International Journal of Pure and Applied Mathematics
Volume 76 No. 5 2012, 669-698
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

ASSET PRICING MODEL WITH HETEROGENEOUS AGENTS: THE WEALTH DYNAMICS

Serena Brianzoni[1], Cristiana Mammana[2], Elisabetta Michetti[3] §
[1]Università Politecnica delle Marche
Ancona, ITALY
[2,3]Università di Macerata
Macerata, ITALY

Abstract: We develop an adaptive model which characterizes the evolution of wealth distribution when agents switch between different trading strategies. The wealth of each group is updated not only as a consequence of portfolio growth of agents adopting the relative strategy, but also due to the flow of agents coming from the other group. This switching mechanism is investigated in a Walrasian scenario and under a growing dividend process. A stationary dynamic model is obtained in terms of excess return, wealth and agent proportions, able to explain wealth distribution among agents in the long run.

AMS Subject Classification: 39A28, 39A30, 39A33, 39A60
Key Words: asset pricing, wealth dynamics, switching mechanisms, long run dynamics, trapping sets

1. Introduction

In the growing field of heterogeneous agent models (see [20] and [23] for an extensive survey), several works have focused on the study of the market equilibrium price and wealth distribution when agents' demand exhibits constant

Received: October 1, 2011

§Correspondence author

relative risk aversion (CRRA). In fact, recent literature leans towards a framework in which investors' optimal decisions depend on their wealth, which is in agreement with the assumption of CRRA utility function, see e.g. [24], [25], [26] and [13]. Differently, in the setting with constant absolute risk aversion (CARA) utility function the wealth dynamics does not affect agents' demand.

Recently, some works have explored the CRRA framework with heterogeneous agents in order to take into account the wealth dynamics of agents. Examples which document this progess in the literature are: [15], [16], [14], [3], [2], [4] and [10].

[15] study an asset pricing model with two groups of agents and fixed population fractions. The model shows the volatility clustering as well as other anomalies observed in financial market data. In order to obtain a more appealing framework, [16] allow agents to switch between different trading strategies and show profitability of momentum trading strategies over short time intervals and of contrarian trading strategies over long time intervals.
[3] develops a more general framework. They obtain analytical results for a market populated by an arbitrarily large number of technical traders whose demand functions belong to a relatively large set. [2] applies this general agent-based model to special cases of optimizing behavior.
All these studies are based on the assumption that the dividend yield is an i.i.d. process. On the contrary, [14] and [4] investigate the CRRA framework with exogenously growing dividends. More in detail, [14] consider a market maker scenario and two specific types of traders. Differently, [4] keep the investment strategies as general as possible and make use of the market clearing condition to determine the asset price.
In spite of the wide progress in the literature, many contributions to the development and analysis of financial models with heterogeneous agents and CRRA utility consider fixed proportions of agents. Moreover, models which allow agents to switch between different trading strategies (such as [16]) make the following assumption: when agents switch from an old strategy to a new strategy, they agree to accept the average wealth level of agents using the new strategy.
An important step towards the analysis of the evolution of wealth has been made in [10]. In fact, when agents switch between different prediction strategies, the wealth of the new group takes into account the wealth coming from the group of origin. As a consequence, the wealth of each group is updated from period t to $(t+1)$ not only as a consequence of portfolio growth of agents adopting the relative strategy, but also due to the flow of agents coming from the other group. [10] investigate this framework with an i.i.d. dividend process

and under a market-maker scenario.

In our paper, we consider the same switching mechanism as in [10] but we allow for a growing dividend process and assume a Walrasian scenario. Even if the analytical exploration of the CRRA framework with heterogeneous agents and switching mechanism is difficult, we obtain a tractable three-dimensional dynamical system. Moreover, we can investigate the role of the switching mechanism in determing the long-run wealth evolution.
Since we are focusing on the role of market force as wealth-driven selection and empirical implication of the model (stylized empirical findings), our paper takes place in the realm of heterogeneous agent models. By looking at agents' survival in a financial market, our work belongs to the field of *evolutionary finance* literature as well (see e.g. [17], [18], [8], [9] and [19]). In contrast with the evolutionary finance approach, we incorporate the feedback on past prices with the investment strategies, as in the recent contributions of [4] and [10]. Our market is populated by two groups of agents, where type-1 are rational traders while type-2 are chartists. For a long time, chartists have been viewed as irrational and they would be driven out of the market by rational traders. We will see that both types of agents can survive in the market in the long run.

The paper is organized as follows. Section 2 presents the general framework describing an asset pricing model where agents use different beliefs about future price. We obtain a three-dimensional dynamical system where wealth distribution can be explicitly considered. We focus on the case in which the market is populated by rational traders and chartists. The resulting map has a particular structure, being piecewise smooth, although we analytically find the steady states and we prove the existence of trapping regions in Section 3.

In order to consider the possibility of complex dynamics to be exhibited, in Section 4 we perform a series of numerical simulations showing periodic or even chaotic fluctuations in prices and a great variety of qualitative behaviors. In addition, the new switching mechanism involves complexity in the long run wealth distribution. Section 5 offers a comparision between the model with switching and the model without switching. It will inform us on the role played by the switching mechanism in the long-run wealth evolution. Section 6 concludes.

2. The Model

Consider an economy composed of one risky asset paying a random dividend y_t at time t and one risk free asset with constant risk free rate $r = R - 1 > 0$. We denote by p_t the price (ex dividend) per share of the risky asset at time t.

In order to describe wealth's dynamics, we assume that all agents belonging to the same group agree to share their wealth whenever an agent joins the group (or leaves it).[1] According to such an assumption, the wealth of agent type h at time t, denoted by $\bar{w}_{h,t}$, is given by the total wealth of group h in the fraction of agents belonging to this group. As a consequence, the dynamics of the wealth of investor h is described by the following equation:

$$W_{h,t+1} = (1 - z_{h,t})\bar{w}_{h,t}R + z_{h,t}\bar{w}_{h,t}(1 + \rho_{t+1}) = \bar{w}_{h,t}[R + z_{h,t}(\rho_{t+1} - r)] \quad (1)$$

where $z_{h,t}$ is the fraction of wealth that agent-type h invests in the risky asset and $\rho_{t+1} - r = \frac{p_{t+1}+y_{t+1}-(1+r)p_t}{p_t}$ is the excess return in period $t+1$, and $W_{h,t+1}$ represents the wealth earned by agent h at time $t+1$ later on the investment made at time t.

The individual demand function $z_{h,t}$ derives from the maximization problem of the expected utility of $W_{h,t+1}$, i.e. $z_{h,t} = \max_{z_{h,t}} E_{h,t}[u_h(W_{h,t+1})]$, where $E_{h,t}$ is the belief of investor-type h about the conditional expectation, based on the available information set of past prices and dividends. Since each agent is assumed to have a CRRA utility function, investors' optimal decisions depend on their wealth. Following [15], the optimal (approximated) solution is given by:

$$z_{h,t} = \frac{E_{h,t}[\rho_{t+1} - r]}{\lambda_h \sigma_h^2} \quad (2)$$

where λ_h is the relative risk aversion coefficient and $\sigma_h^2 = Var_{h,t}[\rho_{t+1} - r]$ is the belief of investor h about the conditional variance of excess returns.

2.1. Population Distribution and Adaptiveness

We consider a market populated by two groups of agents which revise their beliefs in a boundedly rational way in the sense that, at each time, most agents choose the predictor generating the highest performance. In other words, the fraction $n_{h,t}$ of traders using strategy h at time t will be updated according to a performance measure $\phi_{h,t}$. Hence, as in [12], the adaptation of beliefs, i.e. the dynamics of the fractions $n_{h,t}$ of different trader types, is given by:

$$n_{h,t+1} = \frac{exp[\beta(\phi_{h,t} - C_h)]}{Z_{t+1}}, \quad Z_{t+1} = \sum_h exp[\beta(\phi_{h,t} - C_h)] \quad (3)$$

where the parameter β is the *intensity of choice* measuring how fast agents choose between different predictors and $C_h \geq 0$ are the costs for strategy h.

[1] This assumption was introduced by [16].

When β increases, more and more agents use the predictor with the highest fitness. In the extreme case $\beta = +\infty$ all agents choose the strategy with the highest fitness, in the other extreme case $\beta = 0$ no switching at all takes place and both fractions are equal to $\frac{1}{2}$.

Let us define the performance measure $\phi_{h,t}$. [10] observe that at time t agent h measures the performance he has achieved and then chooses whether to stay in group h or switch to another one. With this consideration, the authors measure the past performance as the personal wealth coming from the investment in the risky asset with respect to $\bar{w}_{h,t}$, i.e. $\phi_{h,t} = z_{h,t}(\rho_{t+1} - r)$. Differently, in this framework we assume forward looking agents with strategy selection: at time t the fitness measure used for strategy selection is given by:

$$\phi_{h,t} = z_{h,t}E_{h,t}[\rho_{t+1} - r]. \tag{4}$$

In making this assumption we observe that agents are forward looking in their investment decision, since they maximize the expected utility of wealth. In our model agents are forward looking with both investment decisions and selection of prediction strategies ([11] introduce forward looking agents in the cobweb model).

