$T_{\frac{1}{2}}$ space is pairwise s^*gw-$T_{\frac{1}{2}}$ space.*

Concerning the continuity and composition of functions, we observe the following.

Definition 5.22. A map $f : X \to Y$ is called

(a) pairwise s^*gw-continuous if the inverse image of σ_j-w closed set in Y is $\tau_i\tau_j$-s^*gw closed in X, i, j = 1,2, $i \neq j$.

(b) pairwise s^*gw-irresolute if the inverse image of $\sigma_i\sigma_j$-s^*gw closed set in Y is $\tau_i\tau_j$-s^*gw closed in X, i, j = 1,2, $i \neq j$.

Theorem 5.23. *(a) The composition of two pairwise s^*gw-irresolute functions is pairwise s^*gw-irresolute.*

Equivalently, If f, g are pairwise s^*gw-irresolute, then gof is also pairwise s^*gw-irresolute.

(b) If f is pairwise s^*gw-irresolute and g is pairwise s^*gw-continuous, then gof is also pairwise s^*gw-continuous.

The composition of two pairwise s^*gw-continuous functions is not pairwise s^*gw-continuous.

Theorem 5.24. *a) Every pairwise s^*gw-continuous function is pairwise gw-continuous, and*

b) Every pairwise s^*gw-continuous function is pairwise rgw-continuous.

The next theorem shows that pairwise s^*gw-$T_{\frac{1}{2}}$ spaces are preserved under pairwise s^*gw-irresolute map if it is also a pairwise pre w-closed map.

Theorem 5.25. *Let $f : X \to Y$ be onto pairwise s^*gw-irresolute and pairwise pre w-closed map. If X is pairwise s^*gw-$T_{\frac{1}{2}}$ then Y is also pairwise s^*gw-$T_{\frac{1}{2}}$.*

Theorem 5.26. *a) Every pairwise s^*gw-closed function is pairwise gw-closed,*

b) Every pairwise s^*gw-closed function is pairwise rgw-closed.

Since every τ_j-w closed set is $\tau_i\tau_j$-s^*gw closed, we have the following theorem.

Theorem 5.27. *Every pairwise s^*gw-irresolute map is pairwise s^*gw-continuous map.*

6. Pairwise Complemented Spaces

Now we characterize complemented spaces using generalized sets and some separation axioms in bitopological spaces [Chandrasekhara Rao, K and Narasimhan, D., Characterizations of pairwise complemented spaces, Antarctica J. Math., No.3(1), (2006), 17-27.]. In addition pairwise $T_{\frac{1}{2}}$-spaces are also discussed with pairwise complemented spaces. In particular, a necessary and sufficient condition for a bitopological space to be a pairwise $T_{\frac{1}{2}}$- spaces is obtained.

Definition 6.1. A space (X, τ_1, τ_2) is said to be pairwise complemented if every τ_1-open set is τ_2-closed and every τ_2-open set is τ_1-closed.

Example 6.2. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$ $\tau_2 = \{\phi, X, \{b, c\}\}$, then (X, τ_1, τ_2) is pairwise complemented space.

Theorem 6.3. *If X is pairwise complemented T_S-space, then X is pairwise a $T_{\frac{1}{2}}$-space*

Lemma 6.4. *(a) Every $\tau_1\tau_2$-gs closed set in a pairwise T_b-space is $\tau_1\tau_2$-s^*g closed,*
(b) Every $\tau_1\tau_2$-sg closed set in a pairwise T_b-space is $\tau_1\tau_2$-s^*g closed.

Theorem 6.5. *(a) Every subset of a pairwise complemented T_b-space is $\tau_1\tau_2$-s^*g closed,*
(b) Every subset of a pairwise complemented $T_{\frac{1}{2}}$-space is $\tau_1\tau_2$-s^*g closed.

Lemma 6.6. *Every $\tau_1\tau_2$-αg closed set in a pairwise ${}_\alpha T_b$-space is $\tau_1\tau_2$-s^*g closed.*

Theorem 6.7. *Every subset of a pairwise complemented ${}_\alpha T_b$-space is $\tau_1\tau_2$-s^*g closed.*

Theorem 6.8. If X is a pairwise complemented space, then every $\tau_i\tau_j$-δg closed set is $\tau_i\tau_j$-s^*g closed in X, $i, j = 1{,}2$ and $i \neq j$.

Theorem 6.9. *Let (X, τ_1, τ_2) be a pairwise complemented space. Then*

(a) Every $\tau_1\tau_2$ -semi closed subset is τ_1-open and every $\tau_2\tau_1$ -semi closed subset is τ_2-open.

(b) Every subset of X is both $\tau_1\tau_2$-pre closed and $\tau_2\tau_1$-pre closed.

(c) Every subset of X is both $\tau_1\tau_2$-semi pre closed and $\tau_2\tau_1$-semi pre closed.

