International Journal of Pure and Applied Mathematics
Volume 76 No. 5 2012, 647-659
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

ERROR ANALYSIS, STABILITY, AND NUMERICAL SOLUTIONS OF FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we investigate the use of spline functions of integral form to approximate the solution of differential equations fractional order. The proposed spline approximation method is first introduced, and then error analysis and stability are theoretically investigated. A numerical example is given to illustrate the applicability, accuracy and stability of the suggested method.

AMS Subject Classification: 65L20
Key Words: spline functions, differential equations fractional order

1. Introduction

In the last few years there has been growing interest in the use of various types of spline functions for numerical treatments of ordinary differential equations [10, 11, 12, 14]. Recently, fractional order differential equations have found interesting applications in the area of mathematical biology [1, 2]. Also, mathematical models of numerous engineering and physical phenomena involve either ordinary or partial differential equations of fractional order.

Received: July 25, 2011

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The analysis of fractional differential equations of the form

$$y^{(\alpha)}(x) = f(x, y(x)), \quad y(0) = y_0 \tag{1}$$

is studied by Kia Dithelm and Neville J. Ford [4] and a number of approximation solutions of the initial value problem (1) have been proposed in the literature. The Adams-Bashforth-Moulton method is introduced in [6, 7]. An alternative technique is the backward differentiation formula is presented in [5] which is based on the idea of discretizing the differential operator in Eq. (1) by certain finite difference. The main result of [5] was that under suitable assumption we expect an $o(h^{\alpha-2})$ convergence behavior. In [8] an improvement of the performance of the method presented in [5] is achieved by applying extrapolation principles. Kia Dithelm et al. [9] considered a fast algorithm for a numerical solution of Eq. (1) in the sense of Caputo. More recently, Lagrange multiplier and homotopy perturbation methods are numerically considered for multi-order fractional differential equation see [13].

The purpose of this paper is to introduce the concept of definition of spline functions for solving the fractional ordinary differential equation of the form:

$$\begin{aligned} &y^{(\alpha)}(x) = f(x, y(x)), \qquad a \le x \le b, \\ &y(a) = y_0, \quad \alpha > 0. \end{aligned} \tag{2}$$

where f is known function, y is the unknown function need to be found for $x > a$.

The rest of the paper is organized as follows: In Section 3, we discuss the existence and uniqueness of the solution as in [4]. In Section 4, we estimate the error and convergence analysis of the approximation solutions and define upper bound to the error. In Section 5, we study the stability analysis of the proposed method. In Section 6, a numerical example is given to illustrate the accuracy and stability of the proposed method. In Section 7, a conclusion of our paper is given.

2. Existence and Uniqueness of the Solution

Looking at the questions of existence and uniqueness of the solution, we can present the following results that are very similar to the corresponding classical theorems known in the case of first-order equation. Only the scalar setting will be discussed explicitly; the generalization to vector-valued functions is straightforward.

Theorem 3.1. (Existence, see [4]) *Assume that* $D := [0, \chi^*] \times [y_0^{(0)} - \beta, y_0^{(0)} + \beta]$ *with some* $\chi^* > 0$ *and some* $\beta > 0$, *and let the function* $f : D \to \Re$ *be continuous. Furthermore, define* $\chi := \min\{\chi^*, (\beta\ \Gamma(\alpha+1)/\ \|f\|_\infty)^{1/\alpha}\}$. *Then, there exists a function* $y : [0, \chi] \to \Re$ *solving the equation* (2).

Theorem 3.2. (Uniqueness, see [4]) *Assume that* $D := [0, \chi^*] \times [y_0^{(0)} - \beta, y_0^{(0)} + \beta]$ *with some* $\chi^* > 0$ *and some* $\beta > 0$, *Furthermore, let the function* $f : D \to \Re$ *be bounded on* D *and fulfill a Lipschitz condition with respect to the second variable, i.e.*

$$| f(x, y) - f(x, z)| \le L\ |y - z|$$

with some constant $L > 0$ *independent of* x, y *and* z . *Then, denoting* χ *as in theorem 3.1, there exists at most one function* $y : [0, \chi] \to \Re$ *solving the equation* (2).

The proof is identical with proof in [4].

