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ODD DIMENSIONAL CIRCULATE GEOMETRY

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Abstract: Following the paper [1] from G. Stanilov, in this paper we consider odd dimensional circulate geometry. At first we prove that the set of the non-degenerated circulate matrices is a group and find the corresponding transformations, which arise the odd dimensional circulate geometry. Then we find some numerical invariants.

AMS Subject Classification: circulate matrix, circulate geometry, Lie group, Klein geometry, numerical invariant
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1. Algebraic Consideration

A circulate matrix of odd order $2m+1$ is the matrix of the following type

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdot & \cdot & a_{2m-1} & a_{2m} & a_{2m+1} \\ a_{2m+1} & a_1 & a_2 & \cdot & \cdot & a_{2m-2} & a_{2m-1} & a_{2m} \\ a_{2m} & a_{2m+1} & a_1 & \cdot & \cdot & a_{2m-3} & a_{2m-2} & a_{2m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_4. & a_5 & a_6 & \cdot & \cdot & a_{2m} & a_{2m+1} & a_3 \\ a_3 & a_4 & a_5 & \cdot & \cdot & a_{2m+1} & a_1 & a_2 \\ a_2 & a_3 & a_4 & \cdot & \cdot & .a_{2m} & a_{2m+1} & a_1 \end{bmatrix}$$

Its elements are real numbers. We shall use the following fact from the linear

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algebra: Its determinant can be written in the following way

$$\det(A) = F(\varepsilon_0)F(\varepsilon_1)...F(\varepsilon_{2m-1})F(\varepsilon_{2m}),$$

Here$F(x)$ is the polynomial of order $2m$ with coefficients$a_1, a_2, a_3, ...a_{2m}, a_{2m+1}$:

$$F(x) = a_1 + a_2x + a_3x^2 + ... + a_{2m}x^{2m-1} + a_{2m+1}x^{2m}$$

and$\varepsilon_0, \varepsilon_1, \varepsilon_2, ..., \varepsilon_{2m-1}, \varepsilon_{2m}$ are the roots of the equation

$$x^{2m+1} - 1 = 0.$$

All these roots are different, more precisely, they are:

$$\varepsilon j := \cos\frac{j\pi}{2m+1} + i\sin\frac{j\pi}{2m+1}, \quad j = 0, 1, 2, .., 2m-1, 2m.$$

Evidently$\varepsilon_0 = 1$. It is well known the others roots are complex conjugate; indeed if $(i = \sqrt{-1})$

$$\varepsilon_k := \cos\frac{k\pi}{2m+1} + i\sin\frac{k\pi}{2m+1},$$

for $k = 1, 2, ..., m$ and if$\delta_k = \varepsilon_{2m-k+1}$, for $k = 1, 2, ..., m$, then

$$\varepsilon_k + \delta_k = 2\cos\frac{2k\pi}{2m+1}.$$

So ε_k, δ_k are complex conjugate numbers.

From the complex analysis is well known that the numbers:

$$F(\varepsilon_k) = \sum_{i=1}^{2m+1} a_i\varepsilon_k^{i-1}, F(\delta_k) = \sum_{i=1}^{2m+1} a_i\delta_k^{i-1}, \quad k = 1, ..., m$$

are also complex conjugate. Then their product

$$J_k = F(\varepsilon_k)F(\delta_k)$$

is a real positive number.

Denote

$$J_0 = F(\varepsilon_0) = \sum_{i=1}^{2m+1} a_i = a_1 + a_2 + a_3 + ... + a_{2m-1} + a_{2m} + a_{2m+1}$$

we can formulate the following

Theorem 1.1. *The* $\det(A)$ *can be decomposed in real multipliers in the following way:*

$$\det(A) = J_0 \prod_{k=1}^{m} J_k.$$

2. The Circulate Group

The following important theorem holds:

Theorem 2.1. *The set of all non-degenerated circulate matrices of order* $2m+1$ *is a group.*

At first we remark that any circulate matrix of order $2m+1$ can be characterized by the relations in

Lemma 2.2. *Putting* $a_{ij} = a_{(i,j)}$, *then* $a_{(i,j)} = a_{(i+1,j+1)}$, $a_{(i+1,1)} = a_{(i,2m+1)}$.

Consequence. *For any circulate matrix its transpose matrix is again such matrix.*

The following assertion gives a representation for all elements of any such circulate matrix:

Lemma 2.3. $a_{(i,j)} = a_{1+j-i}$ *for* $1 \leq i \leq j$; *2.* $a_{(i,j)} = a_{2m+1+j+1-i}$ *for* $i > j = 1, 2, 3, ..., 2m+1$.

To prove Theorem 2.1 one can verify as in the paper [1] the following two lemmas:

Lemma 2.4. *The product of any two circulate matrices of order* $2m$ *is again such matrix.*

Lemma 2.5. *The inverse matrix of any non-degenerated circulate matrix is such a matrix.*

The group of the no degenerated circulate matrices of order $2m+1$ arises in Klein geometry sense the so called *odd dimensional circulate geometry*. In the next part we find some invariants of this geometry.

