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ON $(\in, \in \vee q)$-FUZZY ESSENTIAL IDEALS OF RINGS

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Abstract: In this paper, our attempt is to define fuzzy essential ideals and fuzzy annihilators using the notions of belongingness $(\in)$ and quasi-coincidence(q) of fuzzy points of sets. We study $(\in, \in \vee q)$-fuzzy essential ideals of rings and investigate different characteristics of such ideals. Various properties on $(\in, \in \vee q)$-fuzzy annihilators of fuzzy subsets of rings are also established.

1. Introduction

Fuzzy sets were introduced by Zadeh [12] in 1965. Since then this theory has been applied to many branches of Mathematics. Algebraic structures play a vital role in Mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information science, coding theory and so on. This provides motivation to researchers to review various concepts of algebraic structures in fuzzy setting. Fuzzy subgroup of a group was introduced by Rosenfeld [11] in 1971. Since then many generalization of this fundamental concept have been done. Bhakat and Das in [1], redefined fuzzy subgroups of a group using the notion of belongings to $(\in)$ and

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quasi-coincident of a fuzzy point to a fuzzy set of the group and in [3], fuzzy subring and ideal are redefined. Dudek et al. [8] characterized different types of (α, β)-fuzzy ideals of a hemiring. Davvaz et al. in [7] generalized this concept to Hv-submodules and redefined fuzzy Hv-submodules by applying many valued implication operators.

In this paper, our attempt is to define fuzzy essential ideals and fuzzy annihilators using the notions of belongingness ($\in$) and quasi-coincidence(q) of fuzzy points of sets. We study $(\in, \in \vee q)$-fuzzy essential ideals of rings and investigate different characteristics of such ideals. Various properties on $(\in, \in \vee q)$-fuzzy annihilators of fuzzy subsets of rings are also established. Some interesting properties of $(\in, \in \vee q)$-fuzzy essential ideals are being established which give the idea of similar properties of crisp set theory. In this discussion, we consider $(\in, \in \vee q)$-fuzzy left ideals only. Similar results of $(\in, \in \vee q)$-fuzzy right ideals can be obtained.

In Section 2, we give some definitions and notations. Some basic resuts and properties of $(\in, \in \vee q)$-fuzzy annihilators and $(\in, \in \vee q)$-fuzzy essential ideals are established in Section 3 and Section 4. Let μ, ν, σ be $(\in, \in \vee q)$-fuzzy left ideals of R. Then μ is an $(\in, \in \vee q)$-fuzzy essential left ideal in $\nu(\mu \subseteq_e \nu)$, if and only if μ_t (level subset of μ), for every $t \in (0, T]$ is an essential left ideal of ν_t, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ. Suppose A and B are two non-zero left ideals of R. Then A is essential in R if and only if χ_A(characteristics function on A) is an $(\in, \in \vee q)$-fuzzy essential left ideal of R. Again A is an essential left ideal of B if and only if χ_A is an $(\in, \in \vee q)$-fuzzy essential left ideal in χ_B. Every non-zero $(\in, \in \vee q)$-fuzzy left ideal of R is an $(\in, \in \vee q)$-fuzzy essential left ideal of itself. If $\mu \subseteq_e \sigma$. Then for any $(\in, \in \vee q)$-fuzzy left ideal θ of R, $\mu \cap \theta \subseteq_e \sigma \cap \theta$. Also for any non-zero $a_t \in \vee q\sigma$, $t \in (0, T], a \in R$, there exists an $(\in, \in \vee q)$-fuzzy left ideal γ of R with $\gamma \subseteq_e R(\gamma$ is an $(\in, \in \vee q)$-fuzzy essential left ideal of $R)$ such that γa_t is non-zero and $\gamma a_t \subseteq \mu$, where $T(\leq 0.5)$ is the supremum among all non-zero values of μ. Again if $\mu \subseteq_e R$, then for any non-zero $a \in R$, there exists an $(\in, \in \vee q)$-fuzzy left ideal γ of R with $\gamma \subseteq_e R$ such that γa is non-zero. Lastly, suppose that $\mu \subseteq \nu \subseteq \sigma$. Then $\mu \subseteq_e \sigma$ if and only if $\mu \subseteq_e \nu \subseteq_e \sigma$. There are two examples in Section 5, which assert the concept of $(\in, \in \vee q)$-fuzzy essential left ideals of non-commutative rings.

2. Definitions and Notations

Throughout this paper R denotes a non-commutative ring with unity and X is any non-empty set.

Definition 2.1. A map $\lambda : X \to [0,1]$ is called a fuzzy subset of X.

Definition 2.2. [10] A fuzzy subset λ of X of the form

$$\lambda(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Definition 2.3. [10] A fuzzy point x_t is said to belong to (respectively be quasi-coincident with) a fuzzy set λ, written as $x_t \in \lambda$ (respectively $x_t q\lambda$) if $\lambda(x) \geq t(resp.\lambda(x) + t > 1)$.

$x_t \in \lambda$ or $x_t q\lambda$ will be denoted by $x_t \in \vee q\lambda$. $x_t \overline{\in \vee q}\lambda$ means $x_t \in \vee q\lambda$ does not hold.