Finally, as in [12], we define the difference in fractions at time t, i.e. $m_t = n_{1,t} - n_{2,t}$, so that $n_{1,t} = \frac{1+m_t}{2}$ and $n_{2,t} = \frac{1-m_t}{2}$. As a consequence:

$$m_{t+1} = \tanh\left[\frac{\beta}{2}\left(\phi_{1,t} - \phi_{2,t} - C_1 + C_2\right)\right]. \tag{5}$$

2.2. Wealth Dynamics

Following [10], we consider a market populated by two groups of agents, where agents can move from group i to group j while both movements are not simultaneously possible. Note that the unilateral switching is in agreement with the fact that agents choose the predictor generating the highest past performance. As a consequence, we define the difference in the fraction of agents of type h from time t to time $t+1$:

$$\Delta n_{h,t+1} = n_{h,t+1} - n_{h,t}, \quad h = 1, 2$$

hence $\Delta n_{1,t+1} = -\Delta n_{2,t+1}$.
Then, two possible cases may occur, depending on the strategy with the better performance:

1) $\Delta n_{1,t+1} \geq 0$ (i.e. $m_{t+1} \geq m_t$), iff $\Delta n_{1,t+1}$ fraction of agents moves from group 2 to group 1 at time $t+1$,

2) $\Delta n_{1,t+1} < 0$ (i.e. $m_{t+1} < m_t$), iff $\Delta n_{1,t+1}$ fraction of agents moves from group 1 to group 2 at time $t+1$.

In order to describe the wealth dynamics of each group, we define the *wealth of group h* as:

$$\tilde{W}_{h,t} = n_{h,t}\bar{w}_{h,t}, \quad h = 1, 2,$$

then in our framework $\tilde{W}_{h,t}$ is the share of the wealth produced by group h to the total wealth.
Clearly, in defining the evolution of the wealth of each group, we have to take into account the switching. Let us go to focus on the wealth of group 1:

1) if $\Delta n_{1,t+1}$ fraction of agents moves from group 2 to group 1, then $\tilde{W}_{1,t+1}$ is given by the wealth coming from group 2 and the wealth generated by traders of type 1,

2) if $\Delta n_{1,t+1}$ fraction of agents moves from group 1 to group 2, the wealth $\tilde{W}_{1,t+1}$ is simply given by the wealth of agents which do not leave the group.

Summarizing:

$$\tilde{W}_{1,t+1} = \begin{cases} \Delta n_{1,t+1} W_{2,t+1} + n_{1,t} W_{1,t+1} = \\ = n_{1,t}\left(W_{1,t+1} - W_{2,t+1}\right) + n_{1,t+1} W_{2,t+1}, \quad \text{if} \quad m_{t+1} \geq m_t \\ n_{1,t+1} W_{1,t+1}, \quad \text{if} \quad m_{t+1} < m_t \end{cases}. \tag{6}$$

In a similar way we can derive the wealth of group 2:

$$\tilde{W}_{2,t+1} = \begin{cases} n_{2,t+1} W_{2,t+1} \quad \text{if} \quad m_{t+1} \geq m_t \\ n_{2,t}\left(W_{2,t+1} - W_{1,t+1}\right) + n_{2,t+1} W_{1,t+1} \quad \text{if} \quad m_{t+1} < m_t \end{cases}. \tag{7}$$

Finally, we define the relative wealth of each group as the wealth of group h in the total wealth:

$$w_{h,t} = \tilde{W}_{h,t} / \sum_h \tilde{W}_{h,t}$$

where $\tilde{W}_{h,t} = n_{h,t}\bar{w}_{h,t}$ and $h = 1, 2$.
In the following, we will consider the dynamics of the difference in the relative wealths:

$$w_t := w_{1,t} - w_{2,t}.$$

To this end, we recall (6) and (7) and analyze both the cases $m_{t+1} \geq m_t$ and $m_{t+1} < m_t$.

1. **Case** $m_{t+1} \geq m_t$

   From (6) and (7) and after some algebra we obtain:

   $w_{t+1} = w_{1,t+1} - w_{2,t+1} = \frac{\tilde{W}_{\mathbf{1},t+\mathbf{1}} - \tilde{W}_{\mathbf{2},t+\mathbf{1}}}{\tilde{W}_{\mathbf{1},t+\mathbf{1}} + \tilde{W}_{\mathbf{2},t+\mathbf{1}}} = \frac{-2n_{\mathbf{2},t+\mathbf{1}} W_{\mathbf{2},t+\mathbf{1}} + n_{\mathbf{2},t} W_{\mathbf{2},t+\mathbf{1}} + n_{\mathbf{1},t} W_{\mathbf{1},t+\mathbf{1}}}{n_{\mathbf{2},t} W_{\mathbf{2},t+\mathbf{1}} + n_{\mathbf{1},t} W_{\mathbf{1},t+\mathbf{1}}}$

   where we have made use of relation $n_{1,t} + n_{2,t} = 1$.

   Considering equation (1), it follows:

   $w_{t+1} = \frac{-2n_{2,t+1}\bar{w}_{2,t}[R+z_{2,t}(\rho_{t+1}-r)] + n_{2,t}\bar{w}_{2,t}[R+z_{2,t}(\rho_{t+1}-r)] + n_{1,t}\bar{w}_{1,t}[R+z_{1,t}(\rho_{t+1}-r)]}{n_{2,t}\bar{w}_{2,t}[R+z_{2,t}(\rho_{t+1}-r)] + n_{1,t}\bar{w}_{1,t}[R+z_{1,t}(\rho_{t+1}-r)]}.$

   Remembering that $\bar{w}_{h,t} = \frac{\tilde{W}_{h,t}}{n_{h,t}}$ and $w_{h,t} = \frac{\tilde{W}_{h,t}}{\tilde{W}_{1,t}+\tilde{W}_{2,t}}$, hence $\bar{w}_{h,t} = (\tilde{W}_{1,t} + \tilde{W}_{2,t})\frac{w_{h,t}}{n_{h,t}}$, we divide both numerator and denominator for $\tilde{W}_{1t} + \tilde{W}_{2t}$ to obtain:

   $w_{t+1} = \frac{-2n_{2,t+1}\frac{w_{2,t}}{n_{2,t}}[R+z_{2,t}(\rho_{t+1}-r)] + n_{2,t}\frac{w_{2,t}}{n_{2,t}}[R+z_{2,t}(\rho_{t+1}-r)] + n_{1,t}\frac{w_{1,t}}{n_{1,t}}[R+z_{1,t}(\rho_{t+1}-r)]}{n_{2,t}\frac{w_{2,t}}{n_{2,t}}[R+z_{2,t}(\rho_{t+1}-r)] + n_{1,t}\frac{w_{1,t}}{n_{1,t}}[R+z_{1,t}(\rho_{t+1}-r)]}.$

   Finally, recalling that $w_{1,t} = \frac{1+w_t}{2}$, $w_{2,t} = \frac{1-w_t}{2}$ and $n_{1,t} = \frac{1+m_t}{2}$, $n_{2,t} = \frac{1-m_t}{2}$, we have the following equation:

   $w_{t+1} = \frac{-\frac{1-m_{t+1}}{1-m_t}(1-w_t)[R+z_{2,t}(\rho_{t+1}-r)] + \frac{1-w_t}{2}[R+z_{2,t}(\rho_{t+1}-r)] + \frac{1+w_t}{2}[R+z_{1,t}(\rho_{t+1}-r)]}{\frac{1-w_t}{2}[R+z_{2,t}(\rho_{t+1}-r)] + \frac{1+w_t}{2}[R+z_{1,t}(\rho_{t+1}-r)]}$

   $$= \frac{-2\frac{1-m_{t+1}}{1-m_t}(1-w_t)[R+z_{2,t}(\rho_{t+1}-r)]}{(1-w_t)[R+z_{2,t}(\rho_{t+1}-r)] + (1+w_t)[R+z_{1,t}(\rho_{t+1}-r)]} + 1.$$

2. **Case** $m_{t+1} < m_t$

   By following the same steps, we obtain:

   $w_{t+1} = w_{1,t+1} - w_{2,t+1} = \frac{\tilde{W}_{\mathbf{1},t+\mathbf{1}} - \tilde{W}_{\mathbf{2},t+\mathbf{1}}}{\tilde{W}_{\mathbf{1},t+\mathbf{1}} + \tilde{W}_{\mathbf{2},t+\mathbf{1}}} = \frac{2n_{\mathbf{1},t+\mathbf{1}} W_{\mathbf{1},t+\mathbf{1}} - (n_{\mathbf{2},t} W_{\mathbf{2},t+\mathbf{1}} + n_{\mathbf{1},t} W_{\mathbf{1},t+\mathbf{1}})}{n_{\mathbf{2},t} W_{\mathbf{2},t+\mathbf{1}} + n_{\mathbf{1},t} W_{\mathbf{1},t+\mathbf{1}}}$,

   and by using equation (1):

   $w_{t+1} = \frac{2n_{1,t+1}\bar{w}_{1,t}[R+z_{1,t}(\rho_{t+1}-r)] - (n_{2,t}\bar{w}_{2,t}[R+z_{2,t}(\rho_{t+1}-r)] + n_{1,t}\bar{w}_{1,t}[R+z_{1,t}(\rho_{t+1}-r)])}{n_{2,t}\bar{w}_{2,t}[R+z_{2,t}(\rho_{t+1}-r)] + n_{1,t}\bar{w}_{1,t}[R+z_{1,t}(\rho_{t+1}-r)]}.$

   Dividing both numerator and denominator for $\tilde{W}_{1t} + \tilde{W}_{2t}$ and recalling that $\bar{w}_{h,t} = (\tilde{W}_{1,t} + \tilde{W}_{2,t})\frac{w_{h,t}}{n_{h,t}}$:

   $w_{t+1} = \frac{2n_{1,t+1}\frac{w_{1,t}}{n_{1,t}}[R+z_{1,t}(\rho_{t+1}-r)] - (n_{2,t}\frac{w_{2,t}}{n_{2,t}}[R+z_{2,t}(\rho_{t+1}-r)] + n_{1,t}\frac{w_{1,t}}{n_{1,t}}[R+z_{1,t}(\rho_{t+1}-r)])}{n_{2,t}\frac{w_{2,t}}{n_{2,t}}[R+z_{2,t}(\rho_{t+1}-r)] + n_{1,t}\frac{w_{1,t}}{n_{1,t}}[R+z_{1,t}(\rho_{t+1}-r)]}.$

   Finally, we arrive at the following equation:

   $w_{t+1} = \frac{\frac{1+m_{t+1}}{1+m_t}(1+w_t)[R+z_{1,t}(\rho_{t+1}-r)] - (\frac{1-w_t}{2}[R+z_{2,t}(\rho_{t+1}-r)] + \frac{1+w_t}{2}[R+z_{1,t}(\rho_{t+1}-r)])}{\frac{1-w_t}{2}[R+z_{2,t}(\rho_{t+1}-r)] + \frac{1+w_t}{2}[R+z_{1,t}(\rho_{t+1}-r)]}$

$$= \frac{2\frac{1+m_{t+1}}{1+m_t}(1+w_t)[R+z_{1,t}(\rho_{t+1}-r)]}{(1-w_t)[R+z_{2,t}(\rho_{t+1}-r)]+(1+w_t)[R+z_{1,t}(\rho_{t+1}-r)]}-1.$$

As a consequence, the dynamics of the state variable w_t can be described by the following system:

$$w_{t+1} = \begin{cases} \frac{F_1}{G}+1 & if \;\; m_{t+1} \geq m_t \\ \\ \frac{F_2}{G}-1 & if \;\; m_{t+1} < m_t \end{cases} \tag{8}$$

where:
$F_1 = -2\frac{1-m_{t+1}}{1-m_t}(1-w_t)[R+z_{2,t}(\rho_{t+1}-r)]$,
$F_2 = 2\frac{1+m_{t+1}}{1+m_t}(1+w_t)[R+z_{1,t}(\rho_{t+1}-r)]$,
$G = (1-w_t)[R+z_{2,t}(\rho_{t+1}-r)]+(1+w_t)[R+z_{1,t}(\rho_{t+1}-r)]$.

The switching mechanism leads to a continuous piecewise smooth function.

The number of shares at price p_t that investor h wishes to hold is given by $N^D_{h,t} = \frac{z_{h,t}\bar{w}_{h,t}}{p_t}$. Summing the demands of all investors gives the aggregate demand:

$$N^D_t = \frac{n_{1,t}z_{1,t}\bar{w}_{1,t}+n_{2,t}z_{2,t}\bar{w}_{2,t}}{p_t}.$$

The supply of shares is assumed to be fixed $N^S_t = N$. Hence the Market Clearing Equation (MCE) at time t, $N^D_t = N^S_t$, is given by:

$$\frac{n_{1,t}z_{1,t}\bar{w}_{1,t}+n_{2,t}z_{2,t}\bar{w}_{2,t}}{p_t} = N.$$

It is well-known that the CRRA framework with heterogeneous agents is difficult to be explored. In order to obtain an analytically-tractable model we focus on the case of zero net supply of shares.[2] Nevertheless, it is possible to work with non-zero supply as well. In this last case one obtain a higher-dimensional system more difficult to be handled. However, numerical simulations confirm the analytical results reached in the present study.

In the case of zero net supply of shares the MCE becomes:

$$n_{1,t}z_{1,t}\bar{w}_{1,t}+n_{2,t}z_{2,t}\bar{w}_{2,t} = 0$$

[2] In making this assumption we follow [12] and [15].

after dividing for $\tilde{W}_{1,t} + \tilde{W}_{2,t}$ we arrive at the following equation:

$$z_{1,t}w_{1,t} + z_{2,t}w_{2,t} = 0,$$

which can be rewritten in terms of the state variable w_t:

$$z_{1,t}(1 + w_t) + z_{2,t}(1 - w_t) = 0. \tag{9}$$

Specializing the MCE to the case of one type of agents, it is possible to get a bench mark notion of the rational expectations "fundamental solution" $p_t^\star$, i.e. a long-run market clearing price path which would be obtained under homogeneous beliefs on expected excess return (see [12] and [14]).

Differently from [10], in which a market-maker scenario is investigated under the assumption of an i.i.d. dividend process, we allow for a growing dividend process described by:

$$E_t(y_{t+1}) = (1 + g)y_t,$$

with $0 \leq g < r$. In this case, the fundamental solution is:

$$p_t^\star = \frac{(1 + g)y_t}{r - g}$$

and the fundamental evolves over time according to $E_t(p_{t+1}^\star) = (1 + g)p_t^\star$.

Notice that *along the fundamental path* the expected yield and the capital gain are given by $E_t[y_{t+1}/p_t^*] = r-g$ and $E_t[(p_{t+1}^* - p_t^*)/p_t^*] = g$, hence $E_t[\rho_{t+1} - r] = 0$.

2.3. Trading Strategies

In order to derive the final system, let us go to specify the type of beliefs we will investigate. In this work we focus on the case of a market populated by two groups of agents, where type 1 are rational agents and type 2 chartists.[3]

In the following we will consider the dynamics of the excess return:

$$x_{t+1} = \rho_{t+1} - r.$$

Rational agents have perfect foresight and they are able to compute x_{t+1} correctly. Hence, the rational predictor is defined as:

$$E_{1,t}[x_{t+1}] = x_{t+1}.$$

[3] [12] consider perfect foresight versus trend chaser in the CARA framework.

Differently, type-2 agents are chartists, who use simple linear trading rules. More specifically, we adopt the following chartists' expectation scheme:

$$E_{2,t}[x_{t+1}] = ax_t$$

with $a \in \mathbb{R}$.

We assume that rational agents pay a cost $C > 0$ because they are more sophisticated as they perfectly compute x_{t+1}, while chartists' beliefs are freely available.

Considering our expectation schemes, the MCE (9) becomes:

$$\frac{x_{t+1}}{\lambda_1 \sigma_1^2}(1 + w_t) + \frac{ax_t}{\lambda_2 \sigma_2^2}(1 - w_t) = 0. \tag{10}$$

Let us go to distinguish two cases: $w_t \neq -1$ and $w_t = -1$.

1. ***Case*** $w_t \neq -1$**.**

Formula (10) implies that the state variable x_{t+1} evolves according to the following equation:

$$x_{t+1} = -\frac{\lambda_1 \sigma_1^2}{\lambda_2 \sigma_2^2} \frac{1 - w_t}{1 + w_t} a x_t. \tag{11}$$

Consider now the wealth dynamics in the case $m_{t+1} \geq m_t$. After substituting (11) into the first equation of System (8)[4] we obtain:

$$w_{t+1} = -\frac{1}{R} \frac{1 - m_{t+1}}{1 - m_t}(1 - w_t) \left[R + \frac{ax_t}{\lambda_2 \sigma_2^2} \left(-\frac{\lambda_1 \sigma_1^2}{\lambda_2 \sigma_2^2} \frac{1 - w_t}{1 + w_t} a x_t \right) \right] + 1.$$

Also, using relation $1 - \tanh(x) = \frac{2}{e^{2x}+1}$ and Equation (5),

$$w_{t+1} = \frac{2(1-w_t)}{R(1-m_t)\{exp[\beta(\phi_{1,t}-\phi_{2,t}-C)]+1\}} \left[\left(\frac{a}{\lambda_2 \sigma_2^2} \right)^2 \lambda_1 \sigma_1^2 \frac{1-w_t}{1+w_t} x_t^2 - R \right] + 1.$$

Consider now the case $m_{t+1} < m_t$ of System (8):

$$w_{t+1} = \frac{1}{R} \frac{1 + m_{t+1}}{1 + m_t}(1 + w_t) \left[R + \frac{x_{t+1}^2}{\lambda_1 \sigma_1^2} \right] - 1.$$

Using of (11) and simplification of the corresponding expression lead to equation:

[4]Observe that, under the market clearing equilibrium condition, function G reduces to $G = 2R$.

$$w_{t+1} = \frac{1}{R}\frac{1+m_{t+1}}{1+m_t}(1+w_t)\left[R + \frac{\lambda_1\sigma_1^2}{(\lambda_2\sigma_2^2)^2}a^2\left(\frac{1-w_t}{1+w_t}\right)^2 x_t^2\right] - 1.$$

Finally, using relation $1+\tanh(x) = \frac{2}{e^{-2x}+1}$ and Equation (5), one obtains the following equation for the wealth w_{t+1}:

$$w_{t+1} = \frac{2(1+w_t)}{R(1+m_t)\{exp[-\beta(\phi_{1,t}-\phi_{2,t}-C)]+1\}}\left[R + \left(\frac{a}{\lambda_2\sigma_2^2}\right)^2 \lambda_1\sigma_1^2\left(\frac{1-w_t}{1+w_t}\right)^2 x_t^2\right] - 1.$$

2. ***Case*** $w_t = -1$**.**

From Equation (10) it immediately follows $x_t = 0$, as a consequence $z_{2,t} = 0$ and the demand of the second group is zero. On the other hand, the wealth of the first group at time t is zero. This implies that the demand of the first group is zero as well. Being both the aggregate demand and the supply of shares zero at time t, it follows:

$$x_{t+1} = x_t,$$

and from Equation (5) we obtain:

$$m_{t+1} = \tanh\left\{-\frac{\beta}{2}C\right\}.$$

Consider now the wealth dynamics. Remembering that $w_t = -1$ and $x_{t+1} = x_t = 0$, from System (8) we immediately obtain:

$$w_{t+1} = \begin{cases} -2\frac{1-m_{t+1}}{1-m_t} + 1 & if \;\; m_{t+1} \geq m_t \\ -1 & if \;\; m_{t+1} < m_t \end{cases}.$$