Suppose that $f : [a, b] \times \Re \to \Re$ is continuous and satisfies the Lipsechitz condition

$$| f(x, y_1) - f(x, y_2)| \le L\ |y_1 - y_2| \tag{3}$$

for all $(x, y_1), (x, y_2) \in [a, b] \times \Re$

These conditions assure the existence of unique solution y of equation (2).

Let Δ be a uniform partition to the interval $[a, b]$ defined by the nodes

$$\Delta : a = x_0 < x_1 < ... < x_k < x_{k+1} < ... < x_n = b,\ \ x_k = x_0 + kh,\ h = \frac{b-a}{n}$$
$$\text{and } k = 0, 1, ..., n-1.$$

Define the spline function $S(x)$ approximating the exact solution y by:

$$S(x) = S_\Delta(x) \tag{4}$$

Assume the function $y^{(\alpha)}$ has a modulus of continuity $\omega(y^{(\alpha)}, h) = \omega(h)$. Choosing the required positive integer m, we define $S_\Delta(x)$ by:

$$S_\Delta(x) = S_k^{[m]}(x) = S_{k-1}^{[m]}(x_k) + \underset{x_k\ \ x}{\overset{-\alpha}{D}} f(x, S_k^{[m-1]}(x)) \tag{5}$$

where $S_{-1}^{[m]}(x_0) = y_0,$.

In (5), we use the following m iterations for $x \in [x_k, x_{k+1}]$, *k=0, 1,. . . , n-1* and *j=1, 2,. . . , m*

$$S_k^{[j]}(x) = S_{k-1}^{[m]}(x_k) + \underset{x_k\ \ x}{\overset{-\alpha}{D}} f(x, S_k^{[j-1]}(x)) \tag{6}$$

$$S_k^{[0]}(x) = S_{k-1}^{[m]}(x_k) + M_k(x - x_k)$$

$$M_k = f(x_k, S_{k-1}^{[m]}(x_k)).$$

From above it is obviously such $S_\Delta(x)$ exists and unique see [4].

3. Error Estimation and Convergence Analysis

To estimate the error, it is convenient to represent the exact y solution in the form given below described by the following scheme [3].

$$y(x) = \sum_{i=-n}^{n-1} \frac{D^{(\alpha+k)}y(x_0)(x-x_k)^{\alpha+k}}{\Gamma(\alpha+k+1)} + R_n \ , \ \ n \in Z^+ \tag{7}$$

for all $a \le x_0 < x \le b$, where $R_n(x)$ is the reminder where $\zeta_k \in (x_k, x_{k+1})$, $y(x_k) = y_k$.

For *i=0,1, 2,* $\ldots$ *,m* we write:

$$y^{[j]}(x) = y(x_k) + \underset{x_k \ \ x}{\overset{-\alpha}{D}} f(x, y_k^{[j-1]}(x)) \tag{8}$$

Moreover, we denote to the estimated error of $y(x)$ at any point $x \in [a,b]$ by

$$e(x) = |y(x) - S_\Delta(x)| \ \ , \ \ e_k = |y_k - S_\Delta(x_k)| \tag{9}$$

Lemma 1. (see [14]) *Let α and β be non negative real numbers and $\{A_i\}_{i=0}^m$ be a sequence satisfying $A_i \le \alpha + \beta A_{i+1}$ for i=1,2,...,m-1 then: $A_1 \le \beta^{m-1}A_m + \alpha\sum_{i=0}^{m-2}\beta^i$.*

Lemma 2. (see [14]) *Let α and β be non negative real numbers, $\beta \ne 1$ and $\{A_i\}_{i=0}^k$ be a sequence satisfying $A_0 \ge 0$ and $A_{i+1} \le \alpha + \beta A_i$ for i=0,1,...,k then:*

$$A_{k+1} \le \beta^{k+1}A_0 + \alpha\frac{(\beta^{k+1}-1)}{(\beta-1)}$$

Definition 1. For any $x \in [x_k, x_{k+1}]$, *k=0,1,...,n-1* and *j=1,2,...,m* we define the operator $T_{kj}(x)$ by: $T_{kj}(x) = \left|y^{[m-j]}(x) - S_k^{[m-j]}(x)\right|$ whose norm is defined by: $\|T_{kj}\| = \max_{x\in[x_k,x_{k+1}]}\{T_{kj}(x)\}$.