Definition 2.4. Let λ be a fuzzy subset of X. Then $\forall t \in (0,1]$, the set $\lambda_t = \{x \in X : \lambda(x) \geq t \; or \lambda(x) + t > 1\}$ is called $(\in, \in \vee q)$-level subset of λ.

Definition 2.5. [3] A fuzzy subset λ of R is called an $(\in, \in \vee q)$-fuzzy subring of R if $\forall x, y \in R$ and $t, r \in (0,1]$,

$(i) x_t, y_r \in \lambda \Rightarrow (x+y)_{M(t,r)} \in \vee q\lambda$, ($M(t,r)$ stands for minimum of $t \; and \; r$)

$(ii) x_t \in \lambda \Rightarrow (-x)_t \in \vee q\lambda$,

$(iii) x_t, y_r \in \lambda \Rightarrow (xy)_{M(t,r)} \in \vee q\lambda$.

Definition 2.6. A fuzzy subset λ of R is called an $(\in, \in \vee q)$-fuzzy left(respectively right) ideal of R if $\forall x \in R$ and $t \in (0,1]$,

$(i)\lambda$ is an $(\in, \in \vee q)-$fuzzy subring of R,

$(ii) x_t \in \lambda$ and $y \in R \Rightarrow (yx)_t \in \vee q\lambda$(respectively $(xy)_t \in \vee q\lambda$).

Definition 2.7. Let λ be a fuzzy subset of R, $r \in R$. Then λr and $\overline{\lambda r}$ are two fuzzy subsets of R are defined as $(\lambda r)(x) = M(\bigvee\{\lambda(y) : y \in R, yr = x\}, 0.5)$ and $\overline{(\lambda r)}(x) = Max(\bigvee\{\lambda(y) : y \in R, yr = x\}, 0.5)$ respectively, where $Max(s,t)$ stands for maximum of s and t.

Definition 2.8. Let μ be a fuzzy subset of R. Then the left $(\in, \in \vee q)$-fuzzy left and right annihilator of μ, denoted by $l(\mu)$ and $r(\mu)$ respectively, are fuzzy subsets of R and are defined as follows:

$$l(\mu)(x) = \begin{cases} M(\bigvee_{t \in Im\mu} \{t : x \in l(\mu_t)\}, 0.5) \\ 0, \quad \text{if } x \notin l(\mu_t) \; for \; any \; t \in Im\mu \end{cases}$$

and

$$r(\mu)(x) = \begin{cases} M(\bigvee_{t \in Im\mu} \{t : x \in r(\mu_t)\}, 0.5) \\ 0, \quad \text{if } x \notin r(\mu_t) \; for \; any \; t \in Im\mu \end{cases}.$$

Similarly we define $\bar{l}(\mu)$ and $\bar{r}(\mu)$ as follows:

$$\bar{l}(\mu)(x) = \begin{cases} Max(\bigvee_{t \in Im\mu} \{t : x \in l(\mu_t)\}, 0.5) \\ 0, \quad \text{if } x \notin l(\mu_t) \; for \; any \; t \in Im\mu \end{cases}$$

and

$$\bar{r}(\mu)(x) = \begin{cases} Max(\bigvee_{t \in Im\mu} \{t : x \in r(\mu_t)\}, 0.5) \\ 0, \quad \text{if } x \notin r(\mu_t) \; for \; any \; t \in Im\mu \end{cases}$$

Definition 2.9. A left ideal A of R is called an essential left ideal of R, denoted by $A \subseteq_e R$, if for every non-zero left ideal B of R, $A \cap B = (0)$.

Remark 2.1. If μ is non-zero fuzzy subset of X, then there exists a non-zero element x in X such that $\mu(x) = t$, $t \in (0, 1]$.

Definition 2.10. A non-zero $(\in, \in \vee q)$-fuzzy left ideal μ of R is called an $(\in, \in \vee q)$-fuzzy essential left ideal of R, denoted by $\mu \subseteq_e R$, if for every non-zero $(\in, \in \vee q)$-fuzzy left ideal θ of R, there exists $x(\neq 0) \in R$ with $x_t \in \vee q\mu$ and $x_t \in \vee q\theta$, for every $t \in (0, T]$, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ.

Definition 2.11. Let μ and σ be two non-zero $(\in, \in \vee q)$-fuzzy left ideals of R such that $\mu \subseteq \sigma$. Then μ is called an $(\in, \in \vee q)$-fuzzy essential left ideal in σ, denoted by $\mu \subseteq_e \sigma$, if for every non-zero $(\in, \in \vee q)$-fuzzy left ideal θ of R satisfying $\theta \subseteq \sigma$, there exists $x(\neq 0) \in R$ with $x_t \in \vee q\mu$ and $x_t \in \vee q\theta$, for every $t \in (0, T]$, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ.

Definition 2.12. Let μ be an fuzzy subset of R. Then μ is said to have supremum property if for any A of R, there exists $x \in A$ such that $\mu(x) = \bigvee\{\mu(a) : a \in A\}$.