(d) Every subset of X is both $\tau_1\tau_2$-g closed and $\tau_2\tau_1$-g closed.

(e) Every subset of X is both $\tau_1\tau_2$-sg closed and $\tau_2\tau_1$-sg closed.

(f) Every $\tau_1\tau_2$ -semi open subset is τ_1-closed and every $\tau_2\tau_1$ -semi open subset is τ_2-closed.

(g) Every subset of X is both $\tau_1\tau_2$-pre open and $\tau_2\tau_1$-pre open.

(h) Every subset of X is both $\tau_1\tau_2$-semi pre open and $\tau_2\tau_1$-semi pre open.

(i) Every subset of X is both $\tau_1\tau_2$-g open and $\tau_2\tau_1$-g open.

(j) Every subset of X is both $\tau_1\tau_2$-sg open and $\tau_2\tau_1$-sg open.

Theorem 6.10. *If every $\tau_1\tau_2$-closed subset of a bitopological space X is $\tau_1\tau_2$-pre open then X is pairwise complemented.*

Now, we establish a necessary and sufficient condition for a bitopological space to be a pairwise $T_{\frac{1}{2}}$- spaces.

Theorem 6.11. *A bitopological space* (X, τ_1, τ_2) *is pairwise* $T_{\frac{1}{2}}$ *if and only if* $\{x\}$ *is* τ_1*-open or* τ_2*-closed and* $\{x\}$*is* τ_2*-open or* τ_1*-closed .*

Corollary 6.12. X *is pairwise* $T_{1/2}$ *if and only if every subset of* X *is the intersection of all* τ_i*-open and* τ_j*-closed sets containing it,* $i, j = 1, 2$*.*

The condition for a bitopological space X to be a pairwise preregular $T_{\frac{1}{2}}$ through singleton set will be discussed in the following lemma.

Lemma 6.13. *For a bitopological space*(X, τ_1, τ_2), X *is pairwise preregular* $T_{\frac{1}{2}}$ *if every singleton of* X *is either* τ_1*-regular closed or* τ_2*-pre open and either* τ_2*-regular closed or* τ_1 *-pre open.*

Lemma 6.14. *For a space* (X, τ_1, τ_2) *the following are equivalent.*

(a) X *is pairwise pre-regular* $T_{\frac{1}{2}}$

(b) Every nowhere τ_i*-dense singleton of* X *is* τ_j*-regular closed.*

(c) The only nowhere τ_i*-dense subset of* X *is empty set.*

(d) Every subset of X *is both* τ_1*-pre open and* τ_2*-pre open.*

Theorem 6.15. *A pairwise complemented* $T_{\frac{1}{2}}$ *space is pairwise discrete.*

Theorem 6.16. *Let* X *be a pairwise complemented space, then the following are equivalent.*

(a) X *is pairwise semi* $T_{\frac{1}{2}}$

(b) Every $\tau_1\tau_2$*-sg closed set is* τ_1*-open and every* $\tau_2\tau_1$*-sg closed set is* τ_2 *-open*

(c) Every $\tau_1\tau_2$*-sg open set is* τ_1*-closed and every* $\tau_2\tau_1$*-sg open set is* τ_2 *-closed.*

Theorem 6.17. *Let* X *be a pairwise complemented space. Then the following are equivalent.*

(a) X *is pairwise* $T_{\frac{1}{2}}$*,*

(b) X *is pairwise semi* $T_{\frac{1}{2}}$

(c) X *is pairwise discrete.*

Theorem 6.18. *If* X *is pairwise complemented then the space* X *is pairwise semi pre* $T_{\frac{1}{2}}$*.*

7. Conclusion

From the study we observe the following:

$$\begin{array}{ccccc}
{}^*T_p + T_p^* & & \text{door space} & & {}^*T_{\frac{1}{2}} + T^*_{\frac{1}{2}} \\
 & \searrow & \downarrow & \swarrow & \\
{}_\alpha T_b\text{-space} & \rightarrow & T_S\textbf{-space} & \leftarrow & T_b\text{-space} \\
 & \nearrow & \uparrow & \nwarrow & \\
T_d + T_{\frac{1}{2}} & & {}_\alpha T_c + T^*_{\frac{1}{2}} & & T_{\alpha gs}
\end{array}$$

Several spaces that fail to be T_1 are important in the study of the geometric and topological properties of digital images in the study of digital topology. Although the digital line is neither a T_1-space nor an R_0-space, it satisfies a couple of separation axioms which are a bit weaker than T_1 and R_0, that is, the digital line is both a $T_{\frac{3}{4}}$-space (hence a semi-T_1-space) and a semi-R_0-space.

All the separation axioms (spaces) introduced from the research are weaker separation axioms. So, in future there is a possibility for applications of these separation axioms in digital topology. Further research may be undertaken in this direction.

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