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A POINTWISE APPROXIMATION FOR INDEPENDENT GEOMETRIC RANDOM VARIABLES

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Abstract: The Stein-Chen method is used to determine a non-uniform bound on pointwise approximation of the distribution of a sum of n independent geometric random variables by the Poisson distribution with mean $\lambda = \sum_{i=1}^{n} q_i$, in terms of the point metric of two such distributions.

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1. Introduction

Let $Y_1, ..., Y_n$ be n independent geometric random variables with, for each $i \in \{1, ..., n\}$, $P(Y_i = k) = p_i q_i^k$, $k = 0, 1, ...,$ and let $X = \sum_{i=1}^{n} Y_i$. Then X is the number of failures before the n^{th} success in n sequences of independent Bernoulli trials, where, in the sequence i^{th}, success occurs on each trial with a probability of p_i and failure occurs on each trial with a probability of $q_i = 1 - p_i$. If p_i's are identical to p, then X has the Pascal distribution with parameters n and p. It is well known that if all q_i are small, then the distribution of X can be approximated by a Poisson distribution. In this case, for $A \subseteq \mathbb{N} \cup \{0\}$, Teerapabolarn and Wongkasem [5] used the Stein-Chen method to obtain a

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uniform bound

$$\left|P(X\in A)-\sum_{k\in A}\frac{\lambda^k e^{-\lambda}}{k!}\right|\leq\sum_{i=1}^{n}\min\left\{\frac{\lambda^{-1}(1-e^{-\lambda})}{p_i},1\right\}\frac{q_i^2}{p_i}\qquad(1.1)$$

for the difference of the distribution of X and the Poisson distribution with mean $\lambda=\sum_{i=1}^{n}\frac{q_i}{p_i}$, and for $A=\{0,...,x_0\}$, $x_0\in\mathbb{N}\cup\{0\}$, they also gave a non-uniform bound for the difference of the distribution function of X and the Poisson distribution function with the same mean as follows:

$$\left|P(X\leq x_0)-\sum_{k=0}^{x_0}\frac{\lambda^k e^{-\lambda}}{k!}\right|\leq\lambda^{-1}(e^{\lambda}-1)\sum_{i=1}^{n}\min\left\{\frac{1}{p_i(x_0+1)},1\right\}\frac{q_i^2}{p_i}.\qquad(1.2)$$

For the Poisson mean $\lambda=\sum_{i=1}^{n}q_i$, Vellaisamy and Upadhye [6] used Kerstan's method to obtain a uniform bound, in the same form as in (1.1), as follows:

$$\left|P(X\in A)-\sum_{k\in A}\frac{\lambda^k e^{-\lambda}}{k!}\right|\leq\sum_{i=1}^{n}\min\left\{\frac{0.42888}{\sqrt{\lambda}},1\right\}\frac{q_i^2}{p_i},\qquad(1.3)$$

and Teerapabolarn [4] used the Stein-Chen method to obtain a better result of (1.2) in the following:

$$\left|P(X\leq x_0)-\sum_{k=0}^{x_0}\frac{\lambda^k e^{-\lambda}}{k!}\right|\leq\lambda^{-1}(e^{\lambda}-1)\sum_{i=1}^{n}\min\left\{\frac{1}{p_i(x_0+1)},1\right\}q_i^2,\qquad(1.4)$$

where $x_0\in\mathbb{N}\cup\{0\}$.

Consider the results in (1.1) and (1.3), when $A=\{x_0\}$ for $x_0\in\mathbb{N}\cup\{0\}$, the results become

$$\left|P(X=x_0)-\frac{\lambda^{x_0}e^{-\lambda}}{x_0!}\right|\leq\sum_{i=1}^{n}\min\left\{\frac{\lambda^{-1}(1-e^{-\lambda})}{p_i},1\right\}\frac{q_i^2}{p_i}\qquad(1.5)$$

for $\lambda=\sum_{i=1}^{n}\frac{q_i}{p_i}$, and for $\lambda=\sum_{i=1}^{n}q_i$,

$$\left|P(X=x_0)-\frac{\lambda^{x_0}e^{-\lambda}}{x_0!}\right|\leq\sum_{i=1}^{n}\min\left\{\frac{0.42888}{\sqrt{\lambda}},1\right\}\frac{q_i^2}{p_i},\qquad(1.6)$$

which are uniform bounds, with respect to x_0, for the point metric of two such distributions. In this situation, a non-uniform bound with respect to x_0 is more appropriate for measuring the accuracy of the approximation. In this study, we use the Stein-Chen method to determine a non-uniform bound for the point metric of the distribution of X and the Poisson distribution with mean $\lambda=\sum_{i=1}^{n}q_i$.

