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CONNECTEDNESS OF ENDO-CAYLEY DIGRAPHS OF RIGHT(LEFT) ZERO UNION OF SEMIGROUPS

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Abstract: Let S be a finite semigroup, A a subset of S and f an endomorphism on S. The *endo-Cayley digraph of S corresponding to a connecting set A and an endomorphism f*, denoted by $endo - Cay_f(S, A)$ is a digraph whose vertex set is S and a vertex u is adjacent to vertex v if and only if $v = f(u)a$ for some $a \in A$.

In this paper, we study about the connected properties of endo-Cayley digraphs of cartesian product between semigroups and right(left) zero semigroups. We show the type of connected that they can be. Moreover, we also generalize endo-Cayley digraphs of that product into tensor product resulting graphs.

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1. Introduction

A set S and an operator $\cdot$ is called a *semigroup* if it preserve closed and associative properties. Note that we use ab as $a \cdot b$. If S is a semigroup that $ab = a$ for all $a, b \in S$, then we refer S as *left zero semigroup*. Dually, if $ab = b$ for any $a, b \in S$, we call S as *right zero semigroup*. For any two semigroup S_1 and S_2, a *Cartesian product* of S_1 and S_2, denoted by $S_1 \times S_2$, is also a semigroup by an operator $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$ for all $(a_1, a_2), (b_1, b_2) \in S_1 \times S_2$. Let f_1 and f_2 be endomorphisms on semigroups S_1 and S_2, respectively. We define $f_1 \times f_2 : S_1 \times S_2 \to S_1 \times S_2$ as $(f_1 \times f_2)(s_1, s_2) = (f_1(s_1), f_2(s_2))$. It is easy to check that $f_1 \times f_2$ is well defined and also an endomorphism on $S_1 \times S_2$. *A left zero union of semigroup* is a result of the Cartesian product between a left zero semigroup and any semigroup. Similarly, the Cartesian product of right zero semigroup with any semigroup is refered as *a right zero union of semigroup*.

Let S be a finite semigroup, A a subset of S and f an endomorphism on S. The *endo-Cayley digraph of S corresponding to a connecting set A and an endomorphism f*, denoted by $endo - Cay_f(S, A)$ is a digraph whose vertex set is S and a vertex u is adjacent to a vertex v if and only if $v = f(u)a$ for some $a \in A$.

We classify digraphs into 4 classes. A digraph is *strongly connected* if every pair of points are mutually reachable by directed paths. It is *unilateral* if, for every pair of points, at least one is reachable from the other. It is *weak* if its underlying is connected. If a digraph is not even weakly connected, we call it as *disconnected*. Let C_0 be the class of all disconnected digraphs, C_1 the class of all strictly weak digraphs(weak but not unilateral), C_2 the class of all strictly unilateral digraphs(unilateral but not strong), and C_3 the class of all strong digraphs,

The objective of this paper is to study connected properties of endo-Cayley graphs of right(left) zero union of semigroups. In the second section, we present the connectedness of endo-Cayley graphs of right groups. The connectness of endo-Cayley graphs of left groups is showed in the third section. The main results are obtained by using properties of tensor product presented in the last section.

2. Endo-Cayley Graph of Right Zero Semigroup

Let $R_n = \{r_1, r_2, \ldots, r_n\}$ be a right zero semigroup of order n where $n \in \mathbb{N}$. First, we show the connected properties of endo-Cayley graphs of right zero semigroups.

Remark 1. Let R_n be a right zero semigroup, A a nonempty subset of R_n and f an endomorphism on R_n. Then a subgraph of $endo{-}Cay_f(R_n, A)$induced by A forms a complete graph with $|A|$ loops.

Theorem 2. *Let R_n be a right zero semigroup, $A \subseteq R_n$ and f an endomorphism on R_n. Then $endo - Cay_f(R_n, A)$ is alway weakly connected. Moreover, $endo - Cay_f(R_n, A)$ is strongly connected if and only if $R_n = A$ and $n \geq 2$.*

Proof. Let $a \in A$. We know that $f(r)a = a$ for any $r \in R_n$. So a is a successor of all vertices in R_n. It means that its underlying is connected. Hence $endo - Cay_f(R, A_n)$ is a weakly connected graph.