3. $(\in, \in \vee q)$-Fuzzy Essential (Left) Ideal

In this section, we give some basic resuts, which we are going to use to discuss some properties, related with the $(\in, \in \vee q)$-fuzzy essential left(right) ideals of the rings. Interesting results, which establish the necessary and sufficient conditions of $(\in, \in \vee q)$-fuzzy set theory and crisp set theory, are being discussed.

Lemma 3.1. *[3] λ is an $(\in, \in \vee q)$-fuzzy subring if and only if $\lambda(x-y), \lambda(xy) \geq M(\lambda(x), \lambda(y), 0.5), \forall x, y \in R$.*

Lemma 3.2. *λ is an $(\in, \in \vee q)$-fuzzy left (respectively right) ideal of R if and only if*
$(i)\lambda(x-y) \geq M(\lambda(x), \lambda(y), 0.5)$,
$(ii)\lambda(yx)(respectively\ xy) \geq M(\lambda(x), 0.5), \forall x, y \in R$.

Lemma 3.3. *Let $\{\lambda_i : i \in J, J \text{ is an index set}\}$ be any family of $(\in, \in \vee q)$-fuzzy left (right) ideals of R. Then $\lambda = \bigcap \lambda_i$ is a $(\in, \in \vee q)$-fuzzy left (right) of R.*

Lemma 3.4. *Let λ be a nonzero $(\in, \in \vee q)$-fuzzy left ideal of R and $r \in R$. Then:*
(i) $\lambda_t r \subseteq (\lambda r)_t, t \in (0, \lambda(0)]$.
(ii) $\lambda_t r = (\lambda r)_t, t \in (0, \lambda(0)]$, if λ has supremum property.

Proof. (i) Let $x \in \lambda_t r$. Then there exists $y \in \lambda_t$ with $x = yr$. Since $y \in \lambda_t$ implies $\lambda(y) \geq t$ or $\lambda(y) + t > 1$. So, if $\lambda(y) \geq t$, then $(\lambda r)(x) = M(\bigvee\{\lambda(y) : y \in R, yr = x\}, 0.5) \geq M(t, 0.5)$. Now if $t \leq 0.5$, then $M(t, 0.5) = t$ implies $(\lambda r)(x) \geq t$. Also if $t > 0.5$ then $(\lambda r)(x) + t > 1$. This gives $x \in (\lambda r)_t$. Similarly, for $\lambda(y)+t > 1$ i. e. $\lambda(y) > t-1$, we have $(\lambda r)(x) = M(\bigvee\{\lambda(y) : y \in R, yr = x\}, 0.5) > M(1-t, 0.5)$. If $0.5 \leq 1-t$, then $(\lambda r)(x) > 0.5 \geq t$. Also for $0.5 > 1-t$, we get $(\lambda r)(x) + t > 1$. This gives $x \in (\lambda r)_t$ i. e. $\lambda_t r \subseteq (\lambda r)_t$.

(ii) Let $x \in (\lambda r)_t$. Then $(\lambda r)(x) \geq t$ or $(\lambda r)(x) + t > 1$. Since λ has supremum property, so there exists an element $y \in R$ with $x = yr$ and $\lambda(y) \geq t$. Therefore $M(\bigvee\{\lambda(y) : y \in R, yr = x\}, 0.5) \geq t$ gives $M(\lambda(y), 0.5) \geq t$. If $\lambda(y) < 0.5$, then $M(\lambda(y), 0.5) = \lambda(y)$. This implies $\lambda(y) \geq t$. So $x = yr \in \lambda_t r$. Also, if $\lambda(y) \geq 0.5$, then $M(\lambda(y), 0.5) = 0.5$. Therefore $t \leq 0.5$ gives $\lambda(y) \geq t$. Which implies $x \in \lambda_t r$. Again $M(\lambda(y), 0.5) + t > 1$ gives $t > 0.5$ for $\lambda(y) \geq 0.5$. From this $\lambda(y) + t > 1$ i. e. $x = yr \in \lambda_t r$. Similarly, for $\lambda(y) < 0.5$ we get $\lambda(y) + t > 1$ and this implies $x = yr \in \lambda_t r$. Hence $(\lambda r)_t \subseteq \lambda_t r$. □

Lemma 3.5. *Let μ and σ be two non-zero $(\in, \in \vee q)$-fuzzy left ideals of R and $r \in R$. Then $\mu r \subseteq \sigma r$.*

Proof. It is clear that if $x \neq yr$ for any $y \in R$, then $(\mu r)(x) = 0 = (\sigma r)(x)$. So let there exists $y \in R$ with $x = yr$. Then $(\mu r)(x) = M(\bigvee\{\mu(y) : y \in R, yr = x\}, 0.5) \leq M(\bigvee\{\sigma(y) : y \in R, yr = x\}, 0.5) = (\sigma r)(x)$. Hence $\mu r \subseteq \sigma r$. □

Lemma 3.6. *A non-zero $(\in, \in \vee q)$-fuzzy left ideal μ of R is an $(\in, \in \vee q)$-fuzzy essential left ideal of R, if and only if for every non-zero $(\in, \in \vee q)$-fuzzy* *left ideal θ of R, there exists $x(\neq 0) \in R$ with $\mu(x) \geq M(t, 0.5)$ and $\theta(x) \geq M(t, 0.5)$, for every $t \in (0, T]$, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ.*

