International Journal of Pure and Applied Mathematics
Volume 76 No. 5 2012, 699-708
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

ASYMPTOTIC PROPERTY OF GLOBAL SOLUTION FOR SOME MULTIDIMENSIONAL BBM EQUATIONS

Yaojun Ye
Department of Mathematics and Information Science
Zhejiang University of Science and Technology
Hangzhou, 310023, P.R. CHINA

Abstract: The initial-boundary value problem of a class of multidimensional inhomogeneous GBBM equations is studied. The asymptotic property and the continuous dependence on initial value of the solutions for this problem are proved by establishing some estimates of global solutions by means of Gronwall inequality.

AMS Subject Classification: 35B40, 35Q53
Key Words: multidimensional GBBM equations, initial-boundary value problem, asymptotic property, the solution's continuous dependence on initial value

1. Introduction

BBM equation

$$u_t + (1+u)u_x - u_{xxt} - u_{xx} = 0 \tag{1}$$

is a class of nonlinear evolution equation which has been proposed and studied by Benjamin-Bona-Mahony [1, 2] in specific physical situations under long wave limitation in nonlinear dispersive media. The discussion of the initial boundary value problem of equation (1) has a strong physical meaning.

Received: December 3, 2011

The existence and uniqueness of global solutions to the initial-boundary value problem and Cauchy problem for equation (1) or similar to equation (1) have been proved by many people through different methods and techniques under various assumptions [3, 4, 5, 6]. Papers [7, 8, 9, 10] have proposed and studied the existence and uniqueness of the global smooth solutions for a generalized BBM equation in higher dimensions. Literatures [11, 12] have dealt with the existence and uniqueness of global generalized solutions of initial-boundary value proplem for the multidimensional nonlinear BBM equation systems in higher order by using priori estimates and Leray-Schauder fixed point theorem.

when the BBM equation is defined in a bounded domain, there exists finite dimensional global attractor, see [4]. Note when the domain of the equation is unbounded there are additional difficulties when proving the existence of attractors because, in this case, the Sobolev embeddings are not compact. M. Stanislavova [13] first present a new necessary and sufficient condition to verify the asymptotic compactness of an evolution equation defined in an unbounded domain, which involves the Littlewood-Paley projection operators. He then use this condition to prove the existence of an attractor for the damped Benjamin-Bona-Mahony equation in the phase space $H^1(R)$ by showing the solutions are point dissipative and asymptotically compact. Initial boundary problems for $u_t - u_{xxt} + uu_x = 0$ (or its multi-dimensional versions) were considered in [3, 7, 8] and proved existence of local solutions to a corresponding mixed problem. It is easy to see that mixed problem for $u_t - u_{xxt} + uu_x = 0$ with Dirichlet boundary conditions implies conservation of the energy

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_\Omega \{|u(x,t)|^2 + |\nabla u(x,t)|^2\}dx = 0.$$

It means that the energy can not decay with time. N.A. Larkin and M.P. Vishnevskii [14] concerns a dissipative initial boundary value problem for the BBM equation, and he prove the existence and uniqueness of global solutions and the decay of the energy as time tends to infinity.

There are a lot of papers containing similar results on large time behavior of solutions to the Cauchy problem of BBM (or GBBM) equations, cf. e.g. [14, 15, 16, 17] and the references given there. In all these publications, the asymptotic of solutions is given by a self-similar diffusive profile. Our list of such papers is by no mean exhaustive; moreover, we prove the decay estimates of the initial boundary value problem for some inhomogeneous BBM equations in this paper by using the different methods as theirs.

This paper is devoted to the study of the asymptotic stability of global solutions for the nonlinear multidimensional BBM equation

$$u_t - \Delta u_t - \beta \Delta u + u = div f(u),\ (x,t) \in \Omega \times R^+, \tag{2}$$

with the initial-boundary value condition

$$u(x,0) = \varphi(x),\ x \in \Omega, \tag{3}$$

$$u(x,t) = 0,\ (x,t) \in \Omega \times R^+. \tag{4}$$

where $\beta > 0$ is a constant, $\Omega \subset R^n$ is a bounded open domain with smooth boundary $\partial\Omega$, $R^+ \equiv [0, +\infty)$ and $f(u) = (f_1(u), f_2(u), \cdots, f_n(u))$ is a nonlinear vector valued function from R to R^n, where $f_i(u) \in C^1(\Omega),\ i = 1, 2, \cdots, n$.