We show the rest part here. If $n = 1$, then $R_n = A$ and $endo - Cay_f(R_n, A)$ is just a one loop which is not connected. Suppose $R_n \neq A$. Let $r \in R_n \backslash A$. So $d^-(r) = 0$ and there is no $x - r$ directed path for any $x \in R_n$. Hence $endo - Cay_f(R_n, A)$ is not strongly connected. Conversely, assume that $R_n = A$ and $n \geq 2$. We have that $endo - Cay_f(R_n, A)$ is a complete graph with n loops. Hence $endo - Cay_f(R_n, A)$ is strongly connected. □

Theorem 3. *$endo - Cay_f(R_n, A)$is unilateral connected if and only if $|R_n \backslash A| \leq 1$.*

Proof. Suppose that $|R_n \backslash A| > 1$. Let $u, v \in R_n \backslash A$. Then $d^+(u) = 0 = d^-(v)$. Hence there is no directed path $u - v$ and $v - u$. Therefore $endo - Cay_f(R_n, A)$is not unilateral connected. Conversely, we may assume that $|R_n \backslash A| \leq 1$. Let $u, v \in R_n$. So there is at least one vertex in A, we may assume as v. Since $f(u)v = v$, we have that u is adjacent to v. Therefore $endo - Cay_f(R_n, A)$is unilateral connected. □

3. Endo-Cayley Graph of Left Zero Semigroup

Let $L_n = \{l_1, l_2, \ldots, l_n\}$ be a left zero semigroup of order n where $n \in \mathbb{N}$.

Remark 4. 1. Every mapping f on L_n are endomorphisms, since for any $l_1, l_2 \in L_n$, $f(l_1 l_2) = f(l_1) = f(l_1)f(l_2)$.

2. There is an arc $(l, f(l))$ in $endo - Cay_f(L_n, A)$ for any connecting set A where $l \in L_n$ and $|A|$ is a number of multiple arc. To avoiding multiple arc, we set $A = \{a\}$.

Next, we give a necessary condition to make sure that $endo - Cay_f(L_n, A)$ is strongly connected.

Theorem 5. *If $endo - Cay_f(L_n, A)$is strongly connected, then f is an injective function with no fixed point.*

Proof. Assume that f is not injective or has a fixed point. We consider in 2 cases.

- Case I f is not injective: Then f is not surjective. So there is $l \in L_n$ such that $f(l_i) \neq l$ for all $l_i \in L_n$. Therefore there is no arc to l.
- Case II f has a fixed point, say l. So $f^n(l) = l$ for any natural number n. Hence there is no arc from l to other vertices in L_n.

From above cases, we can conclude that $endo - Cay_f(L_n, A)$is not strongly connected. □

Theorem 6. *$endo-Cay_f(L_n, A)$is strong if and only if $endo-Cay_f(L_n, A)$is a directed cycle.*

Proof. Assume that $endo-Cay_f(L_n, A)$is strong. Let $l \in L_n$. We know that $d^+(l) = 1$. By Theorem 5, f is a bijection. So there is exactly one vertex $l_1 \in L_n$ such that $f(l_1) = l$. Thus $d^+(l) = 1 = d^-(l)$. Since $endo - Cay_f(L_n, A)$is connected, we can conclude that $endo - Cay_f(L_n, A)$is a directed cycle. □

Let $l \in L_n$ and f be a function on L_n. We define a set $P(l)$ as $P(l) = \{f^n(l), n \in \mathbb{N}\}$. We note here that induced subgraph generated by $P(l)$ contains directed path with the beginning vertex l and an induce subgraph generated by $\bigcap_{l \in Ln} P(l)$ is a directed cycle.