3. The System

Our model is given by a noisy nonlinear system, since dividends and fundamental price are stochastic processes. Netherless, in this work we focus on the dynamics of the deterministic skeleton of the model by assuming that dividends evolve according to their expected value. Notice that under the assumption of a growing dividend process y_t such that $E_t(y_{t+1}) = (1+g)y_t = (r-g)p_t^\star$, the excess return in period $t+1$ can be rewritten as:

$$x_{t+1} = \frac{p_{t+1} + y_{t+1} - (1+r)p_t}{p_t} = \frac{p_{t+1} - (1+r)p_t}{p_t} + \frac{(r-g)p_t^\star}{p_t}.$$

Our adaptive asset pricing and wealth dynamics model is written in terms of the state variables x_t, m_t and w_t as follows:

1. if $w_t \neq -1$

$$T_1 : \begin{cases} x_{t+1} = f(x_t, w_t) = -\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2}\frac{1-w_t}{1+w_t} a x_t \\ m_{t+1} = g(x_t, w_t) = \tanh\left[\frac{\beta}{2}\left(\Delta\phi_t - C\right)\right] \\ w_{t+1} = h(x_t, m_t, w_t) = \begin{cases} h_1(x_t, m_t, w_t), & \forall m_{t+1} \geq m_t \\ h_2(x_t, m_t, w_t), & \forall m_{t+1} < m_t \end{cases} \end{cases} \qquad (12)$$

where:

$$h_1(x_t, m_t, w_t) = \frac{2(1-w_t)}{R(1-m_t)\{exp[\beta(\Delta\phi_t - C)]+1\}}\left[\left(\frac{a}{\lambda_2\sigma_2^2}\right)^2 \lambda_1\sigma_1^2 \frac{1-w_t}{1+w_t} x_t^2 - R\right] + 1$$

$$h_2(x_t, m_t, w_t) = \frac{2(1+w_t)}{R(1+m_t)\{exp[-\beta(\Delta\phi_t - C)]+1\}}\left[\left(\frac{a}{\lambda_2\sigma_2^2}\right)^2 \lambda_1\sigma_1^2 \left(\frac{1-w_t}{1+w_t}\right)^2 x_t^2 + R\right] - 1$$

and $\Delta\phi_t = \phi_{1,t} - \phi_{2,t} = \frac{x_{t+1}^2}{\lambda_1\sigma_1^2} - \frac{a^2 x_t^2}{\lambda_2\sigma_2^2} = \left[\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2}\left(\frac{1-w_t}{1+w_t}\right)^2 - 1\right]\frac{a^2 x_t^2}{\lambda_2\sigma_2^2}$

2. if $w_t = -1$

$$T_2 : \begin{cases} x_{t+1} = x_t = 0 \\ m_{t+1} = \tanh\left\{-\frac{\beta}{2}C\right\} \\ w_{t+1} = \begin{cases} -2\frac{1-m_{t+1}}{1-m_t} + 1, & \forall m_{t+1} \geq m_t \\ -1, & \forall m_{t+1} < m_t \end{cases} \end{cases} . \qquad (13)$$

Hence the final dynamical system is given by $T = T_1 \cup T_2$.

3.1. Steady States

In order to find the fixed points owned by the final system, we put $x_t = x$, $m_t = m$ and $w_t = w$, $\forall t$.

- Let $a < 0$, then the MCE (10) implies:

$$\{x = 0\} \cup \left\{w = \frac{-A-B}{A-B}\right\}, \text{ with } A = \frac{1}{\lambda_1\sigma_1^2} \text{ and } B = \frac{a}{\lambda_2\sigma_2^2}.$$ [5]

  Using the remaining equations of System (12), it is easy to see that $x = 0$ implies $m = \tanh\left\{-\frac{\beta}{2}C\right\}$ and $w = \bar{w}$, $\forall \bar{w} \in (-1, 1]$. Similarly, $w = \frac{-A-B}{A-B}$ implies $x = 0$ and $m = \tanh\left\{-\frac{\beta}{2}C\right\}$. Using System (13) one can obtain $x = 0$, $m = \tanh\left\{-\frac{\beta}{2}C\right\}$ and $w = -1$.

- Let $a > 0$, then the MCE admits a unique solution given by $x = 0$.[6] As in the previous case, from the remaining equations of System (12) we obtain $m = \tanh\left\{-\frac{\beta}{2}C\right\}$ and $w = \bar{w}$, $\forall \bar{w} \in (-1, 1]$, while System (13) implies $m = \tanh\left\{-\frac{\beta}{2}C\right\}$ and $w = -1$.

- Let $a = 0$, then the MCE implies:

$$\{x = 0\} \cup \{w = -1\}.$$

  Again, for $x = 0$ we obtain: (a) $m = \tanh\left\{-\frac{\beta}{2}C\right\}$ and $w = w^\star$, $\forall w^\star \in (-1, 1]$ (from System (12)), (b) $m = \tanh\left\{-\frac{\beta}{2}C\right\}$ and $w = -1$ (from System (13)). For $w = -1$ we immediately have $m = \tanh\left\{-\frac{\beta}{2}C\right\}$ and $x = 0$.

Summarizing, the steady states of the system are described by the following lemma.

Lemma 1. *For all $a \in \mathbb{R}$ every point $E(x^\star = 0, m^\star = \tanh\left\{-\frac{\beta}{2}C\right\}, w = w^\star)$ is a fixed point in which the long-run wealth distribution is given by any constant $w^\star \in [-1, 1]$. In other words, the system T admits a continuum of steady states which are located in a one-dimensional subset (a straight line) of the phase space.*

[5] Notice that $a < 0$ implies $-1 < w < 1$.
[6] In this case $w = \frac{-A-B}{A-B} \notin [-1, 1]$.

Observe that any steady state E is characterized by $x^\star = 0$, i.e. $(1+r)p_{t-1} - (r-g)p^\star_{t-1} = p_t$, for all t, hence $(1+r)p_{t-1} - (1+r)p^\star_{t-1} + (1+g)p^\star_{t-1} = p_t$ and, being $p^\star_t = (1+g)p^\star_{t-1}$ we immediately obtain:

$$p_t - p^\star_t = (1+r)(p_{t-1} - p^\star_{t-1}) \quad \forall t. \tag{14}$$

Equation 14 describes the evolution of prices in equilibrium. Note that $p_t = p^\star_t$ $\forall t$ imples $x_t = 0$ $\forall t$, while the converse implication does not necessarly hold. This means that some (infinitely many) steady states defined by Lemma 1 are fundamental equilibria (the price is at the fundamental value).

The presence of a continuum of stady states depends on the fact that the expectation schemes are equivalent in equilibrium. In fact, if $x = 0$ then both the predictors are equal to zero and the performance measures are the same. Moreover, both assets are equivalent in terms of return and all the agents earn the same return independent of their investment shares. In such equilibria, the wealth of all agents is constant over time and any initial distribution of w_0 is an equilibrium level.

All the equilibria are on the boundary of the phase space. In addition, by investigating the eigenvalues of the system, it is possible to see that they are non-hyperbolic. Anyway, they are structurally unstable since system perturbations move such equilibria and their eigenvalues.

3.2. Trapping Sets

Our system is defined by a piecewise function, where the phase space is divided into two regions by the surface $m_{t+1} = m_t$. All the equilibria are non-hyperbolic and belong to the boundary. Given these features, we are looking for some trapping subsets of the phase space. We recall that a set X is trapping for a map T if $T(X) \subseteq X$.

A first trapping region is characterized by $x_t = 0$, $\forall t$, as claimed by the following proposition.

Proposition 2. *For all parameter values the set* $X = \{(x_t, m_t, w_t) : x_t = 0\}$ *is trapping for any initial condition* (x_0, m_0, w_0).

Proof. The proof immediately follows from the fact that $x_t = 0$ implies $x_{t+1} = 0$, $\forall m_t, w_t$. □

As the set X is trapping, then $x_t = 0 \Rightarrow x_{t+1} = 0$ $\forall t$. In other words, our heterogeneous agent model reduces to the homogeneous case being the predictor of the first group equivalent to the predictor of the second group. In this case the system restricted to the subspace X converges to a fixed point and the long-run wealth evolution depends on the initial condition. More in detail, the restriction of the map to the subset X, $T_X : (m_t, w_t) \to (m_{t+1}, w_{t+1})$, is defined by:

$$m_{t+1} = \tanh\left\{-\frac{\beta}{2}C\right\}, \quad w_{t+1} = w_t.$$

For initial conditions belonging to set X the dynamics of the system can be completely described. More precisely, let $(0, m_0, w_0) \in X$ the state of the system at the initial time. Then $T(0, m_0, w_0) = (0, m_1, w_1)$ where $m_1 = m^\star = \tanh\left\{-\frac{\beta}{2}C\right\}$ while $w_1 = 1 - (1 - w_0)\frac{(1-m^\star)}{(1-m_0)}$ iff $m_0 \leq m^\star$ or $w_1 = -1 + (1 + w_0)\frac{(1+m^\star)}{(1+m_0)}$ iff $m_0 > m^\star$. Under the iteration of system T it is trivial to observe that $T^k(0, m_1, w_1) = (0, m_1, w_1)$, $\forall k = 1, 2, ..., n$.

