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ON THE PROPERTY (BAW)

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Abstract: In this paper we introduce and study the property (Baw). We show that if T is a bounded linear operator acting on a Banach space X, then property (Baw) holds for T if and only if generalized a-Browder's theorem holds for T and $\pi^a(T) = E_0^a(T)$, where $\pi^a(T)$ is the set of left poles of T and $E_0^a(T)$ is the set of eigen values of finite multiplicity which are isolated in the approximate spectrum. We explore conditions on Hilbert space operators T and S so that property (Baw) holds for $T \oplus S$.

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Key Words: Weyl's theorem, generalized a-Weyl's theorem, generalized a-Browder's theorem, property (Baw)

1. Introduction

Let $B(X)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex Banach space X. For an operator $T \in B(X)$, we denote by T^{*}, $\sigma(T)$, $\sigma_a(T)$, $N(T)$ and $R(T)$ the adjoint, the spectrum, the approximate

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spectrum, the null space and the range space of T, respectively. Let $\alpha(T)$ and $\beta(T)$ denote the dimension of the kernel $N(T)$ and the codimension of the range $R(T)$, respectively. Let $E(T)$ be the set of all isolated eigen values of T and $E^a(T)$ be the set of all eigenvalues of T which are isolated in $\sigma_a(T)$. Let $\sigma_{iso}(T)$ and $\sigma_a^{iso}(T)$ denote the set of isolated points of $\sigma(T)$ and $\sigma_a(T)$ respectively. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$), then T is called an upper semi-Fredholm (resp., a lower semi-Fredholm) operator.

If T is either an upper or a lower semi-Fredholm then T is called a semi-Fredholm operator, while T is said to be a Fredholm operator if it is both upper and lower semi-Fredholm. If $T \in B(X)$ is semi-Fredholm, then the index of T is defined as $\text{ind}(T) = \alpha(T) - \beta(T)$. An operator $T \in B(X)$ is called Weyl if it is Fredholm operator of index 0. The Weyl spectrum is defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$.

For a bounded linear operator T and a nonnegative integer n, we define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into itself (in particular $T_0 = T$). If for some integer n, the range space $R(T^n)$ is closed and T_n is an upper (resp., a lower) semi-Fredholm operator, then T is called an upper (resp., a lower) semi B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. Moreover, if T_n is a Fredholm operator, then T is called a B-Fredholm operator. From [4, Proposition 2.1] if T_n is a semi-Fredholm operator then T_m is also a semi-Fredholm operator for each $m \geq n$ and ind $(T_m) = \text{ind}(T_n)$. Thus the index of a semi-B-Fredholm operator T is defined as the index of the semi-Fredholm operator T_n (see [3, 4]). An operator $T \in B(X)$ is called a B-Weyl operator if it is a B-Fredholm operator of index 0. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined as $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}$. Denote by $USBF^-(X)$ the class of all upper semi B-Fredholm operators with index less than or equal to 0. Set $\sigma_{usbf^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin USBF^-(X)\}$.

The descent $q(T)$ and the ascent $p(T)$ of T are given by $q(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$ and $p(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$. An operator $T \in B(X)$ is said to be Drazin invertible if it has finite ascent and descent. The Drazin spectrum is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}$. Define the set $LD(X)$ by $LD(X) = \{T \in B(X) : p(T) \prec \infty$ and $R(T^{p(T)+1})$ is closed$\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not in } LD(X)\}$. Following [4], an operator $T \in B(X)$ is said to be left Drazin invertible if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I \in LD(X)$. Let $\pi(T)$ and $\pi^a(T)$ be the set of all poles and set of all left poles of T, respectively.Let $E_0^a(T) = \{\lambda \in \sigma_a^{iso}(T) : 0 < \alpha(T - \lambda I) < \infty\}$, then we say that $T \in B(X)$ satisfies

1. generalized a-Weyl's theorem if $\sigma_{usbf^-}(T) = \sigma_a(T) \setminus E^a(T)$,
2. generalized a-Browder's theorem if $\sigma_{usbf^-}(T) = \sigma_a(T) \setminus \pi^a(T)$.

The single valued extension property was introduced by Dunford ([7], [8]) and it plays an important role in local spectral theory and Fredholm theory ([1], [10]).

The operator $T \in B(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0 \in \mathbb{C}$) if for every open disc U of λ_0 the only analytic function $f : U \to X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$, is the function $f \equiv 0$. An operator $T \in B(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. An operator $T \in B(X)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Every operator T has SVEP at an isolated point of the spectrum. Furthermore, we have

$$\sigma_a(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda.$$

A bounded linear operator $T \in B(X)$ is said to satisfy property (Bab) if $\sigma_a(T) \setminus \sigma_{usbf^-}(T) = \pi_0^a(T)$. Property (Bab) has been introduced and studied in [9].

