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BOUNDED POSITIVE SOLUTIONS FOR A SECOND ORDER NEUTRAL DELAY DIFFERENCE EQUATION

Lijun He[1], Shin Min Kang[2 §], Chahn Yong Jung[3]

[1]Department of Physics and Mathematics
Kunming Medical College
Kunming, Yunnan 650031, P.R. CHINA

[2]Department of Mathematics and RINS
Gyeongsang National University
Jinju, 660-701, KOREA

[3]Department of Business Administration
Gyeongsang National University
Jinju, 660-701, KOREA

Abstract: This paper studies solvability of the second order neutral delay difference equation

$$\Delta^2(x_n + bx_{n-\tau}) + f(n, x_{n-c_n}, x_{n-d_n}) = 0, \quad n \ge n_0.$$

Using the Banach fixed point theorem, we show the existence of a bounded positive solution for the difference equation. Three examples are also included.

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Key Words: second order neutral delay difference equation, bounded positive solution, Banach fixed point theorem

1. Introduction and Preliminaries

Tang [3] studied the existence of a bounded nonoscillatory solution for the

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§Correspondence author

second order linear delay difference equation

$$\Delta^2 x_n = p_n x_{n-k}, \quad n \geq 0. \tag{1.1}$$

Jinfa [1] used the Banach fixed point theorem to study the existence of a nonoscillatory solution for the second order neutral delay difference equation with positive and negative coefficients

$$\Delta^2(x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n \geq n_0 \tag{1.2}$$

under the condition $p \neq -1$. Migda and Migda [2] discussed the asymptotic behavior of the second order neutral difference equation

$$\Delta^2(x_n + px_{n-k}) + f(n, x_n) = 0, \quad n \geq 1. \tag{1.3}$$

The purpose of this note is to investigate solvability of the following second order neutral delay difference equation

$$\Delta^2(x_n + bx_{n-\tau}) + f(n, x_{n-c_n}, x_{n-d_n}) = 0, \quad n \geq n_0, \tag{1.4}$$

where $b \in \mathbb{R}, \tau \in \mathbb{N}, n_0 \in \mathbb{N}_0$, $\{c_n\}_{n\in\mathbb{N}_{n_0}} \cup \{d_n\}_{n\in\mathbb{N}_{n_0}} \subseteq \mathbb{Z}$, $\lim_{n\to\infty}(n - c_n) = \lim_{n\to\infty}(n - d_n) = +\infty$ and $f : \mathbb{N}_{n_0} \times \mathbb{R}^2 \to \mathbb{R}$ is a mapping. It is clear that Eq.(1.4) includes Eqs.(1.1)-(1.3) as special cases. Using the Banach fixed point theorem, we establish three existence results of a bounded positive solution for Eq.(1.4). To illustrate our results, three examples are also included.

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}$ and $\mathbb{N}$ stand for the sets of all integers and positive integers, respectively,

$$\begin{aligned}
&\mathbb{N}_a = \{n : n \in \mathbb{N} \text{ with } n \geq a\}, \quad \mathbb{Z}_a = \{n : n \in \mathbb{Z} \text{ with } n \geq a\}, \quad a \in \mathbb{Z},\\
&\alpha = \min\big\{\inf\{n - c_n : n \in \mathbb{N}_{n_0}\}, \inf\{n - d_n : n \in \mathbb{N}_{n_0}\}\big\},\\
&\beta = \min\{n_0 - \tau, \alpha\},
\end{aligned}$$

l^∞_β denotes the Banach space of all bounded sequences on $\mathbb{Z}_\beta$ with norm

$$\|x\| = \sup_{n\in\mathbb{Z}_\beta} |x_n| \quad \text{for } x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in l^\infty_\beta$$

and

$$A(N, M) = \big\{x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in l^\infty_\beta : N \leq x_n \leq M, \, n \in \mathbb{Z}_\beta\big\} \quad \text{for } M > N > 0.$$

It is easy to see that $A(N, M)$ is a bounded closed subset of l^∞_β.

By a solution of Eq.(1.4), we mean a sequence $\{x_n\}_{n\in\mathbb{Z}_\beta}$ with a positive integer $T \geq n_0 + \tau + |\alpha|$ such that Eq.(1.4) is satisfied for all $n \geq T$. As is customary, a solution of Eq.(1.4) is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is said to be *nonoscillatory.*

2. Existence of Bounded Positive Solutions

Now we investigate the existence of bounded positive solutions for Eq.(1.4).