Theorem 7. *$endo - Cay_f(L_n, A)$is weakly connected if and only if $\bigcap_{l \in L_n} P(l) \neq \phi$.*

Proof. Necessity. Assume that $endo-Cay_f(L_n, A)$is weakly connected. Let $P(l_1, l_2, \ldots, l_i)$ be the longest directed path in $endo - Cay_f(L_n, A)$.

$$l_1 \longrightarrow l_2 \longrightarrow l_3 \longrightarrow \ldots \longrightarrow l_i$$

Figure 1: A directed path P

So $P(l_m) \subseteq P(l_{m-1})$ for all $m = 2, 3, \ldots i$. If $i = n$, we are done. Suppose $i < n$. Then there is $l \in L_n \backslash V(P)$. Since L_n is weakly, we get that there is a semi-directed path between l and l_2. Because outdegree of each vertex is 1, so we can conclude that there is a subdirected path of that semi-directed path with endvertex of that directed path at l_j for some $j = 2, 3, \ldots, i$. Hence $l_i \in P(l)$ and also $P(l) \cap \bigcap_{j=1,2,\ldots i} P(l_j) \neq \phi$. Therefore $\bigcap_{l \in L_n} P(l) \neq \phi$.

Sufficiency. Assume that $\bigcap_{l \in L_n} P(l) \neq \phi$. Let $l \in \bigcap_{l \in L_n} P(l)$. Since for any $u, v \in L_n$, $u \in P(u)$ and $v \in P(v)$. We can conclude that there is a directed path from u to l and a directed path v to l. Hence there is a semi-directed path from u to v. Therefore $endo - Cay_f(L_n, A)$is weakly connected. □

Remark 8. If $\bigcap_{l \in L_n} P(l) \neq \phi$, it generates a cycle of length $\left|\bigcap_{l \in L_n} P(l)\right|$.

Theorem 9. *$endo - Cay_f(L_n, A)$is strongly connected if and only if $\bigcap_{l \in L_n} P(l) = L_n$.*

Proof. For a strongly connected graph $endo - Cay_f(L_n, A)$, it is a cycle with order n. So $P(l) = L_n$ for all $l \in L_n$. Hence $\bigcap_{l \in L_n} P(l) = L_n$. Conversely, we may assume that $endo - Cay_f(L_n, A)$is not strong. Then there are $l_i, l_j \in L_n$ such that there is no directed path from l_i to l_j. So $l_j \notin P(l_i)$. Hence $\bigcap_{l \in L_n} P(l) \neq L_n$. □

Theorem 10. *$endo - Cay_f(L_n, A)$is strictly unilateral connected if and only if it is a directed cycle joining with one directed path.*

Proof. It is clear that a directed cycle joining with one directed path, we call as G, is isomorphic to $endo - Cay_f(L_n, A)$where $n = |V(G)|$ and G is strictly unilateral connected. So sufficiency condition is proved. Next, we suppose that $endo - Cay_f(L_n, A)$ is strictly unilateral connected. So $endo - Cay_f(L_n, A)$is not a cycle and there is a vertex $l \in L_n$ such that $d^-(l) \neq d^+(l) = 1$. Suppose that $d^-(l) > 1$ for all $l \in L_n$. Then $\sum_{l\in L_n} d^-(l) > n = \sum_{l\in L_n} d^+(l)$. This is a contradiction. Clearly that if there are $l_1, l_2 \in L_n$ such that $d^-(l_1) = 0 = d^-(l_2)$, then there is no directed path between l_1 and l_2. It is impossible because $endo - Cay_f(L_n, A)$is unilateral connected. Therefore there is exactly $l \in L_n$ such that $d^-(l) = 0$. Then $P(l) = L_n$ and the end vertex in directed path generated by $P(l)$ must be adjacent to some vertex in $L_n \backslash \{l\}$. Hence $endo - Cay_f(L_n, A)$is a directed cycle joining with one directed path. □

4. Tensor Product

Let G_1 and G_2 be graphs. The *tensor product(Kronecker product)* of G_1 and G_2, denoted by $G_1 \otimes G_2$, is a graph with vertex set $V(G_1) \times V(G_2)$ where (u_1, v_1) is adjacent to (u_2, v_2) if and only if u_1 is adjacent to u_2 and v_1 is adjacent to v_2.

Remark 11. If one of original graphs is disconnected, then the result of tensor product is also disconnected.