Definition 2. (see [14]) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order then we say that $A \le B$, iff:

(i) Both a_{ij} and b_{ij} are non negative

(ii) $a_{ij} \le b_{ij}$ $\forall i, j$.

Lemma 3. *For any $x \in [x_k, x_{k+1}]$, k=0, 1,...,n-1 and j=1, 2,..., m then*

$$\|T_{km}\| \le (1 + hL)e_k + h\omega(h) \tag{10}$$

$$\|T_{k1}\| \le d_1 e_k + d_2 h^m \omega(h) \tag{11}$$

where $d_1 = \sum_{i=0}^{m} L^i$ and $d_2 = L^{m-1}$ are constants independent of h and L is Lipsechitz constant.

Proof. Using the definition of the proposed spline function and Lipsechitz condition and (9), we obtain:

$$T_{km}(x) = \left| y^{[0]}(x) - S_k^{[0]}(x) \right| \\ \le \left| y(x_k) - S_{k-1}^{[m]}(x_k) \right| + \left| y^{(\alpha)}(\zeta_k) - M_k \right| |x - x_k| \tag{12}$$

Since:

$$\left| y^{(\alpha)}(\zeta_k) - M_k \right| \le \left| y^{(\alpha)}(\zeta_k) - y_k^{(\alpha)} \right| + \left| y_k^{(\alpha)} - M_k \right| \\ \le \omega(y^{(\alpha)}, h) + L \left| y(x_k) - S_{k-1}^{[m]}(x_k) \right| = \omega(h) + Le_k \tag{13}$$

where L is a Lipschitz constant independent of h.

Using (13) in (12) we get:

This completes the proof of inequality (10).

To prove inequality (11), we compute $\|T_{kj}\|$ using (8), (12), (3) and (9), respectively we get:

$$+L \underset{x_k}{\overset{-\alpha}{D}}{}_{x} \left| y^{[m-j-1]}(x) - S_k^{[m-j-1]}(x) \right| = e_k + L \underset{x_k}{\overset{-\alpha}{D}}{}_{x} T_{k(j+1)}(x)$$

Thus

$$\|T_{kj}\| = \max_{x \in [x_k, x_{k+1}]} \{T_{kj}(x)\} \le e_k + L \underset{x_k}{\overset{-\alpha}{D}}{}_{x} \max_{x \in [x_k, x_{k+1}]} \{T_{k(j+1)}(x)\} \\ \le e_k + L \frac{(x - x_k)^\alpha}{\alpha\Gamma(\alpha)} \left\| T_{k(j+1)} \right\| \\ \le e_k + L \frac{h^\alpha}{\alpha\Gamma(\alpha)} \left\| T_{k(j+1)} \right\| \le e_k + Lh \left\| T_{k(j+1)} \right\|,$$

where $\frac{h^{\alpha}}{\alpha\Gamma(\alpha)} \leq h$, since $0<\alpha<1$

Using Lemma 1 and inequality (10), we get:

$$\|T_{k1}\| \leq (Lh)^{m-1}\|T_{km}\| + \left[\sum_{i=0}^{m-2}(Lh)^{i}\right] e_k$$

$$\leq (Lh)^{m-1}\left[(1+hL)e_k + h\omega(h)\right] + \left[\sum_{i=0}^{m-2}(Lh)^{i}\right] e_k$$

$$\leq \left[\sum_{i=0}^{m} L^{i}\right] e_k + (L)^{m-1} h^{m}\omega(h) = d_1 e_k + d_2 h^{m}\omega(h)$$

where $d_1 = \left[\sum_{i=0}^{m} L^{i}\right]$ *and* $d_2 = (L)^{m-1}$ are constants independent of h. This completes the proof of the inequality (11).