Hence we can conclude that initial conditions $(0, m_0, w_0) \in X$ generate trajectories converging to the fixed point $E_1 = (0, m^\star, 1 - (1 - w_0)\frac{(1-m^\star)}{(1-m_0)})$ if $m_0 \leq m^\star$ (i.e. at the initial time the difference in fractions of agents is less than its equilibrium value), or to the fixed point $E_2 = (0, m^\star, -1 + (1 + w_0)\frac{(1+m^\star)}{(1+m_0)})$ if $m_0 > m^\star$ (i.e. at the initial time the difference in fractions of agents is greater than its equilibrium value).

Anyway, both types of agents survive in the market, having positive wealth. In our equilibria, agents get the same return (both assets are equivalent in terms of return) and the market is not able to play the role of a selection force. Obviously, exceptions are the cases where one type of traders has the total market wealth. In fact, further trapping subsets of the phase space are characterized by $w_t = 1$ and $w_t = -1$.

Proposition 3. *For all parameter values the set* $W_1 = \{(x_t, m_t, w_t) : w_t = 1\}$ *is trapping for any initial condition* (x_0, m_0, w_0) *such that* $m_0 \leq \tanh\{-\frac{\beta}{2}C\}$.

Proof. Looking at system (12) for $m_{t+1} \geq m_t$, we find that $w_t = 1$ implies $w_{t+1} = 1$ $\forall x_t$. Therefore we require $m_{t+1} \geq m_t$ $\forall t$. From functions f and g of system (12) it is easy to obtain $x_{t+1} = 0$ and $m_{t+1} = \tanh(-(\beta/2)C)$. Hence we get the fixed point characterized by $w^\star = 1$. As a consequence a sufficient condition for the set W_1 to be trapping is $m_0 \leq \tanh(-(\beta/2)C)$. □

The trapping set W_1 defined by Proposition 3 allow us to conclude on the global stability of the steady state $E_1(x^\star, m^\star, 1)$ (with $x^\star = 0, m^\star = \tanh\left\{-\frac{\beta}{2}C\right\}$) when the dynamical system is restricted to the subspace W_1. At the initial time rational traders have the total market wealth, i.e. $w_0 = 1$, then by considering the MCE it follows $x_1 = 0$. Furthermore, being $m_0 \leq \tanh(-(\beta/2)C)$, previous stability results on the restriction X can be applied and the equilibrium E_1 with $w_0 = 1$ is reached. Note that, being $m_0 \leq \tanh(-(\beta/2)C)$, a shift to class 1 takes place.

Following similar arguments, the trapping set characterized by $w_t = -1$ is present.

Proposition 4. *For all parameter values the set* $W_2 = \{(x_t, m_t, w_t) : w_t = -1\}$ *is trapping for any initial condition* $(0, m_0, w_0)$ *such that* $m_0 \geq \tanh\{-\frac{\beta}{2}C\}$.

Proof. Looking at system (13) for $m_{t+1} < m_t$, we find that $w_t = -1$ implies $w_{t+1} = -1 \; \forall x_t$. Therefore we require $m_{t+1} < m_t \; \forall t$. Since $x_{t+1} = 0$ and $m_{t+1} = \tanh(-(\beta/2)C)$. Hence we get the fixed point characterized by $w^\star = -1$. As a consequence a sufficient condition for the set W_2 to be trapping is $m_0 > \tanh(-(\beta/2)C)$. Observe that W_2 is trapping also if $m_0 = \tanh(-(\beta/2)C)$ as function h of system (13) is continuous. □

The trapping set W_2 defined by Proposition 4 allow us to conclude on the global stability of the steady state $E_2(x^\star, m^\star, -1)$ with $x^\star = 0, m^\star = \tanh\left\{-\frac{\beta}{2}C\right\}$ when the dynamical system is restricted to the subspace W_2. If at the initial time the total market wealth is accumulated by chartists ($w_0 = -1$) then, by considering the MCE, $x_1 = 0$. Again, the stability results on the restriction X hold and the fixed point characterized by the total wealth owned by chartists is reached. Since $m_0 \geq \tanh(-(\beta/2)C)$, a fraction of rational traders shift to group 2.

When type-h agents have the total market wealth, type-k agents ($k \neq h$) do not invest in the risky asset. Type $h-$agents will get higher returns and a faster growing wealth. Thus, the dynamics will not leave the steady state where the total market wealth is had by one type of traders.

3.3. Simmetry Properties

Let us focus on the succession $\{x_t\}$, $t \in \mathbb{N}$. A preliminary consideration is that $\forall w_t \in [0, 1]$,

- if $a > 0$ then the sequence $x_0, x_1, x_2, ...$ oscillates;
- if $a < 0$ then the sequence $x_0, x_1, x_2, ...$ has constant sign.

More precisely, assume $a > 0$ then $x_0 > 0$ ($x_0 < 0$) implies $x_t > 0$ ($x_t < 0$) $\forall t = 2k$ and $k \in \mathbb{N}$; differently, assume $a < 0$ then $x_0 > 0$ ($x_0 < 0$) implies

$x_t > 0$ ($x_t < 0$) $\forall t$. We now consider the role of the parameter a in iterating the system for initial conditions $(x_0, m_0, w_0; a)$. Define $\alpha = -|a|$ and $\xi_0 = |x_0|$ so that $T^h(\xi_0, m_0, w_0; \alpha) = (\xi_h, m_h, w_h; \alpha)$ is the h-iterate of $(\xi_0, m_0, w_0; \alpha)$ under the application of T. Then the following proposition holds.

Proposition 5. *(a) If $a < 0$ and $x_0 > 0$ then $T^h(x_0, m_0, w_0; a) = (\xi_h, m_h, w_h; \alpha)$;*

(b) If $a < 0$ and $x_0 < 0$ then $T^h(x_0, m_0, w_0; a) = (-\xi_h, m_h, w_h; \alpha)$;

(c) If $a > 0$ and $x_0 > 0$ then $T^h(x_0, m_0, w_0; a) = ((-1)^h \xi_h, m_h, w_h; \alpha)$;

(d) If $a > 0$ and $x_0 < 0$ then $T^h(x_0, m_0, w_0; a) = ((-1)^{h+1} \xi_h, m_h, w_h; \alpha)$.

Proof. Consider systems (12) and (13). Then it is easy to observe that $g(x_t, m_t, w_t, a) = g(\xi_t, m_t, w_t, \alpha)$ and $h(x_t, m_t, w_t, a) = h(\xi_t, m_t, w_t, \alpha)$. Furthermore the following property holds:

(a) If $a < 0$ and $x_0 > 0$ then $f(x_t, m_t, w_t, a) = f(\xi_t, m_t, w_t, \alpha)$;

(b) If $a < 0$ and $x_0 < 0$ then

$$f(x_t, m_t, w_t, a) = f(-\xi_t, m_t, w_t, \alpha) = -f(\xi_t, m_t, w_t, \alpha);$$

(c) If $a > 0$ and $x_0 > 0$ then

$$f(x_t, m_t, w_t, a) = f(\xi_t, m_t, w_t, -\alpha) = f(\xi_t, m_t, w_t, \alpha)$$

if $t = 2k$ while $f(x_t, m_t, w_t, a) = f(\xi_t, m_t, w_t, -\alpha) = -f(\xi_t, m_t, w_t, \alpha)$ if $t = 2k + 1$, $\forall k \in \mathbb{N}$;

(d) If $a > 0$ and $x_0 < 0$ then

$$f(x_t, m_t, w_t, a) = f(-\xi_t, m_t, w_t, -\alpha) = -f(\xi_t, m_t, w_t, \alpha)$$

if t is pair while $f(x_t, m_t, w_t, a) = f(-\xi_t, m_t, w_t, -\alpha) = f(\xi_t, m_t, w_t, \alpha)$ if $t = 2k + 1$, $\forall k \in \mathbb{N}$.

□

Observe that $a = 0$ or $x_0 = 0$ implies $T^h(x_0, m_0, w_0; a) = (0, m_h, w_h; a)$, $(\forall h = 1, 2, ...)$.

According to the previous proposition, given any initial condition, the evolution of both the fraction of type-h agents and their wealth is not affected by the signs of x_0 and a. More precisely, given the parameter values and other conditions, the evolutions of m_t and w_t only depend on the the distance between x_0 and its equilibrium value $x = 0$. Furthermore, opposite values of the paramer a only affect the sign of the elements of the sequence of iterates $\{x_0, x_1, x_2...\}$.

4. Numerical Simulations

In this section we move to the study of the asymptotic dynamics by using numerical simulations.

In what follows we assume $\lambda_1 = \lambda_2 = \sigma_1^2 = \sigma_2^2 = 1$, $R = 1.02$ and $C = 0.5$ while we let parameters β and a vary.

Being interested in considering the role of the parameter a in the market dynamics, we develop the one-dimensional bifurcation diagram of the state variable x_t with respect to a, see Figure 1. According to the simmetry property previously proved, we can restrict our study to the case $a > 0$. In panel (b) we present an enlargement.
Similarly, Figure 2 shows the one-dimensional bifurcation diagram of the state variables m_t and w_t for positive values of a.

Besides observing the well-known period doubling biforcations, we found border collision bifurcations, i.e. non-canonical bifurcations which occur in piecewise smooth maps.[7] In fact, successions of periodic windows occur. As in [10], this type of bifurcation is involved by the wealth dynamics which is described by a piecewise smooth function. Differently from [10], the width of periodic windows does not reduce monotonically as the parameter a increases and they are not characterized by the same periodicity. Another important difference from the previos work is related to the long-run evolution of the state variables m_t and w_t. More in detail, the model of [10] leads to different success indicators: a certain strategy can be successful in terms of the number of agents using it or in terms of the wealth of the respective group. On the contrary, in our framework the state variables w_t and m_t are characterized by strong correlation. This feature is in agreement with the concept of *survivor*, as the agent with positive wealth share (see [1] and [4] among others).