In 1963, P.M. Weichsel presented a paper named "The Kronecker product of graphs". He showed a condition to make sure that result of tensor product(The Kronecker product in that paper) of graphs is connected. We state that Theorem as follow.

Theorem 12. (see [3]) *For connected graphs G and H, the product $G \otimes H$ is connected if and only if either G or H contains an odd cycle[one of original graphs is connected bipartite].*

We show in next Theorem that endo-Cayley graph of the Cartesian product between two semigruops is isomorphic to a resulting graph of tensor product of their endo-Cayley graphs.

Theorem 13. *Let S_1 and S_2 be semigroups, $A_1 \subseteq S_1$, $A_2 \subseteq S_2$ and f_1, f_2 endomorphisms on S_1 and S_2, respectively. Then* $endo{-}Cay_{f_1\times f_2}(S_1\times S_2, A_1\times A_2) = endo{-}Cay_{f_1}(S_1, A_1)\otimes endo{-}Cay_{f_2}(S_2, A_2)$.

Proof. Let (u_1, v_1) be adjacent to (u_2, v_2) in $endo - Cay_{f_1\times f_2}(S_1 \times S_2, A_1 \times A_2)$. Then

$((u_1, v_1), (u_2, v_2)) \in E(endo - Cay_{f_1\times f_2}(S_1 \times S_2, A_1 \times A_2))$
$\leftrightarrow (u_2, v_2) = (f_1(u_1)a_1, f_2(v_1)a_2)$, $a_1 \in A_1$ and $a_2 \in A_2$
$\leftrightarrow u_2 = f_1(u_1)a_1$ and $v_2 = f_2(v_1)a_2$
$\leftrightarrow (u_1, u_2) \in E(endo - Cay_{f_1}(S_1, A_1))$ and
$(v_1, v_2) \in E(endo - Cay_{f_2}(S_2, A_2))$
$\leftrightarrow ((u_1, v_1), (u_2, v_2)) \in E(endo - Cay_{f_1}(S_1, A_1) \otimes endo - Cay_{f_2}(S_2, A_2))$.

Therefore $endo - Cay_{f_1\times f_2}(S_1 \times S_2, A_1 \times A_2) = endo - Cay_{f_1}(S_1, A_1) \otimes endo - Cay_{f_2}(S_2, A_2)$. □

To reach our goal, we turn to study about properties of tensor product. We find useful Theorem in "Connectedness of products of two directed graphs" by F. Harrary and C.A. Trauth. There are many Theorems that we use in this paper. We post it without prove here.

Theorem 14. (see [1]) *The connectedness category of the tensor product of two digraph does not exceed the category of either.*

Theorem 15. (see [1]) *If D_1 and D_2 are strong, then $D_1 \otimes D_2$ is either disconnected or strong; and if disconnected, every weak component is strong.*

Proposition 16. (see [1]) *If the tensor product of two digraphs is strictly unilateral, then one of them is strong and the other is strictly unilateral.*

Theorem 17. (see [1]) *Let D_1 be unilateral and D_2 strong. Then $D = D_1 \otimes D_2$ is unilateral if and only if for each strong component S of D_1, $S \otimes D_2$ is strong.*

By Theorem 2, we know that endo-Cayley of right zero semigroup is always weakly connected. So if $endo - Cay_{f\times g}(S \times R, A \times B)$ is disconnected, it's because of disconnectedness of $endo - Cay_f(S, A)$. By the way, if $endo - Cay_f(S, A)$ is weakly connected, Is $endo - Cay_{f\times g}(S \times R, A \times B)$ weakly connected? The answer is showed in Theorem 19.

Lemma 18. *If vertex u is adjacent to v in $endo - Cay_f(S, A)$, then (u, r) is adjacent to (v, r) in $endo - Cay_f(S \times R, A \times \{r\})$ where $r \in R$.*

Proof. Let $r \in R$. Assume that u is adjacent to v in $endo - Cay_f(S, A)$. Then $v = f(u)a$ for some $a \in A$. So $(a, r) \in A \times \{r\}$. Since $(f \times g)(u, r)(a, r) = (f(u), g(r))(a, r) = (f(u)a, g(r)r) = (v, r)$, we have that (u, r) is adjacent to (v, r) in $endo - Cay_f(S \times R, A \times \{r\})$. □

Theorem 19. *$endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$ is weakly connected if and only if $endo - Cay_{f_1}(S, A)$ is weakly connected.*

Proof. Let S be a semigroup, $A \subseteq S$, $B \subseteq R_n$ and f_1, f_2 endomorphisms on S and R_n, respectively.