Theorem 2.1. *Let $b \in [0,1)$, M and N be two positive constants with $M > N$. Assume that there exist two sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$|f(n,u,v) - f(n,\bar{u},\bar{v})| \le P_n \max\{|u-\bar{u}|, |v-\bar{v}|\}, \quad n \in \mathbb{N}_{n_0},\ u,\bar{u},v,\bar{v} \in [N,M]; \tag{2.1}$$

$$|f(n,u,v)| \le Q_n, \quad n \in \mathbb{N}_{n_0},\ u,v \in [N,M]; \tag{2.2}$$

$$\sum_{s=n_0}^{\infty}\sum_{t=s}^{\infty} \max\{P_t, Q_t\} < +\infty. \tag{2.3}$$

Then Eq.(1.4) *has a bounded positive solution in $A(N,M)$.*

Proof. Now we construct a contraction mapping $S : A(N,M) \to A(N,M)$ and show that its fixed point is a bounded positive solution of Eq.(1.4). It follows from (2.3) that there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + |\alpha|$ satisfying

$$\theta = b + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} P_t; \tag{2.4}$$

$$\sum_{s=T}^{\infty}\sum_{t=s}^{\infty} Q_t < \frac{(1-b)(M-N)}{2}. \tag{2.5}$$

Define a mapping $S : A(N,M) \to l_\beta^\infty$ by

$$(Sx)_n = \begin{cases} \dfrac{(1+b)(M+N)}{2} - bx_{n-\tau} \\ \quad -\displaystyle\sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}), \quad n \ge T, \\ (Sx)_T, \quad \beta \le n < T \end{cases} \tag{2.6}$$

for all $x \in A(N,M)$. In view of (2.4)-(2.6), we get that for any $x, y \in A(N,M)$

and $n \geq T$

$$\begin{aligned}
&|(Sx)_n - (Sy)_n| \\
&= \Bigg| \frac{(1+b)(M+N)}{2} - bx_{n-\tau} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\
&\quad - \frac{(1+b)(M+N)}{2} + by_{n-\tau} + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, y_{t-c_t}, y_{t-d_t}) \Bigg| \\
&\leq b|x_{n-\tau} - y_{n-\tau}| + \left| \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, y_{t-c_t}, y_{t-d_t}) \right| \\
&\leq b\|x-y\| + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} P_t \max\left\{|x_{t-c_t} - y_{t-c_t}|, |x_{t-d_t} - y_{t-d_t}|\right\} \\
&\leq b\|x-y\| + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} P_t\|x-y\| \leq \theta\|x-y\|,
\end{aligned}$$

$$\begin{aligned}
(Sx)_n &= \frac{(1+b)(M+N)}{2} - bx_{n-\tau} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\
&\leq \frac{(1+b)(M+N)}{2} - bN + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} |f(t, x_{t-c_t}, x_{t-d_t})| \\
&\leq \frac{(1+b)(M+N)}{2} - bN + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} Q_t \\
&< \frac{(1+b)(M+N)}{2} - bN + \frac{(1-b)(M-N)}{2} = M
\end{aligned}$$

and

$$\begin{aligned}
(Sx)_n &= \frac{(1+b)(M+N)}{2} - bx_{n-\tau} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\
&\geq \frac{(1+b)(M+N)}{2} - bM - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} |f(t, x_{t-c_t}, x_{t-d_t})| \\
&\geq \frac{(1+b)(M+N)}{2} - bM - \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} Q_t \\
&> \frac{(1+b)(M+N)}{2} - bM - \frac{(1-b)(M-N)}{2} = N,
\end{aligned}$$

which imply that

$$\|Sx - Sy\| \leq \theta\|x-y\|, \quad S : A(N,M) \to A(N,M), \quad x, y \in A(N,M). \tag{2.7}$$

Clearly (2.7) means that $S : A(N, M) \to A(N, M)$ is a contraction in the closed subset $A(N, M)$ of the Banach space l^∞_β. Consequently the Banach fixed point ensures that S has a unique fixed point $x \in A(N, M)$, which yields that

$$x_n = (Sx)_n = \frac{(1+b)(M+N)}{2} - bx_{n-\tau} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}), \quad n \geq T,$$

that is,

$$x_n + bx_{n-\tau} = \frac{(1+b)(M+N)}{2} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}), \quad n \geq T,$$

which implies that

$$\begin{aligned}\Delta(x_n + bx_{n-\tau}) &= \frac{(1+b)(M+N)}{2} - \sum_{s=n+1}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\ &\quad - \left[\frac{(1+b)(M+N)}{2} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t})\right] \\ &= \sum_{t=n}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}), \quad n \geq T\end{aligned}$$

and

$$\begin{aligned}\Delta^2(x_n + bx_{n-\tau}) &= \sum_{t=n+1}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) - \sum_{t=n}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\ &= -f(t, x_{n-c_n}, x_{n-d_n}), \quad n \geq T,\end{aligned}$$

which means that x is a bounded positive solution of Eq.(1.4). This completes the proof. □

Theorem 2.2. *Let $b \in (-1, 0]$, M and N be two positive constants with $M > N$. Assume that there exist two sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1)-(2.3). *Then Eq.*(1.4) *has a bounded positive solution in $A(N, M)$.*