Lemma 4. *Let* $e(x)$ *be defined as in* (9), *and when* $h < (L)^{\frac{1}{m}-1}$ *then there exist constants* d_3, d_4 *independent of* h *such that the following inequality holds:*

$$e(x) \leq (1+hd_3)e_k + d_4 h^{m+1}\omega(h).$$

Proof. Using (8), (5), (3), and (9) and (11), respectively we get:

$$e(x) = |y(x) - S_{\Delta}(x)| = \left|y^{[m]}(x) - S_k^{[m]}(x)\right| \leq \left|y(x_k) - S_{k-1}^{[m]}(x_k)\right| +$$

$$L \underset{x_k \ \ x}{\overset{-\alpha}{D}} \left|y^{[m-1]}(x) - S_k^{[m-1]}(x)\right|$$

$$= e_k + L \underset{x_k \ \ x}{\overset{-\alpha}{D}} T_{k1}(x) = e_k + L \underset{x_k \ \ x}{\overset{-\alpha}{D}} \max_{x\in[x_k,x_{k+1}]}\{\ \ T_{k1}(x)\}$$

$$= e_k + L\|T_{k1}\| \underset{x_k \ \ x}{\overset{-\alpha}{D}} = e_k + L\|T_{k1}\| \frac{h^{\alpha}}{\alpha\Gamma(\alpha)}$$

$$\leq e_k + hL\|T_{k1}\| \leq e_k + hL[d_1 e_k + d_2 h^{m}\omega(h)]$$

$$= (1+hd_3)e_k + d_4 h^{m+1}\omega(h) \tag{14}$$

where $d_3 = Ld_1$ *and* $d_4 = Ld_2$ are constants independent of h. Since inequality (14) holds for any $x \in [a,b]$, then setting $x = x_{k+1}$, we get:

$$e_{k+1} \leq (1+hd_3)e_k + d_4 h^{m+1}\omega(h)$$

Using Lemma 2 and noting that $e_0 = 0$, we obtain:

$$e(x) \leq d_4 h^m \omega(h) \frac{[1+hd_3]^{k+1}-1}{1+hd_3-1} \leq \frac{d_4}{d_3} h^m \omega(h) \left\{ \left[1 + \left(\frac{b-a}{n} \right) d_3 \right]^n - 1 \right\}$$
$$\leq d_5 h^m \omega(h) \qquad (15)$$

where $d_5 = \frac{d_4}{d_3} [\exp(d_3(b-a)) - 1]$ is a constant independent of h.

4. Stability Analysis of the Method

The stability of the method means that "a small change in the starting values only produces bounded changes in the numerical values provided by the method". To study the stability of the proposed method, we change $S_\Delta(x)$ to $W_\Delta(x)$ where

$$W_\Delta(x) = W_k^{[m]}(x) = W_{k-1}^{[m]}(x_k) + \underset{x_k \quad x}{\overset{-\alpha}{D}} f(x, W_k^{[m-1]}(x)) \qquad (16)$$

with $W_{-1}^{[m]}(x_0) = y_0^*$.

In equation (16), we use the following m iterations. For j=*1, 2,...,m* we have

$$W_k^{[j]}(x) = W_{k-1}^{[m]}(x_k) + \underset{x_k \quad x}{\overset{-\alpha}{D}} f(x, W_k^{[m-1]}(x)) \qquad (17)$$

$$W_k^{[0]}(x) = W_{k-1}^{[m]}(x_k) + N_k(x - x_k)$$

$$N_k = f\left(x_k, W_{k-1}^{[m]}(x_k)\right) ..$$

Moreover, we use the following notation

$$e^*(x) = |S_\Delta(x) - W_\Delta(x)| \quad , \quad e_k^* = |S_\Delta(x_k) - W_\Delta(x_k)| \qquad (18)$$

Definition 4. For any $x \in [x_k, x_{k+1}]$, *k=0,1,...,n-1* and *j=1,2,...,m,* we define the operator $T_{kj}^*(x)$ by: $T_{kj}^*(x) = \left| S_k^{[m-j]}(x) - W_k^{[m-j]}(x) \right|$ whose norm is defined by:

$$\|T_{kj}^*\| = \max_{x \in [x_k, x_{k+1}]} \{T_{kj}(x)\}.$$

Lemma 5. *For any $x \in [x_k, x_{k+1}]$, k=0, 1,..., n-1 and j=1, 2,..., m, then:*

$$\|T_{km}^*\| \leq (1+hL)e_k^* \qquad (19)$$

$$\|T^*_{k1}\| \le \left[\sum_{i=0}^{m} L^i\right] e^*_k \tag{20}$$

where L is a Lipschitz constant independent of h.