2. Method

Stein [3] introduced a powerful and general method for bounding the error in the normal approximation. This method was first developed and applied in the setting of Poisson approximation by Chen [2], which is refer to as the Stein-Chen method. Stein's equation for Poisson distribution with mean $\lambda > 0$ is, for given h, of the form

$$h(x) - \wp_\lambda(h) = \lambda f(x+1) - xf(x), \tag{2.1}$$

where $\wp_\lambda(h) = e^{-\lambda}\sum_{l=0}^{\infty} h(l)\frac{\lambda^l}{l!}$ and f and h are bounded real valued functions defined on $\mathbb{N} \cup \{0\}$.

For $A \subseteq \mathbb{N} \cup \{0\}$, let function $h_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

For $A = \{x_0\}$ where $x_0 \in \mathbb{N} \cup \{0\}$, let $C_x = \{0, ..., x\}$ and $h_{x_0} = h_{\{x_0\}}$, then following Barbour et al. [1], the solution $f_{x_0} = f_{\{x_0\}}$ of (2.1) is of the form

$$f_{x_0}(x) = \begin{cases} -(x-1)!\lambda^{-x}e^{\lambda}[\wp_\lambda(h_{x_0})\wp_\lambda(h_{C_{x-1}})] & \text{if } x \le x_0, \\ (x-1)!\lambda^{-x}e^{\lambda}[\wp_\lambda(h_{x_0})\wp_\lambda(1-h_{C_{x-1}})] & \text{if } x > x_0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.2}$$

The following lemma gives a non-uniform bound for f_{x_0}.

Lemma 2.1. *Let $x_0, x \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{1\}$, then the following inequality holds:*

$$\sup_{x \ge k} |f_{x_0}(x)| \le \begin{cases} \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{2}, \frac{1}{k}\right\} & \text{if } x_0 = 1, \\ \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}, \frac{1}{k}\right\} & \text{if } x_0 \ge 2. \end{cases} \tag{2.3}$$

Proof. For $k \le x \le x_0$, it follows from Barbour et al. [1] that f_{x_0} is negative and decreasing in $x \le x_0$, thus we obtain

$$\begin{aligned} 0 \le -f_{x_0}(x) &\le -f_{x_0}(x_0) \\ &= (x_0-1)!\lambda^{-x_0}e^{\lambda}[\wp_\lambda(h_{x_0})\wp_\lambda(h_{C_{x_0-1}})] \\ &= \frac{1}{x_0}e^{-\lambda}\sum_{i=0}^{x_0-1}\frac{\lambda^i}{i!} \end{aligned}$$

$$= \lambda^{-1} e^{-\lambda} \sum_{i=1}^{x_0} \frac{\lambda^i}{i!} \frac{i}{x_0}$$
$$\leq \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}\right\},$$

this yields $\sup_{x\geq k} |f_{x_0}(x)| \leq \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}, \frac{1}{k}\right\}$.

For $k \leq x$ and $x_0 < x$,

$$\begin{aligned} 0 \leq f_{x_0}(x) &= (x-1)!\lambda^{-x} e^{\lambda}[\wp_\lambda(h_{x_0})\wp_\lambda(1-h_{C_{x-1}})] \\ &= \frac{(x-1)!}{x_0!} e^{-\lambda} \sum_{i=x}^{\infty} \frac{\lambda^{i+x_0-x}}{i!} \\ &= \frac{1}{x} e^{-\lambda} \left\{\frac{\lambda^{x_0}}{x_0!} + \frac{\lambda^{x_0+1}}{x_0!(x+1)} + \frac{\lambda^{x_0+2}}{x_0!(x+1)(x+2)} + \cdots\right\} \\ &\leq \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}, \frac{1}{k}\right\}, \end{aligned}$$

we have $\sup_{x\geq k} |f_{x_0}(x)| \leq \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}, \frac{1}{k}\right\}$.