In order to consider the role of the parameter β, in Figure 3 we present the two-dimensional bifurcation diagram in the parameter plane (a, β) while considering the same initial condition as in the previous picture. Rich dynamics

[7] These bifurcations have mainly been studied in the context of piecewise linear maps. [21] showed, for instance, that a "period three to period two" bifurcation occurs for a class of piecewise linear maps. More recent interesting contributions on this topic are from [22], [5], [6] and [7].

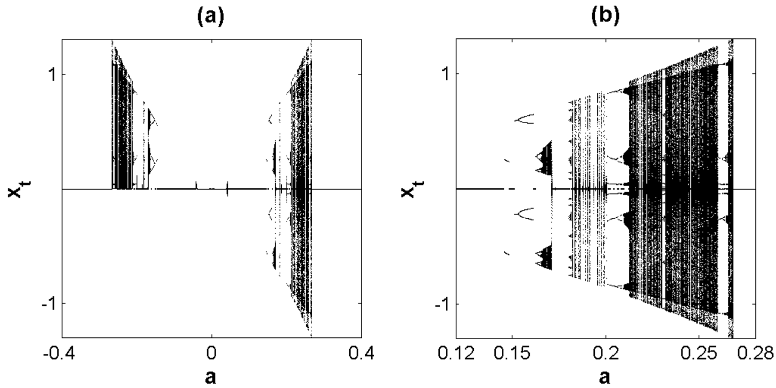

Figure 1: (a) One-dimensional bifurcation diagram of the state variable x_t with respect to a for the initial condition $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = -0.9$ and value $\beta = 6.5$. In panel (b) and enlargement.

is exhibited for intermediate values of the parameters a and β, differently to what happens in heterogeneous agent models. In fact, in models with heterogeneous agents and adaptiveness, complexity increases as the intensity of choice increases.

We can observe similar features after changing the initial conditions. In Figure 4 we present the two-dimensional bifurcation diagrams in the following cases: at the initial time the market is dominated by rational traders (chartists) who have the great fraction of the total wealth, i.e. $m_0 = w_0 = 0.9$ ($m_0 = w_0 = -0.9$).

In order to investigate the role of initial conditions, we make use of the basins of attraction in the plane (x_0, w_0). In Figure 5 we present the basins for increasing values of parameter β, by keeping β low enought.

Similarly, the basins of attraction in the planes (m_0, w_0) and (x_0, m_0) are portrayed in Figure 6.

The main impression gained from the basins of attraction is that the model is able to generate a wide range of different dynamic scenarios, with a strong

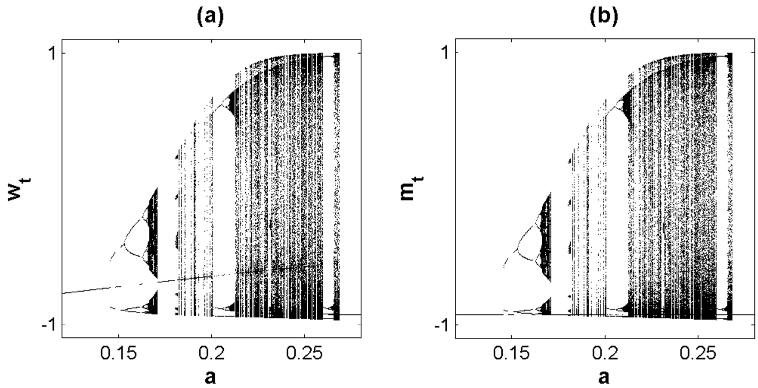

Figure 2: (a) One-dimensional bifurcation diagram of the state variable w_t with respect to a for the initial condition $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = -0.9$ and parameter value $\beta = 6.5$. (b) One-dimensional bifurcation diagram of the state variable m_t with respect to a for the initial condition $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = -0.9$ and value $\beta = 6.5$.

dependence on small changes of the initial conditions: limit cycles, periodic orbits, or more complex dynamics. Moreover, the final dynamics strictly depends on the parameter values.

Finally, it is well-known that models with heterogeneous agents and a switching mechanism are able to reproduce the stylized facts observable in real markets, as periodic or even chaotic fluctuations in prices, excess of volatility, bubbles and crashes.

In order to investigate whether our model can produce some empirical findings, Figure 7 shows the evolutions of x_t and w_t versus time. Time series of the excess return x_t show that return fluctuates. Moreover, with the increase of the parameter a, returns become more volatile (bottom left). Volatility clustering is also observed.

Anyway, return and wealth series are unstable and can generate a-periodic orbits and strange attractors, leading to complex (bounded) dynamics.

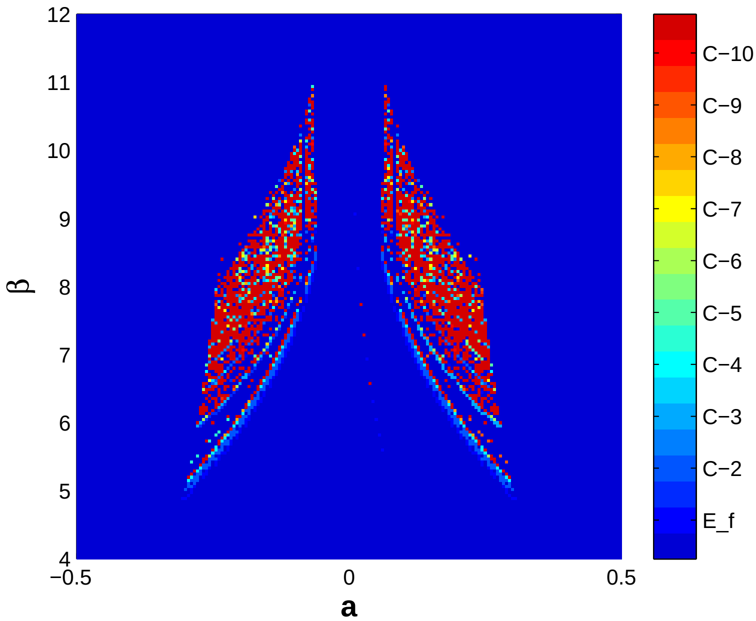

Figure 3: Two dimensional bifurcation diagram in the parameter plane (a, β) for the initial condition $x_0 = 0.5$, $m_0 = 0.4$ and $w_0 = -0.9$.

5. The Role of the Switching Mechanism

Heterogeneous agent models in the CRRA framework suggest that the wealth dynamics will lead to a monomorphic behavior in the long run. In other words, all agents with positive relative wealths invest the same share of their wealth. The selection process leads to monomorphic (i.e. homogeneous) population. Because of the new assumption on the wealth redistribution, our results do not show such convergence. In order to prove that the switching mechanism is crucial for our outcomes, in the following we investigate what happens in the limiting cases, i.e. when the intensity of choice β goes to infinity and when constant proportions of agents are assumed.

For $\beta \to +\infty$, we obtain:

1. if $\Delta\phi_0 - C = \left[\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2}\left(\frac{1-w_0}{1+w_0}\right)^2 - 1\right]\frac{a^2x_0^2}{\lambda_2\sigma_2^2} - C > 0$ then $x_{t+1} = -\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2}\frac{1-w_t}{1+w_t}ax_t$, $m_{t+1} = 1$, $w_{t+1} = 1$. In this case the system will converge to the globally

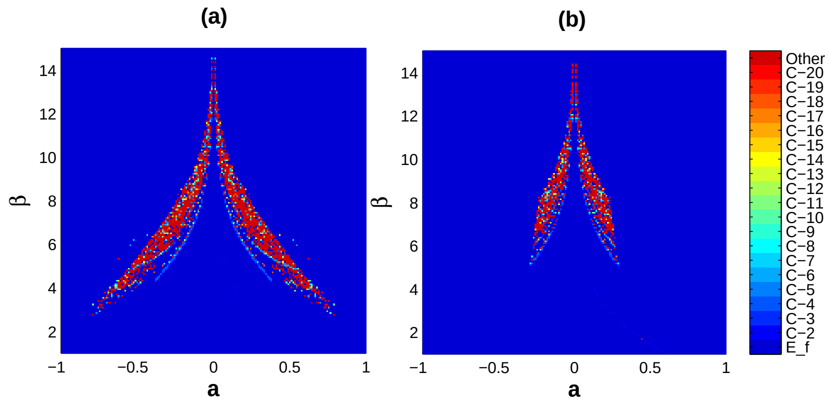

Figure 4: (a) Two dimensional bifurcation diagram in the parameter plane (a, β) for the initial condition $x_0 = 0.5$, $m_0 = 0.9$ and $w_0 = 0.9$. (a) Two dimensional bifurcation diagram in the parameter plane a, β for the initial condition $x_0 = 0.5$, $m_0 = -0.9$ and $w_0 = -0.9$.

stable fixed point $E_1(x^\star = 0, m^\star = 1, w^\star = 1)$,

2. if $\Delta\phi_0 - C = \left[\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2}\left(\frac{1-w_0}{1+w_0}\right)^2 - 1\right]\frac{a^2x_0^2}{\lambda_2\sigma_2^2} - C < 0$ then $x_{t+1} = -\frac{\lambda_1\sigma_1^2}{\lambda_2\sigma_2^2}\frac{1-w_t}{1+w_t}ax_t$, $m_{t+1} = -1$, $w_{t+1} = -1$. In this case the system will converge to the globally stable f. p. $E_2(x^\star = 0, m^\star = -1, w^\star = -1)$.