Proof. Using (6), (17), and (3) and (18), respectively we get:

$$T^*_{kj}(x) = \left|S_k^{[0]}(x) - W_k^{[0]}(x)\right| \le \left|S_{k-1}^{[m]}(x_k) - W_{k-1}^{[m]}(x_k)\right| + |M_k - N_k|\,|x - x_k|$$
$$\le e^*_k + |M_k - N_k|\,|x - x_k|$$

Since $|M_k - N_k| = |f\ (x_k, S_{k-1}^{[m]}(x_k)) -\ f(x_k, W_{k-1}^{[m]}(x_k)\ )|$

$$\le L\ \left|S_{k-1}^{[m]}(x_k) - W_{k-1}^{[m]}(x_k)\right|\ = Le^*_k$$

Thus we obtain:

$$\|T^*_{km}\| \le (1 + hL)e^*_k$$

where L is a constant independent of h , which prove the first part of the lemma.

To prove the second part of the lemma. We compute $\left\|T^*_{kj}\right\|$ using (6), (17), (3), and (18), respectively to get:

$$\begin{aligned} T^*_{kj}(x) &= \left|S_k^{[m-j]}(x) - W_k^{[m-j]}(x)\right| \\ &\le \left|S_{k-1}^{[m]}(x_k) - W_{k-1}^{[m]}(x_k)\right| + L \underset{x_k\quad x}{\overset{-\alpha}{D}} S_k^{[m-j-1]}(x) - W_k^{[m-j-1]}(x) \\ &= e^*_k + L \underset{x_k\quad x}{\overset{-\alpha}{D}} \{T^*_{k(j+1)}(x)\}. \end{aligned} \tag{21}$$

Thus

$$\begin{aligned} \left\|T^*_{kj}\right\| &= \max_{x\in[x_k,x_{k+1}]} \{T^*_{kj}(x)\,\} \\ &\le e^*_k + L \underset{x_k\quad x}{\overset{-\alpha}{D}} \max_{x\in[x_k,x_{k+1}]} \{T^*_{k(j+1)}(x)\}\ \le e^*_k + Lh \left\|T^*_{k(j+1)}\right\|. \end{aligned}$$

Now, using Lemma 1 and inequality (19), we get:

$$\|T^*_{k1}\| \le (Lh)^{m-1} \|T^*_{km}\| + \left[\sum_{i=0}^{m-2} (Lh)^i\right] e^*_k \le (Lh)^{m-1} [(1+hL)e^*_k] + \left[\sum_{i=0}^{m-2} (Lh)^i\right] e^*_k$$
$$\le \left[\sum_{i=0}^{m} L^i\right] e^*_k = d_1 e^*_k$$

where d_1 is defined as in (11). This completes the proof of the lemma.

Lemma 6. *Let* $e^*(x)$ *be defined as in* (18)*, and when* $h < (L)^{\frac{1}{m}-1}$ *then the following inequality holds:*

$$e^*(x) \le (1 + hd_3)\, e_k^* \tag{22}$$

Proof. Using (5), (16), (3), (18), and (20), respectively we get:

$$e^*(x) = |S_\Delta(x) - W_\Delta(x)| = \left|S_k^{[m]}(x) - W_k^{[m]}(x)\right|$$

$$\le \left|S_{k-1}^{[m]}(x_k) - W_{k-1}^{[m]}(x_K)\right| + L \underset{x_k \quad x}{\overset{-\alpha}{D}} \left|S_k^{[m-1]}(x) - W_k^{[m-1]}(x)\right|$$

$$= e_k^* + L \underset{x_k \quad x}{\overset{-\alpha}{D}} \{T_{k1}^*(x)\} = e_k^* + L \underset{x_k \quad x}{\overset{-\alpha}{D}} \max_{x\in[x_k,x_{k+1}]}\{ \quad T_{k1}^*(x)\}$$

$$= e_k^* + L \left\|T_{k1}^*\right\| \underset{x_k \quad x}{\overset{-\alpha}{D}} \le e_k^* + Lh \left\|T_{k1}^*\right\| \le e_k^* + hL[d_1 e_k^*]$$

$$= (1 + hd_3)e_k^* \tag{23}$$

where $d_3 = Ld_1$ is constant independent of h.