Proof. Let μ be a non-zero $(\in, \in \vee q)$-fuzzy essential left ideal of R. If possible suppose that $t < 0.5$ and $\mu(x) < t$ or $\theta(x) < t$. Then $\mu(x) + t < 1$ or $\theta(x) + t < 1$, i. e. $x_t \overline{\in \vee q} \mu$ or $x_t \overline{\in \vee q} \theta$. Also if $t \geq 0.5$ and $\mu(x) < 0.5$ or $\theta(x) < 0.5$. Then $x_{0.5} \overline{\in \vee q} \mu$ or $x_{0.5} \overline{\in \vee q} \theta$. This gives there exists an element $x(\neq 0) \in R$ with $\mu(x) \geq M(t, 0.5)$ and $\theta(x) \geq M(t, 0.5)$, where $t \in (0, T]$. Conversely, we assume that there exists an element $x \in R$ with $\mu(x) \geq M(t, 0.5)$ and $\theta(x) \geq M(t, 0.5)$. If $t \leq 0.5$, then $\mu(x) \geq t$ and $\theta(x) \geq t$. From this, we get $x_t \in \vee q \mu$ and $x_t \in \vee q \theta$. If $t > 0.5$, then $\mu(x) \geq 0.5$ and $\theta(x) \geq 0.5$. This implies $\mu(x) + t > 1$ and $\theta(x) + t > 1$. Therefore $x_t \in \vee q \mu$ and $x_t \in \vee q \theta$. Hence the result. □

Lemma 3.7. *Let μ and σ be two non-zero $(\in, \in \vee q)$-fuzzy left ideals of R such that $\mu \subseteq \sigma$. Then μ is an $(\in, \in \vee q)$-fuzzy essential left ideal in σ, if and only if for every non-zero $(\in, \in \vee q)$-fuzzy left ideal θ of R satisfying $\theta \subseteq \sigma$, there exists $x(\neq 0) \in R$ with $\mu(x) \geq M(t, 0.5)$ and $\theta(x) \geq M(t, 0.5)$, for every $t \in (0, T]$, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ.*

Lemma 3.8. *Let μ be a non-zero $(\in, \in \vee q)$-fuzzy left ideal of R. Then μ is an $(\in, \in \vee q)$-fuzzy essential left ideal of R, if and only if μ_t, for every $t \in (0, T]$ is an essential left ideal of R, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ.*

Proof. Let μ be a non-zero $(\in, \in \vee q)$-fuzzy left ideal of R. Since μ is non-zero, so μ_t is non-zero left ideal of R, for every $t \in (0, T]$. Let A be any non-zero left ideal of R. A fuzzy subset θ of R is defined as

$\theta(x) = \begin{cases} \mu(x), & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$ Then θ is a non-zero $(\in, \in \vee q)$-fuzzy left ideal of R.

So, there exists $x(\neq 0) \in R$ with $\mu(x) \geq M(t, 0.5)$ and $\theta(x) \geq M(t, 0.5)$, for any $t \in (0, T]$. Thus $x \in \mu_t$ and $\theta(x) \geq M(t, 0.5)$, where $t \in (0, \mu(0)]$. Therefore $\theta(x) \geq t$ or $\theta(x) \geq 0.5$. This gives $\theta(x) = \mu(x)$, i. e. $x \in A$. Therefore, μ_t is an essential left ideal of R for any $t \in (0, T]$. Conversely, let, for any $t \in (0, T]$, μ_t be an essential left ideal of R. Then, for any $t \in (0, T]$, μ_t is non-zero for which μ is non-zero. Let θ be any non-zero $(\in, \in \vee q)$-fuzzy left ideal of R with $\mu(0) = \theta(0)$. Thus, there exists a non-zero $x \in R$ with $x \in \mu_t$ and $x \in \theta_t$, where $t \in (0, T]$, as μ_t, for every $t \in (0, T]$ is an essential left ideal of R and θ_t, for $t \in (0, T]$ is non-zero. This implies, there exists $x(\neq 0) \in R$

with $\mu(x) \geq M(t, 0.5)$ and $\theta(x) \geq M(t, 0.5)$, where $t \in (0, T]$. Hence μ is an $(\in, \in \vee q)$-fuzzy left ideal of R. □

Theorem 3.1. *Let μ, ν be two non-zero $(\in, \in \vee q)$-fuzzy left ideals of R. Then μ is an $(\in, \in \vee q)$-fuzzy essential left ideal in ν, if and only if μ_t, for every $t \in (0, T]$ is an essential left ideal of ν_t, where $T(\leq \mu(0))$ is the supremum among all non-zero values.*

Theorem 3.2. *A left ideal A of R is essential in R if and only if χ_A is an $(\in, \in \vee q)$-fuzzy essential left ideal of R.*