In other words, depending on the initial condition, both the fractions of agents and the relative wealths converge to 0 and 1 in the long run, i.e. some classes survive, some classes do not. The surviving group is defined by the trading strategy which performs better at the initial time. In fact, when β goes to infinity, all agents use the predictor with the highest fitness and they accumulate the total market wealth, the two assets become equivalent in terms of return.

We obtain the same distribution of wealth by considering fixed proportions of agents with $m_t = 1$ or $m_t = -1$ $\forall t$, i.e. by considering the homogeneous cases. In fact:

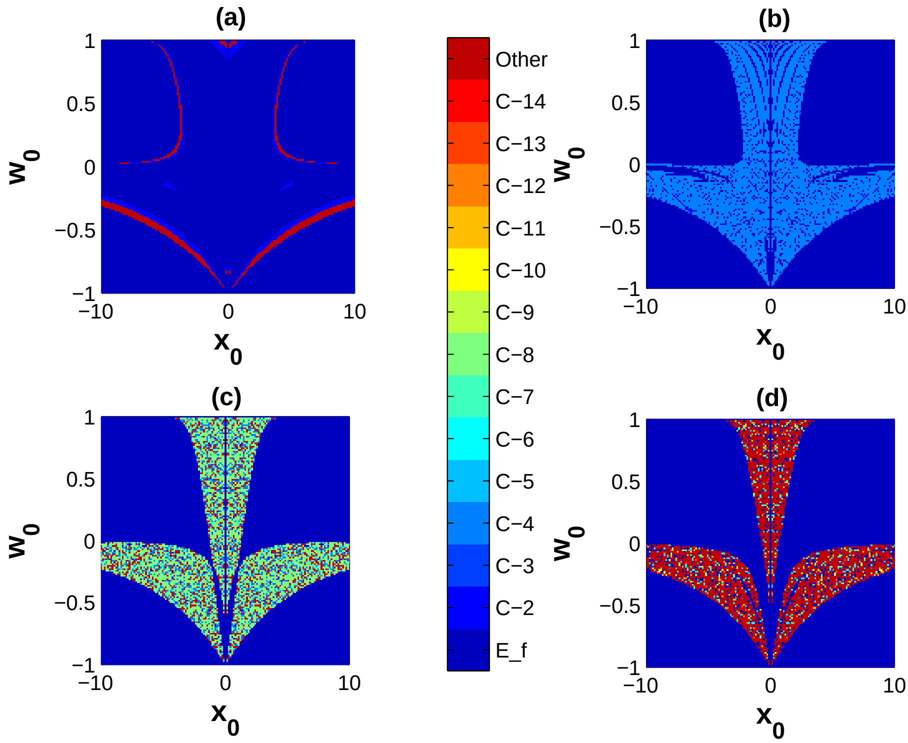

Figure 5: Basins of attraction in the plane (x_0, w_0) for $m_0 = 0.5$ and $a = 0.2$. (a) $\beta = 4$; (b) $\beta = 6$; (c) $\beta = 7$; (d) $\beta = 8$.

1. if $n_1 = 1$ and $n_2 = 0$ then $m_t = 1$, $w_t = 1$ $\forall t > 0$ and $x_t = 0$ $\forall t > 1$,
2. if $n_1 = 0$ and $n_2 = 1$ then $m_t = -1$, $w_t = -1$ $\forall t > 0$ and $x_t = 0$ $\forall t > 1$.

Finally, by assuming constant proportions of agents with $m \neq \pm 1$ one obtain:

$$T : \begin{cases} x_{t+1} = f(x_t, w_t) = -\frac{\lambda_1 \sigma_1^2}{\lambda_2 \sigma_2^2} \frac{1-w_t}{1+w_t} a x_t \\ \\ w_{t+1} = h(x_t, w_t) = \frac{(1-w_t)}{R} \left[\left(\frac{a}{\lambda_2 \sigma_2^2} \right)^2 \lambda_1 \sigma_1^2 \frac{1-w_t}{1+w_t} x_t^2 - R \right] + 1 \end{cases}$$

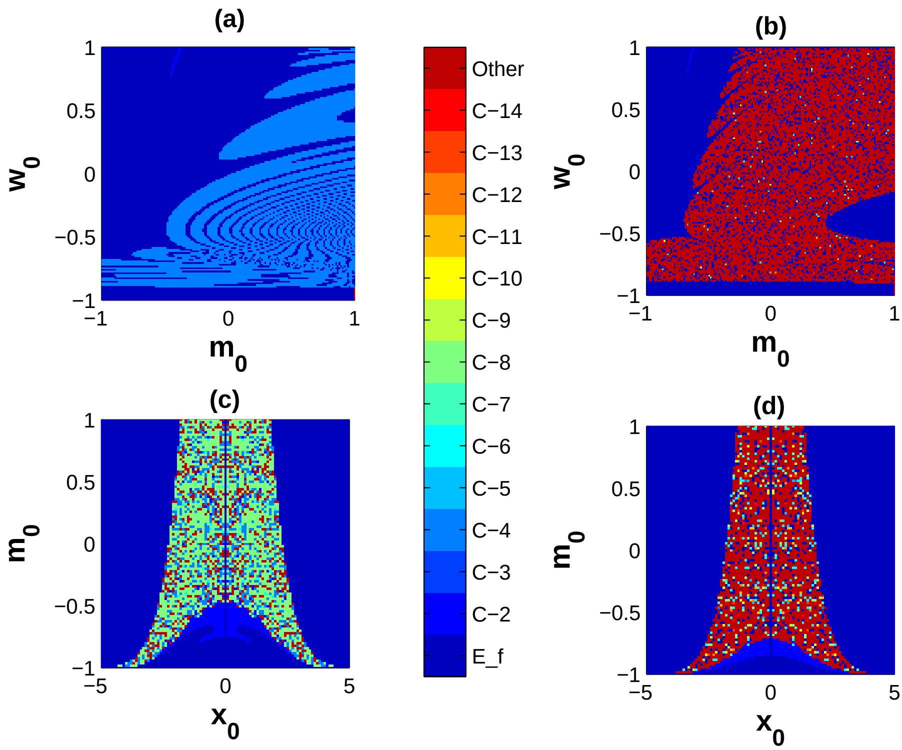

Figure 6: Basins of attraction in the plane (m_0, w_0) for $x_0 = 0.5$ and $a = 0.3$. (a) $\beta = 5$; (b) $\beta = 6$. Basins of attraction in the plane (x_0, m_0) for $w_0 = 0.5$ and $a = 0.2$. (c) $\beta = 7$; (d) $\beta = 8$.

There exists a continuum of steady states defined by: $E = (x^\star = 0, w = w^\star)$, $\forall w^\star \in (-1, 1)$. The expectation schemes are equivalent in all the equilibria and the performance measures are the same. Moreover, both assets are equivalent in terms of return and all the agents earn the same return independent of their investment shares. The lon-run wealth distribution depends on the initial condition. In Figures 8, 9, 10 time series are plotted, showing the convergence of the difference in the relative wealths for different initial conditions.

The study of the limiting cases confirms the main impression gained from the general model: our switching mechanism which allow agents to bring their wealth is responsible for the complexity in the long-run wealth evolution.

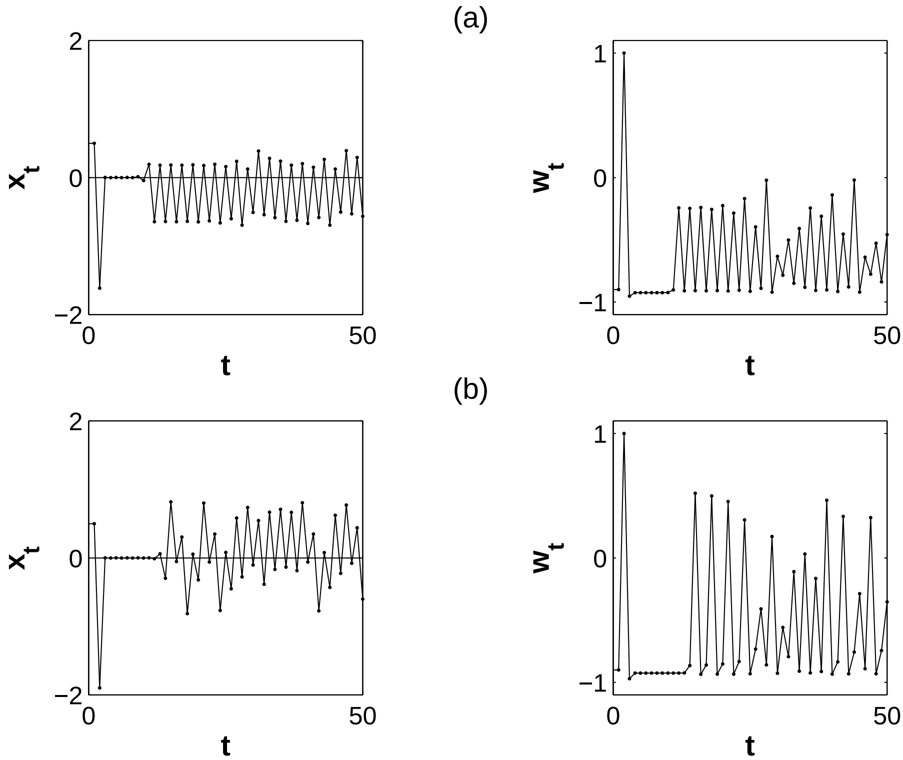

Figure 7: Trajectories versus time of the state variables x_t and w_t for the same initial conditions and β value as in Figure 1. (a) $a = 0.17$; (b) $a = 0.2$.