Now, since inequality (23) holds for any $x \in [a, b]$, then setting $x = x_{k+1}$, we get

$$e_{k+1}^* \le (1 + hd_3)e_k^*$$

using lemma 2, we get:

$$e^*(x) \le \{1 + hd_3\}^{k+1} e_0^* \le \left\{1 + \frac{(b-a)d_3}{n}\right\}^n e_0^* \le d_8 e_0^* \tag{24}$$

where $d_8 = e^{(b-a)d_3}$ is a constant independent of h.

5. Numerical Example

Consider the fractional differential equation

$D^{0.5}[y](x) = -y(x) + x^2 + \frac{2}{\Gamma(??)} x^{1.5}$. The exact solution is $y = x^2$.

The obtained numerical results are summarized in Tables 6.1, 6.2 and 6.3 respectively with iteration $(m = 1, 2, 3)$. The accuracy and stability of the proposed spline method using spline function of integral form are illustrated in these Tables where, the first column represents the values of fractional order

α	x	Appr. solution for the problem	Absolute Error	Appr. solution for the perturbed problem	Absolute diff. between Appr. solutions the two
0.1	0.1	y=0.000940111	8.4×10^{-5}	0.000940127	1.53289×10^{-8}
	0.02	y=0.00292199	2.5×10^{-3}	0.00292201	2.39914×10^{-8}
	0.03	y=0.00569601	4.8×10^{-3}	0.00569604	3.13401×10^{-8}
	0.04	y=0.00916692	7.6×10^{-3}	0.00916696	3.79869×10^{-8}
	0.05	y=0.0132783	1.1×10^{-3}	0.0132783	4.41767×10^{-8}
0.2	0.1	y=0.000548301	4.5×10^{-5}	0.000548311	1.00000×10^{-8}
	0.02	y=0.00182564	1.4×10^{-3}	0.00182566	1.58953×10^{-8}
	0.03	y=0.00370481	2.9×10^{-3}	0.00370483	2.16077×10^{-8}
	0.04	y=0.00613465	4.5×10^{-3}	0.00613467	2.69391×10^{-8}
	0.05	y=0.00908433	6.6×10^{-3}	0.00908436	3.20198×10^{-8}
0.3	0.1	y= 0.000318381	2.2×10^{-4}	0.000318387	5.82450×10^{-9}
	0.02	y= 0.00113567	7.4×10^{-4}	0.00113568	1.04516×10^{-8}
	0.03	y= 0.00239922	1.5×10^{-3}	0.00239923	1.47858×10^{-8}
	0.04	y= 0.00408762	2.5×10^{-3}	0.00408764	1.89618×10^{-8}
	0.05	y=0.00618821	3.7×10^{-3}	0.00618823	2.30361×10^{-8}
0.4	0.1	y= 0.000184096	8.4×10^{-4}	0.000184099	3.55100×10^{-9}
	0.02	y= 0.000703506	3.1×10^{-4}	0.00113568	6.82362×10^{-9}
	0.03	y= 0.00154724	6.5×10^{-3}	0.00239923	1.00467×10^{-8}
	0.04	y= 0.00271233	1.1×10^{-3}	0.00271235	1.32538×10^{-8}
	0.05	y=0.00419792	1.7×10^{-3}	0.00419794	1.55516×10^{-8}
0.5	0.1	y= 0.000106018	6.0×10^{-6}	0.00010602	3.55100×10^{-9}
	0.02	y= 0.000434043	3.4×10^{-5}	0.00113568	$0.000434048 \times 10^{-9}$
	0.03	y= 0.000993812	9.4×10^{-5}	0.00239923	$0.000993818 \times 10^{-9}$
	0.04	y= 0.00179258	1.9×10^{-4}	0.00271235	$0.00179259 \times 10^{-9}$
	0.05	y=0.00283642	3.4×10^{-4}	0.00419794	$0.00283643 \times 10^{-8}$

Table 1: The accuracy and stability of the proposed spline method using spline function of integral form (using $h = 0.01$ and $m = 1$)

α , the second column represents the values of x, the third column gives the approximate solution at the corresponding points while the fourth column gives the absolute error between the exact solution and the obtained approximate numerical solution with the initial conditions $y(0) = 0$. With small change in the initial conditions, $y^*(0) = 0.00001$, an approximate solution, for the perturbed problem, is computed as shown in the fifth column. To test the stability, the difference between the two approximate solutions is computed as shown in the Sixth column.