The existence and uniqueness of global solutions of problem (2)-(4) have been proved by means of the methods and techniques in [5, 6, 7]. The object of this paper is to prove the asymptotic behavior and the continuous dependence on initial value of global solutions for this problem by establishing some estimates of global solutions through Gronwall inequality.

We adopt the usual notation and convention. Let $H^m(\Omega)$ denote the Sobolev space with the norm $\|u\|_{H^m(\Omega)} = (\sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2(\Omega)})^{\frac{1}{2}}$, $H_0^m(\Omega)$ denotes the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, hereafter we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the space $L^2(\Omega)$ inner product and norm respectively, and $\|\cdot\|_{L^\infty(\Omega)}$ indicates the space $L^\infty(\Omega)$ norm. We denote by $L^2(0,+\infty;\ L^2(\Omega))$ the space of strongly measurable functions from $[0,+\infty)$ to $L^2(\Omega)$ with $\|u(t)\| \in L^2(0,+\infty)$ and its norm writes as follows

$$\|u\|_{L^2(0,+\infty;\ L^2(\Omega))} = \left\{ \int_0^{+\infty} \|u(t)\|^2 dt \right\}^{\frac{1}{2}}.$$

Similarly, we can define the space $L^\infty(0,+\infty;\ L^2(\Omega)$ with norm

$$\|u\|_{L^\infty(0,+\infty;\ L^2(\Omega)} = ess \sup_{t\in R^+} \|u(t)\|.$$

Moreover, M denotes various positive constants depending on the known constants and may be different at each appearance, and $Q = \Omega \times R^+$.

2. Main Results

We list up two useful Lemmas in order to investigate main results.

Lemma 1. (Young Inequality) *Let* $a, b \geq 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *for* $1 < p,\ q < +\infty$, *then one has the inequality*

$$ab \leq \delta a^p + C(\delta) b^q,$$

where $\delta > 0$ *is an arbitrary constant, and* $C(\delta)$ *is a positive constant depending on* δ.

Lemma 2. (Gronwall Inequality) *If* $\Phi = \Phi(t)$ *satisfies the inequality*

$$\Phi(t) \leq \Phi_0 + \int_0^t \lambda(\tau)\Phi(\tau)d\tau,$$

where $\Phi_0 = \Phi(0)$ *and* $\lambda(s) \in L^1[0, T]$; *then*

$$\Phi(t) \leq \Phi_0 \exp(\int_0^t \lambda(\tau)d\tau),$$

for any $t \in [0, T]$.

Our main results read as follows:

Theorem 1. *Let*

$$\varphi(x) \in H_0^1(\Omega), \quad f(u) = (f_1(u), f_2(u), \cdots, f_n(u)),\ f_i(u) \in C^1(\Omega),$$

$i = 1, 2, \cdots, n$, *then the global solutions* $u(x, t)$ *of the problem (2)-(4) satisfy*

$$u(x, t),\ \nabla u(x, t) \in L^2(0, +\infty;\ L^2(\Omega));$$

and $\|u\| \to 0,\ \|\nabla u\| \to 0$, *as* $t \to +\infty$.

Theorem 2. *Assume that*

$$\varphi(x) \in H_0^2(\Omega), \quad f(u) = (f_1(u), f_2(u), \cdots, f_n(u)),\ f_i(u) \in C^1(\Omega),$$

$i = 1, 2, \cdots, n$, *then the global solutions* $u(x, t)$ *of the problem (2)-(4) fulfill*

$$\Delta u(x, t) \in L^2(0, +\infty;\ L^2(\Omega)),$$

and $\|\Delta u\| \to 0$, *as* $t \to +\infty$.

Theorem 3. *Supposed that*

$$\varphi(x) \in H_0^1(\Omega), \quad f(u) = (f_1(u), f_2(u), \cdots, f_n(u)), \ f_i(u) \in C^1(\Omega),$$

$i = 1, 2, \cdots, n$, then the solution $u(x,t)$ defined on $Q(T)$ of the problem (2)-(4) is continuous dependence on the initial value $\varphi(x)$ in the following sense: for any $\varepsilon > 0$, there exists a positive number δ, such that

$$\|u(t) - v(t)\| < \varepsilon, \ \|\nabla u(t) - \nabla v(t)\| < \varepsilon$$

is valid when

$$\|\varphi(x) - \psi(x)\| < \delta, \ \|\nabla\varphi(x) - \nabla\psi(x)\| < \delta,$$

where $u(x,t)$ and $v(x,t)$ are respectively solutions of problem (2)-(4) corresponding initial value $\varphi(x)$ and $\psi(x)$, $Q(T) = \{(x,t)| \ x \in \Omega, \ t \in [0,T], \ \forall T > 0\}$.