Hence, from two cases, the inequality (2.3) holds. □

3. Result

We use the Stein-Chen method to determine a result in the Poisson approximation to the distribution of X, in terms of the point metric of two such distributions together with its non-uniform bound, in the following theorem.

Theorem 3.1. *For $x_0 \in \mathbb{N}$, let $\lambda = \sum_{i=1}^{n} q_i$, then we have*

$$\left|P(X=1) - \lambda e^{-\lambda}\right| \leq \sum_{i=1}^{n} \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{2}, p_i\right\} \frac{q_i^2}{p_i} \tag{3.1}$$

and for $x_0 \geq 2$,

$$\left|P(X=x_0) - e^{-\lambda}\frac{\lambda^{x_0}}{x_0!}\right| \leq \sum_{i=1}^{n} \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}, p_i\right\} \frac{q_i^2}{p_i}, \tag{3.2}$$

where $P(X=0) = \prod_{i=1}^{n} p_i$.

Proof. Substituting h by h_{x_0} and x by X and taking expectation in (2.1), it yields

$$\begin{aligned}\left|P(X=x_0)-\frac{e^{-\lambda}\lambda^{x_0}}{x_0!}\right| &= |E[\lambda f(X+1)-Xf(X)]| \\ &\leq \sum_{i=1}^{n}|E[q_i f(X+1)-Y_i f(X)]|, \end{aligned} \tag{3.3}$$

where f is defined as in (2.2).

For $i\in\{1,...,n\}$, let $X_i=X-Y_i$, then by using the proof detailed as in Teerapabolarn [4], it follows that

$$\begin{aligned}|E[q_i f(X+1)-Y_i f(X)]| &= \left|\sum_{k\geq 2}(1-k)p_i q_i^k E[f(X_i+k)]\right| \\ &\leq \sum_{k\geq 2}\left|(1-k)p_i q_i^k\right| E|f(X_i+k)| \\ &\leq \sum_{k\geq 2}(k-1)p_i q_i^k \sup_{x\geq k}|f(x)|,\end{aligned}$$

and from (3.3), we obtain

$$\left|P(X=x_0)-\frac{e^{-\lambda}\lambda^{x_0}}{x_0!}\right| \leq \sum_{i=1}^{n}\sum_{k\geq 2}(k-1)p_i q_i^k \sup_{x\geq k}|f(x)|.$$

Hence, by Lemma 2.1, the theorem is proved. □

Immediately from Theorem 3.1, it is easily obtained a uniform bound of this approximation as follows.

Corollary 3.1. *We have*

$$\left|P(X=x_0)-e^{-\lambda}\frac{\lambda^{x_0}}{x_0!}\right| \leq \sum_{i=1}^{n}\min\left\{\lambda^{-1}(1-e^{-\lambda}),\frac{1}{2},p_i\right\}\frac{q_i^2}{p_i} \tag{3.4}$$

for every $x_0\in\mathbb{N}$.

Corollary 3.2. *For* $x_0\in\mathbb{N}$, *if* $p_i=p$ *for every* $i\in\{1,...,n\}$ *and* $\lambda=nq$, *then*

$$\left|P(X=1)-\lambda e^{-\lambda}\right| \leq \min\left\{1-e^{-\lambda},\frac{\lambda}{2},\lambda p\right\}\frac{q}{p} \tag{3.5}$$

and for $x_0 \geq 2$,

$$\left| P(X = x_0) - e^{-\lambda} \frac{\lambda^{x_0}}{x_0!} \right| \leq \min \left\{ 1 - e^{-\lambda}, \frac{\lambda}{x_0}, \lambda p \right\} \frac{q}{p}, \tag{3.6}$$

where $P(X = 0) = p^n$.

Remark. By simple comparison between the bound in Corollary 3.1 and the bounds in (1.5) and (1.6), it can be seen that the bound in Corollary 3.1 is sharper than the bound in (1.5) and the second bound in (1.6), and it is also sharper than the first bound in (1.6) when $\lambda < 0.72$ or $\lambda > 5.39$.

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