From the obtained results in Tables 6.1, 6.2 and 6.3 respectively. With iteration ($m = 1, 2, 3$) the given test example, we can see that the proposed method using the spline function gives acceptable accuracy and the method is shown to be very efficient where its algorithm has recursive nature which makes it easy and simple to be programmed.

6. Conclusion

In this paper, we investigated the possibility of using the spline functions in fractional form for approximating the solution of fractional ordinary differential

α	x	Appr. solution for the problem	Absolute Error	Appr. solution for the perturbed problem	Absolute diff. between Appr. solutions the two
0.1	0.1	y=0.0000204867	7.9×10^{-5}	0.0000204872	5.37741×10^{-10}
	0.02	y=0.000127615	2.7×10^{-4}	0.000127617	1.69429×10^{-9}
	0.03	y=0.000376302	5.2×10^{-4}	0.000376305	3.36119×10^{-9}
	0.04	y=0.000815843	7.8×10^{-4}	0.000815848	5.00000×10^{-9}
	0.05	y=0.00149358	1.0×10^{-3}	0.00149358	8.12385×10^{-9}
0.2	0.1	y=3.75799	9.6×10^{-5}	3.7581×10^{-6}	1.10999×10^{-10}
	0.02	y=0.0000293943	3.7×10^{-4}	0.0000293948	4.38470×10^{-10}
	0.03	y=0.0000989963	8.0×10^{-4}	0.0000989973	9.92329×10^{-10}
	0.04	y= 0.000235822	1.3×10^{-3}	0.000235824	1.78509×10^{-9}
	0.05	y=0.000464404	2.0×10^{-3}	0.000464407	2.82959×10^{-9}
0.3	0.1	$y = 6.11997 \times 10^{-7}$	9.9×10^{-5}	6.12017×10^{-7}	2.02145×10^{-11}
	0.02	$y = 6.09723 \times 10^{-6}$	3.9×10^{-4}	6.09733×10^{-6}	9.77866×10^{-11}
	0.03	y=0.0000236507	8.7×10^{-4}	0.000023651	2.64521×10^{-10}
	0.04	y= 0.0000622709	1.5×10^{-3}	0.0000622714	5.25506×10^{-10}
	0.05	y= 0.00013252	2.3×10^{-3}	0.000132521	8.99527×10^{-10}
0.4	0.1	$y = 8.83867 \times 10^{-8}$	9.9×10^{-5}	8.8389910-8	3.24742×10^{-12}
	0.02	$y = 1.13768 \times 10^{-6}$	3.9×10^{-4}	1.137710-6	2.10546×10^{-11}
	0.03	$y = 5.12525 \times 10^{-6}$	8.8×10^{-4}	5.1253110-6	6.36236×10^{-10}
	0.04	y= 0.000015004	1.5×10^{-3}	0.0000150041	1.40437×10^{-10}
	0.05	y=0.0000346646	2.5×10^{-3}	0.0000346648	2.60814×10^{-10}
0.5	0.1	$y = 1.13095 \times 10^{-8}$	9.9×10^{-5}	1.13099×10^{-8}	4.59512×10^{-13}
	0.02	$y = 1.90755 \times 10^{-7}$	3.9×10^{-4}	1.90759×10^{-7}	3.90199×10^{-12}
	0.03	$y = 1.0064 \times 10^{-6}$	9.0×10^{-4}	1.00641×10^{-6}	1.37993×10^{-11}
	0.04	$y = 3.29521 \times 10^{-6}$	1.5×10^{-3}	3.29524×10^{-6}	3.40474×10^{-11}
	0.05	$y = 8.30312 \times 10^{-6}$	2.5×10^{-3}	8.30319×10^{-6}	6.89258×10^{-11}

Table 2: The accuracy and stability of the proposed spline method using spline function of integral form (using $h = 0.01$ and $m = 2$)

equation. The error analysis and stability are theoretically investigated. A numerical example is given to illustrate the applicability, accuracy and stability of the proposed method. The obtained numerical results reveal that the method is stable and gives high accuracy.