3. The Proof of Main Results

3.1. The Proof of Theorem 1

Multiplying by u on both sides of the equation (2) and integrating over Ω, we get from (4)

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2 + \beta\|\nabla u(t)\|^2 + \|u(t)\|^2 = \int_\Omega div f(u) \cdot u dx. \quad (5)$$

Since

$$\begin{aligned}\int_\Omega div f(u) \cdot u dx &= \int_\Omega \sum_{i=1}^n \frac{\partial f_i(u)}{\partial x_i} u dx = -\sum_{i=1}^n \int_\Omega f_i(u) \frac{\partial u}{\partial x_i} dx \\ &= -\sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\Big(\int_0^u f(\theta) d\theta\Big) dx = -\sum_{i=1}^n \int_0^u f(\theta) d\theta\Big|_{\partial\Omega} = 0,\end{aligned}$$

Thus, taking integral on both sides of (5) over $[0,t]$, we have

$$\|u(t)\|^2 + \|\nabla u(t)\|^2 + 2\beta \int_0^t \|\nabla u(\tau)\|^2 d\tau + 2\int_0^t \|u(\tau)\|^2 d\tau = \|\varphi\|^2 + \|\nabla\varphi\|^2. \quad (6)$$

It follows that from (6)

$$\|u(t)\|^2 + 2\int_0^t \|u(\tau)\|^2 d\tau \le M, \ \|\nabla u(t)\|^2 + 2\beta \int_0^t \|\nabla u(\tau)\|^2 d\tau \le M. \quad (7)$$

We obtain from Lemma 2 and (7)

$$\|u(t)\|^2 \le Me^{-2t}, \quad \|\nabla u(t)\|^2 \le Me^{-2\beta t}, \quad t \in R^+. \tag{8}$$

which implies that

$$\int_0^{+\infty} \|u(\tau)\|^2 d\tau \le M \int_0^{+\infty} e^{-2\tau} d\tau = \frac{M}{2}, \quad \int_0^{+\infty} \|\nabla u(\tau)\|^2 d\tau \le M \int_0^{+\infty} e^{-2\beta\tau} d\tau = \frac{M}{2\beta}. \tag{9}$$

Accordingly, we have from (8) and (9) that $\|u\| \to 0, \ \|\nabla u\| \to 0$ as $t \to +\infty$ and

$$u(x,t), \ \nabla u(x,t) \in L^2(0,+\infty; \ L^2(\Omega)).$$

The proof of Theorem 1 is completed.

3.2. The Proof of Theorem 2

Multiplying by Δu on both sides of equation (2) and integrating over Ω, we have by (4)

$$\frac{d}{dt}(\|\nabla u\|^2 + \|\Delta u\|^2) = 2(-\|\nabla u\|^2 - \beta\|\Delta u\|^2 - \int_\Omega div f(u) \cdot \Delta u dx), \tag{10}$$

We get from the conditions in Theorem 2 and Lemma 1

$$\left| \int_\Omega div f(u) \Delta u dx \right| = \left| \int_\Omega \sum_{i=1}^n \frac{\partial f_i(u)}{\partial x_i} \Delta u dx \right| = \left| \int_\Omega \sum_{i=1}^n f_i'(u) \frac{\partial u}{\partial x_i} \Delta u dx \right| \le \max_{1\le i\le n} \|f_i(u)\|_\infty \|\nabla u\| \cdot \|\Delta u\| \le \frac{\beta}{2}\|\Delta u\|^2 + \frac{M}{2\beta}\|\nabla u\|^2. \tag{11}$$

We obtain from (10) and (11)

$$\frac{d}{dt}(\|\nabla u(t)\|^2 + \|\Delta u(t)\|^2) \le -\beta\|\Delta u(t)\|^2 + (\frac{M}{\beta} - 2)\|\nabla u(t)\|^2. \tag{12}$$