In this paper we define the property (Baw) and show that property (Baw) implies property (Bab). Also we prove that T satisfies property (Baw) if and only if generalized a-Browder's theorem holds for T and $\pi^a(T) = E_0^a(T)$.

2. Property (Baw)

Definition 2.1. A bounded linear operator $T \in B(X)$ is said to satisfy property (Baw) if $\sigma_a(T) \setminus \sigma_{usbf^-}(T) = E_0^a(T)$.

Example 2.2. Let $R \in B(l^2(N))$ be the unilateral right shift and $P \in B(l^2(N))$ be the operator defined by $P(x_1, x_2, x_3, \ldots) = (0, x_2, x_3, x_4, \ldots)$. Consider $T = R \oplus P$ on $l^2(N) \oplus l^2(N)$,then $\sigma(T) = D(0,1)$ is the closed unit disc in $\mathbb{C}$, so that $\sigma(T)$ has no isolated point. Thus $\sigma_{iso}(T) = \phi$. Furthermore $\sigma_a(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle in $\mathbb{C}$ and $\sigma_{usbf^-}(T) = C(0,1)$. This implies that $\sigma_a(T) \setminus \sigma_{usbf^-}(T) = \{0\}$. Moreover, we have $E_0^a(T) = \{0\}$. Therefore T satisfies property (Baw).

Next is an example of an operator which fails to satisfy property (Baw):

Example 2.3. Let $R \in B(l^2(N))$ be right shift and let L be the weighted unilateral left shift defined by $L(x_1, x_2, \ldots) = (\frac{x_2}{2}, \frac{x_3}{3}, \ldots)$ for all $x = (x_1, x_2, \ldots) \in l^2(N)$. Consider $T = R \oplus L$, then $\sigma(T) = D$. On the other hand $\sigma_a(T) =$

$\sigma_{usbf^-}(T) = C(0,1) \cup \{0\}$. However $E_0^a(T) = \{0\}$. Thus T fails to satisfy property (Baw).

Theorem 2.4. *An operator $T \in B(X)$ possesses property (Baw) if and only if T possesses property (Bab) and $E_0^a(T) = \pi_0^a(T)$.*

Proof. Suppose that T possesses property (Baw) and $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T)$, then $T - \lambda I$ is upper semi B-Fredholm operator of index less than or equal to zero. As $\dim N(T - \lambda I) \prec \infty$, $T - \lambda I$ is upper semi-Fredholm operator of index less than or equal to zero [2, Lemma 2.2]. Now $\lambda \in E_0^a(T)$ implies that $\lambda \in \sigma_a^{iso}(T)$. Hence from [3, Theorem 2.8], we have $\lambda \in \pi_0^a(T)$. If $\lambda \in \pi_0^a(T) \subseteq E_0^a(T)$, then as T possesses property (Baw), we have the result.

Now if T possesses property (Bab) and $E_0^a(T) = \pi_0^a(T)$, then $\sigma_a(T) \setminus \sigma_{usbf^-}(T) = E_0^a(T)$. Thus T possesses property (Baw). □

Property (Baw) may be characterized in the following way:

Theorem 2.5. *Let $T \in B(X)$. Then the following statements are equivalent:*

1. *T satisfies property (Baw);*
2. *generalized a-Browder's theorem holds for T and $\pi^a(T) = E_0^a(T)$.*

Proof. (i)⇒(ii) By Proposition 3.10 of [5], it is sufficient to prove that T has SVEP at every $\lambda \notin \sigma_{usbf^-}(T)$. Let us assume that $\lambda \notin \sigma_{usbf^-}(T)$. If $\lambda \notin \sigma_a(T)$ then T has SVEP at λ. If $\lambda \in \sigma_a(T)$ and suppose that T satisfies property (Baw) then $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T) = E_0^a(T)$.Thus $\lambda \in \sigma_a^{iso}(T)$ which implies T has SVEP at λ. Now to prove the equality $\pi^a(T) = E_0^a(T)$ let us consider $\lambda \in E_0^a(T)$.As T satisfies property (Baw), we have that $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T) = \pi^a(T)$, because generalized a-Browder's theorem holds for T. Hence $E_0^a(T) \subseteq \pi^a(T)$. For the opposite inclusion, assume that $\lambda \in \pi^a(T) = \sigma_a(T) \setminus \sigma_{usbf^-}(T) = E_0^a(T)$. Therefore, the equality $\pi^a(T) = E_0^a(T)$.