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α	x	Appr. solution for the problem	Absolute Error	Appr. solution for the perturbed problem	Absolute diff. between Appr. solutions the two
0.1	0.1	y=4.4201810-8	9.9×10^{-5}	4.4203710-8	1.85807×10^{-12}
	0.02	y=8.2876710-7	3.9×10^{-4}	8.2878510-7	1.76452×10^{-11}
	0.03	y=4.7001610-6	8.8×10^{-4}	4.7002310-6	6.74338×10^{-11}
	0.04	y=0.000016307	1.5×10^{-3}	0.0000163072	1.77166×10^{-10}
	0.05	y=0.0000431872	2.5×10^{-3}	0.0000431876	3.78727×10^{-10}
0.2	0.1	y=7.7651610-10	9.9×10^{-5}	7.76555×10^{-6}	3.90052×10^{-11}
	0.02	y=2.5688310-8	3.9×10^{-4}	2.5689×10^{-8}	6.52083×10^{-11}
	0.03	y=2.0299610-7	8.8×10^{-4}	2.02999×10^{-7}	3.46560×10^{-12}
	0.04	y= 8.9086910-7	1.5×10^{-3}	8.90881×10^{-7}	1.14967×10^{-13}
	0.05	y=2.8303310-6	2.5×10^{-3}	2.83036×10^{-6}	2.94335×10^{-14}
0.3	0.1	$y = 7.8161 \times 10^{-12}$	0.1×10^{-4}	7.81656×10^{-12}	4.64774×10^{-16}
	0.02	$y = 4.90385 \times 10^{-9}$	0.2×10^{-4}	4.904×10^{-10}	1.47123×10^{-14}
	0.03	$y = 5.63465 \times 10^{-10}$	1.5×10^{-3}	5.63476×10^{-9}	1.13533×10^{-13}
	0.04	$y = 3.22462 \times 10^{10-8}$	2.5×10^{-3}	3.22467×10^{-8}	4.90496×10^{-13}
	0.05	$y = 1.25849 \times 10^{-7}$	3.7×10^{-3}	1.2585×10^{-7}	1.54069×10^{-12}
0.3	0.1	$y = 4.33931 \times 10^{-14}$	1.0×10^{-4}	4.33961×10^{-14}	3.02837×10^{-18}
	0.02	$y = 5.57094 \times 10^{-12}$	4.0×10^{-4}	5.57114×10^{-12}	1.95924×10^{-16}
	0.03	$y = 9.73346 \times 10^{-11}$	9.0×10^{-4}	9.73369×10^{-11}	2.29660×10^{-15}
	0.04	$y = 7.49942 \times 10^{10}$	16.0×10^{-4}	7.49956×10^{-10}	1.33453×10^{-14}
	0.05	$y = 3.68599 \times 10^{-9}$	25.0×10^{-4}	3.68605×10^{-9}	5.27430×10^{-14}
0.5	0.1	$y = 1.27912 \times 10^{-16}$	1.0×10^{-4}	1.27923×10^{-16}	1.03947×10^{-20}
	0.02	$y = 3.63972 \times 10^{-14}$	4.0×10^{-4}	3.63987×10^{-14}	1.48915×10^{-18}
	0.03	$y = 1.01345 \times 10^{-12}$	9.0×10^{-4}	1.01348×10^{-12}	2.77964×10^{-17}
	0.04	$y = 1.08702 \times 10^{-11}$	16.0×10^{-4}	1.08705×10^{-11}	2.24694×10^{-16}
	0.05	$y = 6.90614 \times 10^{-11}$	25.0×10^{-4}	6.90625×10^{-11}	1.14708×10^{-15}

Table 3: The accuracy and stability of the proposed spline method using spline function of integral form (using $h = 0.01$ and $m = 3$)

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