3. Some Invariants in the Odd Dimensional Circulate Geometry

At first we need the fundamental transformation formulas for this geometry. We apply for the circulate matrix (1.1) the following short notification

$$A = CirculateMatrix(a_1, a_2, ..., a_{2m+1}).$$

Then we write an arbitrary such matrix

$$x = CirculateMatrix(x_1, x_2, ..., x_{2m+1}).$$

If we multiply these two matrices we get the circulate matrix

$$X = CirculateMatrix(X_1, X_2, ..., X_{2m+1}).$$

Its elements of the first row are:

$$\begin{aligned}
&X_1 = a_1x_1 + a_2x_{2m+1} + a_3x_{2m} + ... + a_{2m+1}x_2,\\
&X_2 = a_1x_2 + a_2x_1 + a_3x_{2m+1} + ... + a_{2m+1}x_3,\\
&X_3 = a_1x_3 + a_2x_2 + a_3x_1 + ... + a_{2m+1}x_4,\\
&\vdots\\
&X_{2m+1} = a_1x_{2m+1} + a_2x_{2m} + a_3x_{2m-1} + ... + a_{2m+1}x_1.
\end{aligned}$$

These are *fundamental transformation formulas* for the odd dimensional circulate geometry.

Now we want to show that any pair of two points has some numerical invariants.

At first on the base of (2) we can write the following representations fordet(A):

$$\det(A) = \begin{vmatrix} J_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F(\varepsilon_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F(\varepsilon_m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F(\delta_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F(\delta_m) \end{vmatrix}.$$

If we denote

$$g(x) = \sum_{i=1}^{2m+1} x_i x^{i-1}, \quad G(x) = \sum_{i=1}^{2m+1} X_i x^{i-1},$$

corresponding to the matrices x, X, we can write

$$\det(x) = \begin{vmatrix} g(\varepsilon_0) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g(\varepsilon_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g(\varepsilon_m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g(\delta_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g(\delta_m) \end{vmatrix},$$

where

$$g(\varepsilon_0) = x_1 + x_2 + x_3 + ... + x_{2m-1} + x_{2m} + x_{2m+1}$$

and

$$\det(X) = \begin{vmatrix} G(\varepsilon_0) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G(\varepsilon_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G(\varepsilon_m) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G(\delta_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G(\delta_m) \end{vmatrix},$$

with

$$G(\varepsilon_0) = X_1 + X_2 + X_3 + ... + X_{2m-1} + X_{2m} + X_{2m+1}.$$

From the evident relation $\det(X) = \det(A)\det(x)$ we get the following relations:

$$G(\varepsilon_0) = J_0 g(\varepsilon_0), G(\varepsilon_i) = F(\varepsilon_i)g(\varepsilon_i), G(\delta_i) = F(\delta_i)g(\delta_i)$$

for $i = 1, 2, ..., m$.

Thus we have proved

Theorem 3. *1 For any point $x(x_1, x_2, x_3, ..., x_{2m-1}, x_{2m}, x_{2m+1})$ the expressions*

$$g(\varepsilon_0), g(\varepsilon_1), ..., g(\varepsilon_m), g(\delta_1), ..., g(\delta_m)$$

are relative invariants under the transformations (11) and the expressions

$$g(\varepsilon_0), g(\varepsilon_1)g(\delta_1), ..., g(\varepsilon_m)g(\delta_m)$$

are real relative invariants.

As a consequence we can formulate the following

Theorem 3.2. *Any two points*

$$x(x_1, x_2, x_3, ..., x_{2m-1}, x_{2m}, x_{2m+1}), y(y_1, y_2, y_3, ..., y_{2m-1}, y_{2m}, y_{2m+1})$$

in the odd dimensional circulate geometry have the following absolute real invariants

$$I_0 = \frac{x_1 + x_2 + x_3 + ... + x_{2m-1} + x_{2m} + x_{2m+1}}{y_1 + y_2 + y_3 + ... + y_{2m-1} + y_{2m} + y_{2m+1})},$$

$$I_k = \frac{\sum_{i=1}^{2m+1} x_i(\cos\frac{k\pi}{2m+1} + i\sin\frac{k\pi}{2m+1})^{i-1} \sum_{i=1}^{2m+1} x_i(\cos\frac{(2m-k-1)\pi}{2m+1} + i\sin\frac{(2m-k-1)\pi}{2m+1})^{i-1}}{\sum_{i=1}^{2m+1} y_i(\cos\frac{k\pi}{2m+1} + i\sin\frac{k\pi}{2m+1})^{i-1} \sum_{i=1}^{2m+1} y_i(\cos\frac{(2m-k-1)\pi}{2m+1} + i\sin\frac{(2m-k-1)\pi}{2m+1})^{i-1}},$$

$$k = 1...m.$$

We remark the results here are generalizations of some results in the papers [3-5].

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