(ii)⇒(i). If $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T)$, then generalized a-Browder's theorem implies that $\lambda \in \pi^a(T) = E_0^a(T)$. Conversely, if $\lambda \in E_0^a(T)$ then $\lambda \in \pi^a(T) = \sigma_a(T) \setminus \sigma_{usbf^-}(T)$. Thus $E_0^a(T) = \sigma_a(T) \setminus \sigma_{usbf^-}(T)$. □

Theorem 2.6. *Let T be a bounded linear operator on X. If T has SVEP at points in $\sigma_a(T) \setminus \sigma_{\mathrm{usbf}^-}(T)$, then T satisfies property (Baw) if and only if $E_0^a(T) = \pi^a(T)$.*

Proof. The hypothesis T has SVEP at points in $\sigma_a(T) \setminus \sigma_{usbf^-}(T)$ implies that T satisfies generalized a-Browder's theorem [5, Proposition 3.10]. Hence, from Theorem 2.5, we have that T satisfies property (Baw) if and only if $E_0^a(T) = \pi^a(T)$. □

Definition 2.7. Operators $S, T \in B(X)$ are said to be injectively interwined, denoted $S \prec_i T$ if there exists an injection $U \in B(X)$ such that $TU = US$.

If $S \prec_i T$, then T has SVEP at a point λ implies S has SVEP at a point λ. To see this, let T have SVEP at λ, let U be an open neighbourhood of λ and let $f : U \to X$ be an analytic function such that $(S - \mu)f(\mu) = 0$ for every $\mu \in U$.Then $U(S - \mu)f(\mu) = (T - \mu)Uf(\mu) = 0 \Rightarrow Uf(\mu) = 0$.Since U is injective, $f(\mu) = 0, i.e.S$ has SVEP at λ.

A straight forward application of Theorem 2.6 gives the following result:

Theorem 2.8. *Let $S, T \in B(X)$. If T has SVEP and $S \prec_i T$. Then S satisfies property (Baw) if and only if $E_0^a(S) = \pi^a(S)$.*

Definition 2.9. An operator $T \in B(X)$ is said to be a-isoloid (resp., finitely a-isoloid) if $\sigma_a^{iso}(T) \subseteq E^a(T)(resp., \sigma_a^{iso}(T) \subseteq E_0^a(T)$ and is called left-polaroid if $\sigma_a^{iso}(T) \subseteq \pi^a(T)$.

Theorem 2.10. *Let $T \in B(X)$ be a left polaroid and satisfy property (Baw). Then generalized a-Weyl's theorem holds for T.*

Proof. T is left polaroid and satisfies property (Baw) $\Longleftrightarrow$.
$\sigma_a(T) \setminus \sigma_{usbf^-}(T) = E_0^a(T) \subseteq E_a(T) = \pi_a(T) = \sigma_a(T) \setminus \sigma_{usbf^-}(T)$ (since T satisfies generalized a-Browder's theorem by Theorem 2.5). □

Theorem 2.11. *Let $T \in B(X)$ be a finitely a-isoloid operator and satisfy generalized a-Weyl's theorem. Then T satisfies property (Baw).*

Proof. As T satisfies generalized a-Weyl's theorem, therefore

$$\sigma_a(T) \setminus \sigma_{usbf^-}(T) = E^a(T).$$

We need to prove that $E_a(T) = E_0^a(T)$. Suppose that $\lambda \in E_a(T)$. It implies $\lambda \in \sigma_a^{iso}(T) \subseteq E_0^a(T)$ as T is finitely a-isoloid. Thus $E^a(T) \subseteq E_0^a(T)$. □

3. Property (Baw) for Direct Sums

Let H and K be infinite-dimensional Hilbert spaces. In this section we show that if T and S are two operators on H and K respectively and at least one of them satisfies property (Baw), then their direct sum satisfies property (Baw).

Theorem 3.1. *Suppose that property (Baw) holds for $T \in B(H)$ and $S \in B(K)$. If T and S are a-isoloid and $\sigma_{\mathrm{usbf}^-}(T \oplus S) = \sigma_{\mathrm{usbf}^-}(T) \cup \sigma_{\mathrm{usbf}^-}(S)$, then property (Baw) holds for $T \oplus S$.*

Proof. We know $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pair of operators. If T and S are a-isoloid, then

$$E_0^a(T \oplus S) = [E_0^a(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap E_0^a(S)] \cup [E_0^a(T) \cap E_0^a(S)] \qquad (1)$$

where $\rho_a(\cdot) = \mathbb{C} \setminus \sigma_a(\cdot)$. If property (Baw) holds for T and S, then the right hand side of (3.1) must be just the set

$$[\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)].$$

Thus if

$$\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S),$$

then $E_0^a(T \oplus S) = \sigma_a(T \oplus S) \setminus \sigma_{usbf^-}(T \oplus S)$, which implies that property (Baw) holds for $T \oplus S$. □

Theorem 3.2. *If $T \in B(H)$ has no isolated point in its approximate spectrum and $S \in B(K)$ satisfies property (Baw). If $\sigma_{\mathrm{usbf}^-}(T \oplus S) = \sigma_a(T) \cup \sigma_{\mathrm{usbf}^-}(S)$, then property (Baw) holds for $T \oplus S$.*