Proof. Let A be an essential left ideal of R. Then χ_A is a non-zero $(\in, \in \vee q)$-fuzzy left ideal of R. Now, if χ_A is not an $(\in, \in \vee q)$-fuzzy essential left ideal of R, then there exists a non-zero $(\in, \in \vee q)$-fuzzy left ideal θ of R, such that for every $x(\neq 0) \in R$, $\chi_A(x) < M(t, 0.5)$ or $\theta(x) < M(t, 0.5)$ where $t \in (0, 1]$. This implies, $x \notin A$ or $x \notin \theta_t$ and θ_t, for $t \in (0, 1]$ is a non-zero left ideal of R. This contradicts that A is an essential left ideal of R. Hence χ_A is an $(\in, \in \vee q)$-fuzzy essential left ideal of R. Conversely, let χ_A be an $(\in, \in \vee q)$-fuzzy essential left ideal of R. Let B be any non-zero left ideal of R. Then χ_B is also a non-zero $(\in, \in \vee q)$-fuzzy left ideal of R. Hence, there exists a non-zero x in R with $\chi_A(x) \geq M(t, 0.5)$ and $\chi_B(x) \geq M(t, 0.5)$ for $t \in (0, 1]$. From this $x \in A \cap B$. This implies, A is an essential left ideal of R. □

Theorem 3.3. *If A and B are two non-zero left ideals of R, then A is an essential left ideal of B if and only if χ_A is an $(\in, \in \vee q)$-fuzzy essential left ideal in χ_B.*

Proof. Let $A \subseteq_e B$. If possible we assume χ_A is not an $(\in, \in \vee q)$-fuzzy essential left ideal in χ_B. Then, there exists a non-zero $(\in, \in \vee q)$-fuzzy left ideal θ of R with $\theta \subseteq \chi_B$, such that $\forall x(\neq 0) \in R$, $\chi_A(x) < M(t, 0.5)$ or $\theta(x) < M(t, 0.5)$, for $t \in (0, 1]$. This gives, $x \notin A$ or $x \notin \theta_t$, where θ_t is a non-zero left ideal of R for $t \in (0, 1]$, which implies A is not an essential left ideal in B. Conversely, if $\chi_A \subseteq_e \chi_B$, then $A \subseteq B$. Let C be any non-zero left ideal of B. Then χ_C is also a non-zero $(\in, \in \vee q)$-fuzzy left ideal of χ_B. So, there exists a non-zero $x \in R$ such that $\chi_A(x) \geq M(t, 0.5)$ and $\chi_C(x) \geq M(t, 0.5)$, $t \in (0, 1]$. From this $x \in A \cap B$ i. e. $A \subseteq_e B$. □

Theorem 3.4. *Every non-zero $(\in, \in \vee q)$-fuzzy left ideal of R is an $(\in, \in \vee q)$-fuzzy essential left ideal of itself.*

Proof. Let μ be a non-zero $(\in, \in \vee q)$-fuzzy left ideal of R. Let θ be any non-zero $(\in, \in \vee q)$-fuzzy left ideal of R such that $\theta \subseteq \mu$. Then for some non-zero

$x(\neq 0) \in R$, $\theta(x) = t(\neq 0)$. So $\mu(x) \geq t$. This gives, there exists $x(\neq 0) \in R$ such that $\theta(x) = t \geq M(t, 0.5)$ and $\mu(x) \geq t \geq M(t, 0.5)$. Hence the result. □

Lemma 3.9. *Let μ and σ be two non-zero $(\in, \in \vee q)$-fuzzy left ideals of R such that $\mu \subseteq \sigma$. Then $\mu \subseteq_e \sigma$ if and only if $\mu_t \subseteq_e \sigma_t$, $\forall t \in (0, T]$, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ.*

Proof. Let $\mu \subseteq_e \sigma$. Since $\mu \subseteq \sigma$, we have $\mu_t \subseteq \sigma_t$, $\forall t \in (0, T]$. Let A be any non-zero left ideal of σ_t. A fuzzy subset θ of R is defined such that
$\theta(x) = \begin{cases} \sigma(x), & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$ Then θ is a non-zero $(\in, \in \vee q)$-fuzzy left ideal of R such that $\theta \subseteq \sigma$. This gives there exists $x(\neq 0) \in R$ with $\mu(x) \geq M(s, 0.5)$ and $\theta(x) \geq M(s, 0.5)$, for any $s \in (0, T]$. In particular, $\mu(x) \geq M(t, 0.5)$ and $\theta(x) \geq M(t, 0.5)$. From this, $x \in \mu_t$ and $x \in \theta_t$, which implies, $x \in \mu_t$ and $x \in A$. Hence $\mu_t \subseteq_e \sigma_t$, $\forall t \in (0, T]$. Conversely, $\mu_t \subseteq_e \sigma_t$, $\forall t \in (0, T]$. To show $\mu \subseteq_e \sigma$. Let θ be a non-zero $(\in, \in \vee q)$-fuzzy left ideal of σ. Then, there exists a non-zero x in R with $x \in \mu_t$ and $x \in \theta_t$ for $t \in (0, T]$ as $\mu_t \subseteq_e \sigma_t$, $\forall t \in (0, T]$ and θ_t, for $t \in (0, T]$ is non-zero. This implies, there exists $x(\neq 0) \in R$ with $\mu(x) \geq M(t, 0.5)$ and $\theta(x) \geq M(t, 0.5)$, for any $t \in (0, T]$. Hence $\mu \subseteq_e \sigma$. □