Necessity. Assume that $endo - Cay_{f_1}(S, A)$ is disconnected. By Remark 11, we have that $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$ is also disconnedted.

Sufficiency. Assume that $endo - Cay_{f_1}(S, A)$ is a weakly connected graph. Since B is not empty, we may let $r \in B$. Let $(s_1.r_1), (s_2, r_2) \in S \times R_n$. We know that, for all $a \in A$, $(s_1.r_1)$ and (s_2, r_2) are adjacent to $(f_1(s_1)a, r)$ and $(f_1(s_2)a, r)$, respectively. We call $f_1(s_1)a$ as u and $f_1(s_2)a$ as v. So $u, v \in S$. Then there is $u - v$ semi-directed path in $endo - Cay_{f_1}(S, A)$ as $u, u_1, u_2, \ldots, u_n, v$. By Lemma 18, we can conclude that P:$(u, r), (u_1, r)$, $(u_2, r), \ldots, (u_n, r), (v, r)$ is a semi-directed path in $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$. Therefore there is a $(s_1, r_1) - (s_2, r_2)$ semi-directed path by adding arcs $((s_1, r_1), (u, r))$ and $((s_2, r_2), (v, r))$ in semi-directed path P. □

We know by Theorem 15 tensor product of two strongly connected graphs can be either disconnected or strong. We will show that $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$ is strong if both $endo - Cay_{f_1}(S, A)$ and $endo - Cay_{f_2}(R_n, B)$ are strong.

Theorem 20. *Let S be a semigroup, $A \subseteq S$, $B \subseteq R_n$ and f_1, f_2 endomorphisms on S and R_n, respectively. Then $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$ is strong if and only if both $endo - Cay_{f_1}(S, A)$ and $endo - Cay_{f_2}(R_n, B)$ are strong.*

Proof. By Theorem 14, we can conclude that a condition for $endo - Cay_{f_1}(S, A)$ and $endo - Cay_{f_2}(R_n, B)$ are strong is a necessity condition to make $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$ be strong. Conversely, we may assume that $endo - Cay_{f_1}(S, A)$ and $endo - Cay_{f_2}(R_n, B)$ are strong. So $R_n = B$ and $endo -$

D	$endo - Cay_f(R_n, A)$		
	1	2	3
0	0	0	0
1	1	1	1
2	1	1	2
3	1	2	3

Table 1: Possible categories $D \otimes endo - Cay_f(R_n, A)$

$Cay_{f_2}(R_n, B)$ is complete with n loops. Let $(s_1.r_1)$ and (s_2, r_2) be vertices in $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$. Then $s_1, s_2 \in S$ and $r_1, r_2 \in R_n$. Because $endo - Cay_{f_1}(S, A)$ is strong, so There are paths $P_1(s_1, u_1, u_2, \ldots, u_n, s_2)$ and $P_2(s_2, v_1, v_2, \ldots, v_n, s_1)$. Since $endo - Cay_{f_2}(R_n, B)$ is a complete graph with n loops, we can conclude that $((s_1, r_1), (u_1, r_1), \ldots, (s_2, r_2))$ and $((s_2, r_2), (v_1, r_2), \ldots (s_1, r_1))$ are path in $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$. Hence $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$ is strong. □

Theorem 21. *$endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$ is strictly unilateral if and only if*

1. *$endo - Cay_{f_1}(S, A)$ is strong and $endo - Cay_{f_2}(R_n, B)$ is strictly unilateral or*

2. *$endo - Cay_{f_1}(S, A)$ is strictly unilateral and $endo - Cay_{f_2}(R_n, B)$ is strong.*