Proof. As $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pair of operators, we have

$$\begin{aligned}
\sigma_a(T \oplus S) \setminus \sigma_{usbf^-}(T \oplus S) &= [\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_a(T) \cup \sigma_{usbf^-}(S)] \\
&= \sigma_a(S) \setminus [\sigma_a(T) \cup \sigma_{usbf^-}(S)] \\
&= [\sigma_a(S) \setminus \sigma_{usbf^-}(S)] \setminus \sigma_a(T) \\
&= E_0^a(S) \cap \rho_a(T)
\end{aligned}$$

where $\rho_a(T) = \mathbb{C} \setminus \sigma_a(T)$.

Let $\sigma_a^{iso}(T \oplus S)$ be the set of isolated points of $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$. If $\sigma_a^{iso}(T) = \phi$, it implies that $\sigma_a(T) = \sigma_a^{acc}(T)$, where $\sigma_a^{acc}(T) = \sigma_a(T) \setminus \sigma_a^{iso}(T)$ is the set of all accumulation points of $\sigma_a(T)$.Thus we have

$$\sigma_a^{iso}(T \oplus S) = [\sigma_a^{iso}(T) \cup \sigma_a^{iso}(S)] \setminus [(\sigma_a^{iso}(T) \cap \sigma_a^{acc}(S)) \cup (\sigma_a^{acc}(T) \cap \sigma_a^{iso}(S))]$$

$$
\begin{aligned}
&= (\sigma_a^{iso}(T) \setminus \sigma_a^{acc}(S)) \cup (\sigma_a^{iso}(S) \setminus \sigma_a^{acc}(T)) \\
&= \sigma_a^{iso}(S) \setminus \sigma_a(T) \\
&= \sigma_a^{iso}(S) \cap \rho_a(T).
\end{aligned}
$$

Let $\sigma_P(T)$ denote the point spectrum of T and $\sigma_{PF}(T)$ denote the set of all eigenvalues of T of finite multiplicity.

We have that $\sigma_p(T \oplus S) = \sigma_p(T) \cup \sigma_p(S)$ and $\dim N(T \oplus S) = \dim N(T) + \dim N(S)$ for every pair of operators, so that

$$
\sigma_{PF}(T \oplus S) = \{\lambda \in \sigma_{PF}(T) \cup \sigma_{PF}(S) : \dim N(\lambda l - T) + \dim N(\lambda l - S) \prec \infty\}
$$

Therefore

$$
\begin{aligned}
E_0^a(T \oplus S) &= \sigma_a^{iso}(T \oplus S) \cap \sigma_{PF}(T \oplus S) \\
&= \sigma_a^{iso}(S) \cap \rho_a(T) \cap \sigma_{PF}(S) \\
&= E_0^a(S) \cap \rho_a(T).
\end{aligned}
$$

Thus $\sigma_a(T \oplus S) \setminus \sigma_{usbf^-}(T \oplus S) = E_0^a(T \oplus S)$. Hence $T \oplus S$ satisfies property (Baw). □

Let $\sigma_1(T)$ be the compliment of $\sigma_{usbf^-}(T)$ in $\sigma_a(T)$ i.e. $\sigma_1(T) = \sigma_a(T) \setminus \sigma_{usbf^-}(T)$.A straight forward application of Theorem 3.2 leads to the following corollaries.

Corollary 3.3. *Suppose $T \in B(H)$ is such that $\sigma_a^{\text{iso}}(T) = \phi$ and $S \in B(K)$ satisfies property (Baw) with $\sigma_a^{\text{iso}}(S) \cap \sigma_{\text{PF}}(S) = \phi$ and $\sigma_1(T \oplus S) = \phi$, then $T \oplus S$ satisfies property (Baw).*

Proof. As S satisfies property (Baw), therefore given condition $\sigma_a^{iso}(S) \cap \sigma_{PF}(S) = \phi$ implies that $\sigma_a(S) = \sigma_{usbf^-}(S)$. Now $\sigma_1(T \oplus S) = \phi$ gives that

$$
\sigma_{usbf^-}(T \oplus S) = \sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_{usbf^-}(s).
$$

Thus from Theorem 3.2 we have that $T \oplus S$ satisfies property (Baw). □

Corollary 3.4. *Suppose $T \in B(H)$ is such that $\sigma_1(T) \cup \sigma_a^{\text{iso}}(T) = \phi$ and $S \in B(K)$ satisfies property (Baw). If $\sigma_{\text{usbf}^-}(T \oplus S) = \sigma_{\text{usbf}^-}(T) \cup \sigma_{\text{usbf}^-}(S)$, then property (Baw) holds for $T \oplus S$.*

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