Theorem 3.5. *Let μ and σ be two non-zero $(\in, \in \vee q)$-fuzzy left ideals of R such that $\mu \subseteq_e \sigma$. Then for any non-zero $(\in, \in \vee q)$-fuzzy left ideal θ of R, $\mu \cap \theta \subseteq_e \sigma \cap \theta$.*

Proof. Since $\mu \subseteq \sigma$, so $\mu \cap \theta \subseteq \sigma \cap \theta$, for any $(\in, \in \vee q)$-fuzzy left ideal θ of R. Let η be any non-zero $(\in, \in \vee q)$-fuzzy left ideal of R such that $\eta \subseteq \sigma \cap \theta$. This gives η is a non-zero $(\in, \in \vee q)$-fuzzy left ideal of R such that $\eta \subseteq \sigma$ and $\eta \subseteq \theta$. This implies, there exists $x(\neq 0) \in R$ with $\mu(x) \geq M(t, 0.5)$ and $\eta(x) \geq M(t, 0.5)$, for any $t \in (0, T]$, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ. Then $(\mu \cap \theta)(x) \geq M(t, 0.5) = M$, (say). As $(\mu \cap \theta)(x) = \mu(x) \wedge \theta(x)$. So, $\eta(x) \geq M$ implies $\theta(x) \geq M$, also $\mu(x) \geq M$. Which gives $(\mu \cap \theta)(x) = \mu(x) \wedge \theta(x) \geq M = M(t, 0.5)$. This implies, there exists $x(\neq 0) \in R$ such that $(\mu \cap \theta)(x) \geq M(t, 0.5)$ and $\eta(x) \geq t$ with $t \in (0, T]$. Hence $\mu \cap \theta \subseteq_e \sigma \cap \theta$. □

Lemma 3.10. *If A is an essential left ideal of R, then the fuzzy subset μ of R defined by*
$\mu(x) = \begin{cases} t, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$ *is an $(\in, \in \vee q)$-fuzzy essential left ideal of R, $\forall t \in (0, T]$, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ.*

Proof. Let $x, y \in R$. Then $\mu(x-y) \geq M(\mu(x), \mu(y), 0.5)$, $\mu(yx) \geq M(x, 0.5)$. i. e. μ is an $(\in, \in \vee q)$-fuzzy left ideal of R. Let $\mu_t = A$, for $t \in (0, T]$. Then μ_t, $\forall t \in (0, T]$ is an essential left ideal of R. Hence, by lemma 3. 8 μ is an $(\in, \in \vee q)$-fuzzy essential left ideal of R. □

4. $(\in, \in \vee q)$-Fuzzy Annihilators and Further Properties of $(\in, \in \vee q)$-Fuzzy Essential Ideals

This section is the continuation of our investigations for further $(\in, \in \vee q)$-fuzzy essential ideals. Some properties of $(\in, \in \vee q)$-fuzzy annihilators are being discussed.

Theorem 4.1. *Let μ and σ be two non-zero $(\in, \in \vee q)$-fuzzy left ideals of R such that $\mu \subseteq_e \sigma$. Then for any non-zero $a_t \in \vee q \sigma$, $t \in (0, T], a \in R$, there exists an $(\in, \in \vee q)$-fuzzy left ideal γ of R with $\gamma \subseteq_e R$ such that γa_t is non-zero and $\gamma a_t \subseteq \mu$, where $T(\leq 0.5)$ is the supremum among all non-zero values of μ.*

Proof. Since $\mu \subseteq_e \sigma$, so $\mu_t \subseteq_e \sigma_t$, $\forall t \in (0, T]$, by lemma 3.9. Thus, by lemma 1. 1 [6] for any $a(\neq 0) \in \sigma_t$, there exists a left ideal A of R with $A \subseteq_e R$ such that $Aa \subseteq \mu_t$ and $Aa \neq (0)$. Now we define a fuzzy subset γ of R such that $\gamma(x) = \begin{cases} t & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$. From this $\gamma \subseteq_e R$, by lemma 3.10. Next, let $x \in R$. Then $(\gamma a_t)(x) = \bigvee_{x \in pq} \{\gamma(p) \wedge a_t(q)\} = \gamma(c) \wedge a_t(d)$, for some $x = cd$, where $c, d \in R$. But if $c \notin A$ or $a \neq d$, then $(\gamma a_t)(x) = 0 \leq \mu(x)$. So, let $c \in A$ and $a = d$. Then $(\gamma a_t)(x) = t$, where $c \in A$ and $a = d$. This gives $(\gamma a_t)(x) = t$, where $x \in Aa \subseteq \mu_t$. Now $x \in \mu_t$, implies $\mu(x) \geq t$ or $\mu(x) + t > 1$. If $\mu(x) \geq t$, then $\gamma a_t \subseteq \mu$. Again, if $\mu(x) + t > 1$, then also we get $\gamma a_t \subseteq \mu$. Since $Aa \neq (0)$, so there exists a non-zero element y in Aa with $y = na$, for some $n \in A$. Therefore $(\gamma a_t)(y) = \bigvee_{y \in pq} \{\gamma(p) \wedge a_t(q)\} \geq \nu(n) \wedge a_t(a) \neq 0$, as $n \in A$. Hence the result. □

Lemma 4.1. *Let μ and θ be two $(\in, \in \vee q)$-fuzzy subsets of R. Then:*
(i) $\mu \subseteq \theta$ implies $l(\theta) \subseteq l(\mu)$ and $r(\theta) \subseteq r(\mu)$.
(ii) $l(\mu_t) \subseteq [l(\mu)]_t$ and $r(\mu_t) \subseteq [r(\mu)]_t$.
(iii) $\overline{l}(\chi_A) = \chi_{l(A)}$ and $\overline{r}(\chi_A) = \chi_{r(A)}$, where A is a subset of R.