Taking integral on both sides of (12) over $[0,t]$ to yield

$$\|\nabla u(t)\|^2 + \|\Delta u(t)\|^2 + \beta \int_0^t \|\Delta u(\tau)\|^2 d\tau$$

$$\leq (\frac{M}{\beta} - 2)\int_0^t \|\nabla u(\tau)\|^2 d\tau + \|\nabla\varphi\|^2 + \|\Delta\varphi\|^2. \quad (13)$$

(i) If $M > 2\beta$, then It follows from $\nabla u \in L^2(0, +\infty;\ L^2(\Omega))$ and (13),

$$\|\nabla u(t)\|^2 + \|\Delta u(t)\|^2 + \beta \int_0^t \|\Delta u(\tau)\|^2 d\tau \leq M,$$

Consequently,

$$\|\Delta u(t)\|^2 + \beta \int_0^t \|\Delta u(\tau)\|^2 d\tau \leq M. \quad (14)$$

(ii) If $M < 2\beta$, then $(\frac{M}{\beta} - 2)\int_0^t \|\nabla u(\tau)\|^2 d\tau < 0$. So, the formula (14) is still valid by (13).

We have from Lemma 2 and (14),

$$\|\Delta u(t)\|^2 \leq M e^{-\beta t},\ t \in R^+, \quad (15)$$

which implies that

$$\int_0^{+\infty} \|\Delta u(\tau) d\tau\|^2 \leq M \int_0^{+\infty} e^{-\beta\tau} d\tau = \frac{M}{\beta} < +\infty. \quad (16)$$

we conclude that from (15) and (16) $\|\Delta u(t)\| \to 0$ as $t \to +\infty$ and $\Delta u \in L^2(0, +\infty;\ L^2(\Omega))$. This complete the proof of Theorem 2.

3.3. The Proof of Theorem 3

Assume that $u(x,t)$ and $v(x,t)$ are respectively the solutions of the problem (2)-(4) corresponding initial value $\varphi(x)$ and $\psi(x)$, Let $\omega(x,t) = u(x,t) - v(x,t)$, then $\omega(x,t)$ satisfies

$$\omega_t - \Delta\omega_t - \beta\Delta\omega + \omega = div(f(u) - f(v)),\ (x,t) \in \Omega \times [0,T], \quad (17)$$

$$\omega(x,0) = \omega_0(x),\ x \in \Omega, \quad (18)$$

$$\omega(x,t) = 0,\ (x,t) \in \partial\Omega \times [0,T]. \quad (19)$$

Multiplying by ω on both sides of (17) and integrating over Ω to yield from (19)

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\omega(t)\|^2 + \beta\|\nabla\omega(t)\|^2 + \|\omega(t)\|^2$$

$$= \int_\Omega div[f(u) - f(v)]\omega dx, \quad (20)$$

we obtain from Lemma 1

$$\begin{aligned} \left| \int_\Omega div[f(u) - f(v)]\omega dx \right| &= \left| \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}[f_i(u) - f_i(v)]\omega dx \right| \\ &= \left| \sum_{i=1}^n \int_\Omega [f_i(u) - f_i(v)] \frac{\partial \omega}{\partial x_i} dx \right| \\ &\le \max_{1\le i\le n} \|f_i'(u + (v-u)\theta)\|_\infty \int_\Omega |\omega \cdot \nabla\omega| dx \\ &\le M\|\omega(t)\| \cdot \|\nabla\omega(t)\| \\ &\le \frac{\beta}{2}\|\Delta u\|^2 + \frac{M}{2\beta}\|\nabla u\|^2. \end{aligned} \quad (21)$$

We have from (20) and (21),

$$\frac{d}{dt}(\|\omega(t)\|^2 + \|\nabla\omega(t)\|^2) \le 2M(\|\omega(t)\|^2 + \|\nabla\omega(t)\|^2),\ t \in [0, T],$$

where M is a positive constant independing on t. Thereby, we get from Lemma 2,

$$\|\omega(t)\|^2 + \|\nabla\omega(t)\|^2 \le e^{2MT}(\|\omega(0)\|^2 + \|\nabla\omega(0)\|^2),\ t \in [0, T]. \quad (22)$$

Let $e^{2MT}(\|\omega(0)\|^2 + \|\nabla\omega(0)\|^2) < \varepsilon^2$ for $\forall\varepsilon > 0$, then we have

$$\|\varphi(x) - \psi(x)\|^2 + \|\nabla\varphi(x) - \nabla\psi(x)\|^2 < \varepsilon^2 e^{-2MT}.$$