Proof. It's clearly that if $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$ is strictly unilateral, then condition 1 or 2 hold by Proposition 16. Conversely, assume a condition 1 holds. We know by Theorem 2 that the strong component of $endo - Cay_{f_2}(R_n, B)$ is $endo - Cay_{f_2}(B, B)$. So $endo - Cay_{f_1}(S, A) \otimes endo - Cay_{f_2}(B, B)$ is strong by Theorem 20. Hence $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$ is strictly unilateral by Theorem 17. Next, we may assume a condition 2 holds. Since $endo - Cay_{f_2}(R_n, B)$ is strong, we get that $R_n = B$ and $endo - Cay_{f_2}(R_n, B)$ is complete with n loops. For any strong component H of $endo - Cay_{f_1}(S, A)$. We know by the proof of Theorem 20 that $S \otimes endo - Cay_{f_2}(R_n, B)$ is strong. By Proposition 16, hence $endo - Cay_{f_1 \times f_2}(S \times R_n, A \times B)$ is strictly unilateral. □

We end this study by showing results of connectedness of tensor product between any digraphs and $endo - Cay_f(R_n, A)$.

Now, we turn to consider a connectedness of left zero union of semigroup. The results are different to right zero union of semigroup because $endo - Cay_f(L_n, A)$can be a disconnected digraph. It need some conditions to guarantee that its tensor product is connected. The tools of this study are in *On the product of directed graphs* by M.H. Mcandrew.

Theorem 22. (McAndrew) *Let D_1 and D_2 be digraphs and let d_i be the greatest common divisor of the lengths of all the directed cycles in D_i, $i = 1, 2$. Then their tensor product is strong if and only if both digraphs are strong and d_1 and d_2 are relatively prime.*

We know in previous section that $\left|\bigcap_{l \in L_n} P(l)\right|$ generates a cycle if it not empty. So we have our results by applying Mcandrew's theorem immediately.

Theorem 23. *For a strongly connected digraphs $endo - Cay_{f_1}(S, A)$ and $endo - Cay_{f_2}(L_n, B)$, let d be the greatest common divisor of the lengths of all the directed cycles in $endo - Cay_{f_1}(S, A)$. Then $endo - Cay_{f_1 \times f_2}(S \times L_n, A \times B)$ is strong if and only if $gcd(d, n) = 1$.*

Theorem 24. *$endo - Cay_{f_1 \times f_2}(S \times L_n, A \times B)$ is strictly unilateral connected if and only if*

1. *$endo - Cay_{f_1}(S, A)$ is strong, $endo - Cay_{f_2}(L_n, B)$ is strictly unilateral and $gcd(d, \left|\bigcap_{l \in L_n} P(l)\right|) = 1$ where d be the greatest common divisor of the lengths of all the directed cycles in $endo - Cay_{f_1}(S, A)$ or.*

2. *$endo - Cay_{f_1}(S, A)$ is strictly unilateral, $endo - Cay_{f_2}(L_n, B)$ is strong and $gcd(d, n) = 1$ where d be the greatest common divisor of the lengths of all the directed cycles in strong components of $endo - Cay_{f_1}(S, A)$*

Now, the last part of this paper is to characterize weakly connected of $endo - Cay_{f_1 \times f_2}(S \times L_n, A \times B)$, but the big problem is that characterization for a weakly connected tensor product still open. How can we answer that problem? We have a necessary condition to be weakly connected of tensor product between two directed graphs.

Theorem 25. *For any weak digraphs D_1 and D_2, let G_1 and G_2 be underlying graphs of D_1 and D_2, respectively. If $D_1 \otimes D_2$ is weakly connected, then G_1 or G_2 is bipartite.*

Proof. We mention first that $D_1 \otimes D_2 \subseteq G_1 \otimes G_2$, since both D_1 and D_2 are spanning subgraphs of G_1 and G_2, respectively. Suppose that G_1 and G_2 are not bipartite. By Weichsel' theorem [Theorem 12], we have $G_1 \otimes G_2$ is disconnected. Because $D_1 \otimes D_2$ is a spanning subgraph of $G_1 \otimes G_2$, so we have $D_1 \otimes D_2$ is also disconnected. □

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