Proof. (i) Since $\mu \subseteq \theta$, so $\mu_t \subseteq \theta_t$. This gives $l(\mu)(x) = M(\bigvee_{t\in Im\mu}\{t : x \in l(\mu_t)\}, 0.5) \leq M(\bigvee_{t\in Im\theta}\{t : x \in l(\theta_t)\}, 0.5) = l(\theta)(x)$. Hence $l(\theta) \subseteq l(\mu)$.

Similarly $r(\theta) \subseteq r(\mu)$.

(ii) Let $x \in l(\mu_t)$. Therefore $l(\mu)(x) = M(\bigvee_{t\in Im\mu}\{t : x \in l(\mu_t)\}, 0.5) \geq M(t, 0.5)$. From this $l(\mu_t) \subseteq [l(\mu)]_t$.

Similarly $r(\mu_t) \subseteq [r(\mu)]_t$. □

Theorem 4.2. *Let μ be an $(\in, \in \vee q)$-fuzzy left ideal of R with $\mu(0) = 1$. For $r \in R$, if $\overline{\mu r} = \chi_0$ then $\mu \subseteq \bar{l}(\chi_{(r)})$.*

Proof. Let $\overline{\mu r} = \chi_0$. Let x be any element of R and $\mu(x) = t$. If $\mu(x) = 0$, then we are done. So, let $t \neq 0$. Then $xr = 0$, for if $xr \neq 0$, we have $(\overline{\mu r})(xr) = Max(\bigvee\{\mu(y) : yr = xr\}, 0.5) \geq Max(\mu(x), 0.5) \neq 0$, which contradicts that $\overline{\mu r} = \chi_0$. So $xr = 0$, i. e. $x \in l(r)$. Now $\bar{l}(\chi_{(r)})(x) = Max(\bigvee_{t\in Im\chi_{(r)}}\{t : x \in l(\chi_{(r)})_t\}, 0.5) = 1$. Let $z \in (\chi_{(r)})_t$. Then $\chi_r(z) \geq t$ or $\chi_r(z) + t > 1$. Since $t > 0$, so $r = z$. Thus $l(\chi_{(r)})_t = l(r)$. Hence $\mu \subseteq \bar{l}(\chi_{(r)})$. □

Theorem 4.3. *Let μ be an $(\in, \in \vee q)$-fuzzy left ideal of R with $\mu(0) = 1$. For $r \in R$, if $\mu \subseteq \bar{l}(\chi_{(r)})$ then $\overline{\mu r} = \chi_0$.*

Proof. Let $\mu \subseteq \bar{l}(\chi_{(r)})$. Then $(\overline{\mu r})(0) = Max(\bigvee\{\mu(y) : yr = 0\}, 0.5) \geq Max(\mu(0), 0.5) = 1$. Thus $(\overline{\mu r})(0) = 1$. □

Theorem 4.4. *Let μ be an $(\in, \in \vee q)$-fuzzy left ideal of R. If $r \in R$ and $\mu \subseteq \bar{l}(\chi_{(r)})$, then $(\mu r)(x) = 0$ for any $x \neq 0$.*

Proof. Let $x(\neq 0)$ be any element of R. If $x \notin Rr$, then we are done. So let $x \in Rr$. Then $(\mu r)(x) = M(\bigvee\{\mu(z) : z \in R, zr = x\}, 0.5) = M(\mu(y), 0.5)$, for some $y \in R$ with $x = yr$. If $\mu(y) \neq 0$, then $\bar{l}(\chi_{(r)})(y) > 0$. This gives $\bar{l}(\chi_{(r)})(y) = 1$, by lemma 4. 1. From this, we obtain $\chi_{l(r)}(y) = 1$ i. e. $y \in l(r)$, which implies $yr = 0 = x$, a contradiction. Hence $(\mu r)(x) = 0$, for any $x \neq 0$. □

Theorem 4.5. *If μ and σ are two non-zero $(\in, \in \vee q)$-fuzzy left ideals of R satisfying the supremum property such that $\mu \subseteq_e \sigma$, then $\mu r \subseteq_e \sigma r$ for any $r \in R$.*

Theorem 4.6. *Let μ be a non-zero $(\in, \in \vee q)$-fuzzy left ideal of R with $\mu \subseteq_e R$. Then for any non-zero $a \in R$, there exists an $(\in, \in \vee q)$-fuzzy left ideal ν of R with $\nu \subseteq_e R$ such that νa is non-zero.*