Taking $\delta = \varepsilon e^{-MT}$, then we get from (22) that

$$\|u(t) - v(t)\|^2 + \|\nabla u(t) - \nabla v(t)\|^2 < \varepsilon^2,\ t \in [0, T].$$

when

$$\|\varphi(x) - \psi(x)\|^2 + \|\nabla\varphi(x) - \nabla\psi(x)\|^2 < \delta^2$$

which implies that

$$\|u(t) - v(t)\| < \varepsilon, \quad \|\nabla u(t) - \nabla v(t)\| < \varepsilon,\ t \in [0, T]$$

is valid as

$$\|\varphi(x) - \psi(x)\| < \delta, \quad \|\nabla\varphi(x) - \nabla\psi(x)\| < \delta$$

This finished the proof of Theorem 3.

Acknowledgments

This Research was supported by Natural Science Foundation of Zhejiang Province (No. Y6100016), The Science and Research Project of Zhejiang Province Education Commission (No. Y200803804 and Y200907298), The Middle-aged and Young Leader in Zhejiang University of Science and Technology (2008-2012) and and Interdisciplinary Pre-research Project of Zhejiang University of Science and Technology (2010-2012).

References

[1] T.B. Benjamin, J.L. Bona, J.J. Mahony, Model equation long waves in nonlinear despersive system, *Phil. Trans. Soc. London, ser A*, **272** (1972), 47-87.

[2] J.L. Bona, P.L. Brgant, A mathematical model for long wave generated by wavemake in dispersive systems, *Proc. Phil. Soc.*, **73** (1973), 391-405.

[3] Guo Boling, The global solutions of some problem for a system of equations of Schrödinger-Klein-Gordon field, *Scientia Sinica, Ser A*, **25** (1982), 898-901.

[4] Hong Naiduan, The weak solution for some nonlinear evolution equations, *Journal of Xiamen University (Natural Science)*, **26** (1987), 147-153.

[5] Xing Jiasheng, Global strong solution for a class of Burges-BBM type equation, *Applied Mathematics. A Journal of Chinese Universities*, **6** (1991), 31-36.

[6] J.L. Bona, V.A. Dougalis, An initial boundary value problem for a model equation for propagation of long waves, *J. Math. Anal. Appl.*, **75** (1980), 503-522.

[7] Guo Boling, Initial-boundary value problem for one class of system of multidimensional inhomogeneous GBBM euqations, *Chinese Ann. Math., Ser B*, **8** (1987), 226-236

[8] Guo Boling, Miao Changxing, On inhomogeneous GBBM equation, *J. Partial Diff. Eq.*, **18** (1995), 26-36.

[9] Fang Shaomei, Guo Boling, Long time behavior for solution of initial-boundary value problem of one class of systems with multidimensional inhomogeneous GBBM equations, *Applied Mathematics and Mechanics*, **26** (2005), 716-721.

[10] J.A. Goldstein, R. Kajikiya, S. Oharu, On some nonlinear dispersive equations in several space variables, *Differential and Interal Equations*, **3** (1990), 617-632.

[11] Li Zheshen, Initial-boundary value problem for the system of multidimensional GBBM equations, *Applied Mathematics*, **3** (1990), 71-80.

[12] Xiao Liming, One class of system of multidimensional nonlinear pseudo-parabolic equations, *Acta Mathematica Scientia. Ser.A*, **18** (1998), 303-309.

[13] O. Goubet, R. Rosa, Asymptotic smoothing and the global attractor of a weakly damped KdV equation on the real line, *J. Differential Equations*, **185** (2002), 25-53.

[14] N.A. Larkin, M.P. Vishnevaskii, Dissipative initial boundary value problem for the BBM equations, *Electr. J. Different. Equats.*, **2008** (2008), 1-10.

[15] J.L. Limaco, H.R. Clark, L.A. Medeiros, On equations of Benjamin-Bona-Mahony type, *Nonlinear Anal. TMA*, **59** (2004), 1243-1265.

[16] G. Karch, L^p-decay of solutions to dissipative-dispersive perturbations of conservation laws, *Ann. Polon. Math.*, **67** (1997), 65-86.

[17] G. Karch, Asymptotic behavior of solutions to some pseudoparabolic equations, *Math. Methods Appl. Sci.*, **20** (1997), 271-289.