6. Conclusion

By overcoming models based on fixed proportions of agents, we have established an adaptive model able to characterize the evolution of the distribution of wealth when agents switch between defferent trading strategies. In practice, based on some performance measure, agents can adjust their beliefs from time to time, so that proportions of different beliefs become endogenous state variables. A stationary dynamic model has be written in terms of excess return, wealth and agent proportions.

It is found that the presence of heterogeneous agents and switching mechanism leads the stationary model to have multiple steady-states for wealth distri-

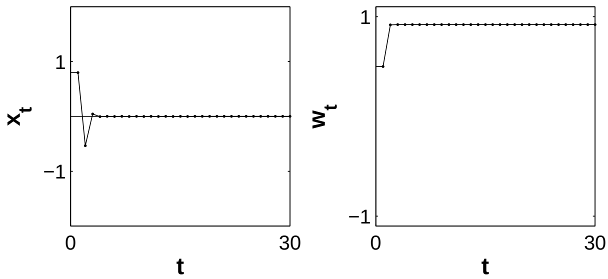

Figure 8: Trajectories versus time of the state variables x_t and w_t being $x_0 = 0.8$, $w_0 = 0.5$ and parameter values $a = 2$, $\lambda_{1,2} = 1$, $\sigma^2_{1,2} = 1$, $R = 1.02$

bution. As far as stability is concerned, the extrapolation rate of the chartists (a) and the intensity of choice (β) play an important role: the dynamics tend to be more complicated for intermediate values of these parameters. This result is unexpected and interesting since, in heterogeneous agent models, complexity increases as the intensity of choice increases. In our framework, a different outcome depends on the new assumption on the wealth redistribution, as proved by studing the limiting cases (β goes to infinity and constant proportions of traders), showing simple dynamics and convergence of relative wealths in the long-run.

The same wealth dynamics has been implemented by [10] in a market maker scenario and under an i.i.d. dividend process. On the contrary, we have made use of the market clearing condition to determine the asset price. Moreover, we have allowed for a growing dividend process. Differently from [10], in our model a strong correlation characterizes the evolution of the difference in the relative wealths and the evolution of the difference in the fractions of agents. This reflects the well-known definition of *survivor*, as the agent with positive wealth share. Furthermore, we have been able to analyze in detail the limiting cases, important in understanding the role of the switching mechanism in the lon-run wealth evolution. Finally, as usual in the realm of heterogeneous agent models,

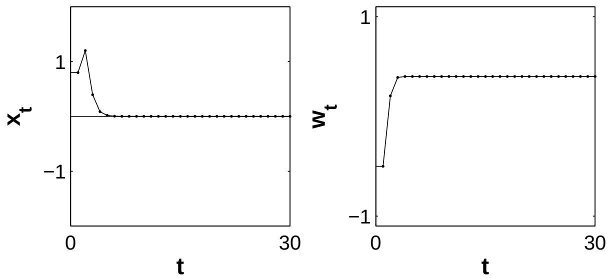

Figure 9: Trajectories versus time of the state variables x_t and w_t being $x_0 = 0.8$, $w_0 = -0.5$ and parameter values $a = -0.5$, $\lambda_{1,2} = 1$, $\sigma^2_{1,2} = 1$, $R = 1.02$

our deterministic framework is able to reproduce the stylized facts observable in financial markets, as a-periodic or even chaotic fluctuations in return and volatility clustering. We have also consider the question whether irrational traders are driven out of the market by rational agents. In fact, by focusing on a market populated by rational traders and chartists, we have proved that both types of agents can survive in the market in the long-run.

By developing a framework where agents tend to switch to the group with the better performance, the switch between strategies is one-way. A further interesting contribution to the literature would be to consider mutual switching between strategies. Moreover, our study show that return and wealth series are unstable and can generate a-periodic orbits and strange attractors, leading to complex (bounded) dynamics. A future research could be devoted to price dynamics, in order to better investigate stylized facts of real markets.

References

[1] R. Amir, T. Hens and K.R. Schenk-Hoppé, Market selection and survival of investment strategies, *Journal of Mathematical Economics, Special Issue on Evolutionary Finance,* **41** (2005), 105-122.

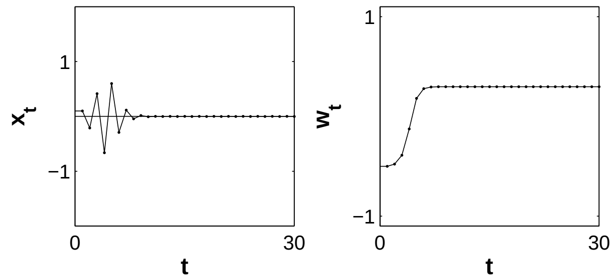

Figure 10: Trajectories versus time of the state variables x_t and w_t being $x_0 = 0.1$, $w_0 = -0.5$ and parameter values $a = 0.7$, $\lambda_{1,2} = 1$, $\sigma^2_{1,2} = 1$, $R = 1.02$

[2] M. Anufriev, Wealth-driven competition in a speculative financial market: examples with maximizing agents, *Quantitative Finance,* **8** (2008), 363-380.

[3] M. Anufriev, G. Bottazzi and F. Pancotto, Equilibria, stability and asymptotic dominance in a speculative market with heterogeneous agents, *Journal of Economic Dynamics and Control,* **30** (2006), 1787-1835.

[4] M. Anufriev, and P. Dindo, Wealth-driven selection in a financial market with heterogeneous agents, *Journal of Economic Behavior and Organization,* **73** (2010), 327-358.

[5] V. Avrutin, B. Eckstein and M. Schanz, The bandcount increment scenario. I. Basic structures, *Proceedings of the Royal Society* A. Vol. 464(2095). (2008a).

[6] V. Avrutin, B. Eckstein and M. Schanz, The bandcount increment scenario. II. Interior structures, *Proceedings of the Royal Society* A. Vol. 464(2097). (2008b).

[7] V. Avrutin, B. Eckstein and M. Schanz, The bandcount increment scenario. III. Deformed structures, *Proceedings of the Royal Society* A. Vol. 465(2101). (2009).

[8] L. Blume and D. Easley, Evolution and market behavior, *The Journal of Economic Theory,* **58** (1992), 9-40. 1992.

[9] L. Blume and D. Easley, The market organism: long-run survival in markets with heterogeneous traders, *Journal of Economic Dynamics and Control,* **33** (2009), 1023-1035.

[10] S. Brianzoni, C. Mammana and E. Michetti, Updating wealth in an asset pricing model with heterogeneous agents, *Discrete Dynamics in Nature and Society,* **2010** (2010), 1-27,

[11] W.A. Brock, P. Dindo and C.H. Hommes, Adaptive rational equilibrium with forward looking agents, *International Journal of Economic Theory,* **2** (2006), 241-278.

[12] W.A. Brock and C.H. Hommes, Heterogeneous beliefs and routes to chaos in a simple asset pricing model, *Journal of Economic Dynamics and Control,* **22** (1998), 1235-1274.

[13] J. Y. Campbell, and L. M. Viceira, *Strategic asset allocation: portfolio choice for long-term investors*, Clarendon Lectures in Economics, Oxford University Press, Oxford (2002).

[14] C. Chiarella, R. Dieci and L. Gardini, Asset price and wealth dynamics in a financial market with heterogeneous agents, *Journal of Economic Dynamics and Control,* **30** (2006), 1755-1786.

[15] C. Chiarella and X. He, Asset price and wealth dynamics under heterogeneous expectations, *Quantitative Finance,* **1** (2001), 509-526.

[16] C. Chiarella and X. He, An adaptive model on asset pricing and wealth dynamics with heterogeneous trading strategies, *Working paper no. 84, School of Finance and Economics*, (2002) UTS.

[17] I. Evstigneev, T. Hens and K. Schenk-Hoppé, Globally evolutionarily stable portfolio rules, *Journal of Economic Theory,* **140** (2008), 197-228.

[18] I. Evstigneev, T. Hens and K. Schenk-Hoppé, Evolutionary Finance, In: *Handbook of financial markets: dynamics and evolution,* ed. by T. Hens and K. Schenk-Hoppé, North-Holland, Amsterdam (2009), 507-566.

[19] T. Hens and K. Schenk-Hoppé, Evolutionary stability of portfolio rules in incomplete markets, *Journal of Mathematical Economics,* **41** (2005), 43-66.

[20] C. H. Hommes, Heterogeneous agent models in economics and finance, In: *Handbook of Computational Economics,* L. Tesfatsion and K. L. Judd, eds., vol. 2, chap. 23. Elsevier, North Holland (2006), 1109-1186.

[21] C.H. Hommes and H. E. Nusse, Period three to period two bifurcation for piecewise linear model, *Journal of Economics*, **54** (1991), 157-169.

[22] P. Jain and S. Banerjee, Border-collision bifurcations in one-dimensional discontinuous maps, *International Journal of Bifurcation and Chaos*, **13** (2003), 3341-3351.

[23] B. LeBaron, Agent based model in finance, In: *Handbook of Computational Economics,* L. Tesfatsion and K. L. Judd, eds., vol. 2, chap. 24. Elsevier, North Holland (2006), 1187-1234.

[24] M. Levy, H. Levy and S. Solomon, A microscopic model of the stock market, *Economics Letters*, **45** (1994), 103-111.

[25] M. Levy, H. Levy and S. Solomon, Microscopic simulation of the stock market: the effect of microscopic diversity, *Journal de Physique I*, **5** (1995), 1087-1107.

[26] M. Levy, H. Levy and S. Solomon, *Microscopic simulation of financial markets - from investor behavior to market phenomena.* Academic Press, Sydney (2000).