Proof. Since $\mu \subseteq_e R$, so $\mu_t \subseteq_e R$, for any $t \in (0, T]$, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ. This gives, for any $a(\neq 0) \in R$, there exists a left ideal N of R with $N \subseteq_e R$ such that $Na \subseteq \mu_t$, $\forall t \in (0, T]$ and $Na \neq (0)$. Now we define an $(\in, \in \vee q)$-fuzzy left ideal ν of R such that $\nu(x) = \begin{cases} t, & \text{for } x \in N, t \in (0, T] \\ 0, & \text{if } x \notin N. \end{cases}$. Then $\nu \subseteq_e R$, by lemma 3. 10. Since $Na \neq (0)$, there exists $y(\neq 0) \in Na$ such that $y = na$ for $n \in N$. Hence, we have $(\nu a)(y) = M(\bigvee\{\nu(z) : z \in N, za = y\}, 0.5) \geq M(\nu(m), 0.5) \neq 0$ as $n \in N$. □

Theorem 4.7. *Let μ, ν and σ be non-zero $(\in, \in \vee q)$-fuzzy left ideals of R such that $\mu \subseteq \nu \subseteq \sigma$. Then $\mu \subseteq_e \sigma$ if and only if $\mu \subseteq_e \nu \subseteq_e \sigma$.*

Proof. Let $\mu \subseteq_e \sigma$. Let θ be any non-zero $(\in, \in \vee q)$-fuzzy left ideal of ν. Then θ is also non-zero $(\in, \in \vee q)$-fuzzy left ideal of σ. This implies, there exists a non-zero x in R, with $\mu(x) \geq M(t, 0.5)$ and $\theta(x) \geq M(t, 0.5)$, for any $t \in (0, T]$, where $T(\leq \mu(0))$ is the supremum among all non-zero values of μ. Which gives $\mu \subseteq_e \nu$. Again, let ξ be any non-zero $(\in, \in \vee q)$-fuzzy left ideal in σ. Since $\mu \subseteq_e \sigma$. Then there exists a non-zero x in R, with $\mu(x) \geq M(t, 0.5)$ and $\xi(x) \geq M(t, 0.5)$, for any $t \in (0, T]$. From this $\nu \subseteq_e \sigma$. Conversely, we assume that $\mu \subseteq_e \nu \subseteq_e \sigma$. Let η be a non-zero $(\in, \in \vee q)$-fuzzy left ideal in σ. So, there exists a non-zero x in R, with $\nu(x) \geq M(t, 0.5)$ and $\eta(x) \geq M(t, 0.5)$, for any $t \in (0, T]$, where $T(\leq \nu(0))$ is the supremum among all non-zero values of ν. Also $\eta \cap \nu$ is a non-zero $(\in, \in \vee q)$-fuzzy left ideal of ν and $\mu \subseteq_e \nu$ implies for any $s \in (0, T^*]$, $\mu(y) \geq M(s, 0.5)$ and $(\eta \cap \nu)(y) \geq M(s, 0.5)$ for some $y(\neq 0) \in R$, where $T^*(\leq \mu(0))$ is the supremum among all non-zero values of μ. From this, there exists a non-zero y in R such that $\mu(y) \geq M(s, 0.5)$ and $\eta(y) \geq M(s, 0.5)$, for any $s \in (0, T^*]$. This gives $\mu \subseteq_e \sigma$. □

5. Examples on $(\in, \in \vee q)$-Fuzzy Essential Ideals

Example. 1. Consider the non-commutative ring $R = \{(a_{ij})_{2\times 2} : a_{ij} \in \mathbb{Z}_2\} = \{A_1, A_2, ..., A_8, B_1, B_2, ..., B_8\}$, where

$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$

$$A_5 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, A_7 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_8 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$
$$B_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B_6 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B_7 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, B_8 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Take $\mu(A_1) = k$, $\mu(A_2) = \mu(A_3) = \mu(A_4) = ... = \mu(B_8) = l$, where $k, l \in (0, 1]$, $l \leq k$. Then $\mu \subseteq_e R$

Example. 2. Consider the ring $R = \{0, a, b, c\}$ with addition and multiplication operations defined as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

and

$.$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	0	b
c	0	c	0	c

Two fuzzy subsets μ and σ of R are defined as follows:

$\mu(0) = k$, $\mu(a) = \mu(b) = \mu(c) = l$

and

$\sigma(0) = k$, $\sigma(a) = \sigma(b) = \sigma(c) = m$, where $k, l, m \in (0, 1]$, $l \leq m \leq k$.

Then $\mu \subseteq_e \sigma$.

6. Conclusion

In this article, we have defined a new kind of fuzzy ideal namely $(\in, \in \vee q)$-fuzzy essential ideal. We have also defined fuzzy annihilators using the notion of belongingness and quasi-coincidence of fuzzy points of sets. We have investigated various properties of $(\in, \in \vee q)$-fuzzy essential ideals. In our opinion this is an opening for investigations of fuzzy aspects of singular ideal of rings, rings with chain conditions on annihilators, fuzzy Goldie dimension and fuzzy dual Goldie dimension.

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