Proof. In terms of (2.3), we choose $\theta \in (0, 1)$ and $T \geq n_0 + \tau + |\alpha|$ satisfying

$$\theta = |b| + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} P_t; \tag{2.8}$$

$$\sum_{s=T}^{\infty}\sum_{t=s}^{\infty} Q_t < \frac{(1+b)(M-N)}{3}. \tag{2.9}$$

Define a mapping $S : A(N,M) \to l_\beta^\infty$ by

$$(Sx)_n = \begin{cases} \dfrac{(1+b)(2M+N)}{3} - bx_{n-\tau} \\ \quad -\displaystyle\sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}), \quad n \geq T, \\ (Sx)_T, \quad \beta \leq n < T \end{cases} \tag{2.10}$$

for all $x \in A(N,M)$. Using (2.1) and (2.8)-(2.10), we conclude that for any $x, y \in A(N,M)$ and $n \geq T$

$$\begin{aligned} &|(Sx)_n - (Sy)_n| \\ &= \Bigg| \frac{(1+b)(2M+N)}{3} - bx_{n-\tau} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\ &\quad - \frac{(1+b)(2M+N)}{3} + by_{n-\tau} + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, y_{t-c_t}, y_{t-d_t}) \Bigg| \\ &\leq |b||x_{n-\tau} - y_{n-\tau}| + \left| \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, y_{t-c_t}, y_{t-d_t}) \right| \\ &\leq |b|\|x-y\| + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} P_t \max\left\{|x_{t-c_t} - y_{t-c_t}|, |x_{t-d_t} - y_{t-d_t}|\right\} \\ &\leq |b|\|x-y\| + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} P_t\|x-y\| \leq \theta\|x-y\|, \end{aligned}$$

$$\begin{aligned} (Sx)_n &= \frac{(1+b)(2M+N)}{3} - bx_{n-\tau} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\ &\leq \frac{(1+b)(2M+N)}{3} - bM + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} |f(t, x_{t-c_t}, x_{t-d_t})| \\ &\leq \frac{(1+b)(2M+N)}{3} - bM + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} Q_t \\ &< \frac{(1+b)(2M+N)}{3} - bM + \frac{(1+b)(M-N)}{3} = M \end{aligned}$$

and

$$
\begin{aligned}
(Sx)_n &= \frac{(1+b)(2M+N)}{3} - bx_{n-\tau} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\
&\geq \frac{(1+b)(2M+N)}{3} - bN - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} |f(t, x_{t-c_t}, x_{t-d_t})| \\
&\geq \frac{(1+b)(2M+N)}{3} - bN - \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} Q_t \\
&> \frac{(1+b)(2M+N)}{3} - bN - \frac{(1+b)(M-N)}{3} > N,
\end{aligned}
$$

which imply (2.7). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

Theorem 2.3. *Let $|b| \in [0, \frac{1}{2})$, M and N be two positive constants with $M(\frac{1}{2} - |b|) > N$. Assume that there exist two sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.1)-(2.3). *Then Eq.*(1.4) *has a bounded positive solution in $A(N, M)$.*

Proof. It follows from (2.3) that there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + |\alpha|$ satisfying (2.8) and

$$
\sum_{s=T}^{\infty}\sum_{t=s}^{\infty} Q_t < \left(\frac{1}{2} - |b|\right)M - N. \tag{2.11}
$$

Define a mapping $S : A(N, M) \to l_\beta^\infty$ by

$$
(Sx)_n = \begin{cases} \dfrac{M+N}{2} - bx_{n-\tau} \\ \quad - \displaystyle\sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}), \quad n \geq T, \\ (Sx)_T, \quad \beta \leq n < T \end{cases} \tag{2.12}
$$

for all $x \in A(N, M)$. By virtue of (2.1), (2.8), (2.11) and (2.12), we deduce that for any $x, y \in A(N, M)$ and $n \geq T$

$$
\begin{aligned}
&|(Sx)_n - (Sy)_n| \\
&= \Bigg| \frac{M+N}{2} - bx_{n-\tau} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\
&\quad - \frac{M+N}{2} + by_{n-\tau} + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, y_{t-c_t}, y_{t-d_t}) \Bigg| \\
&\leq |b||x_{n-\tau} - y_{n-\tau}| + \left| \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, y_{t-c_t}, y_{t-d_t}) \right| \\
&\leq |b|\|x-y\| + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} P_t \max\left\{|x_{t-c_t} - y_{t-c_t}|, |x_{t-d_t} - y_{t-d_t}|\right\} \\
&\leq |b|\|x-y\| + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} P_t\|x-y\| \leq \theta\|x-y\|,
\end{aligned}
$$

$$
\begin{aligned}
(Sx)_n &= \frac{M+N}{2} - bx_{n-\tau} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\
&\leq \frac{M+N}{2} + |b|M + \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} |f(t, x_{t-c_t}, x_{t-d_t})| \\
&\leq \frac{M+N}{2} + |b|M + \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} Q_t \\
&< \frac{M+N}{2} + |b|M + \left(\frac{1}{2} - |b|\right)M - N = M - \frac{N}{2} < M
\end{aligned}
$$

and

$$
\begin{aligned}
(Sx)_n &= \frac{M+N}{2} - bx_{n-\tau} - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} f(t, x_{t-c_t}, x_{t-d_t}) \\
&\geq \frac{M+N}{2} - |b|M - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} |f(t, x_{t-c_t}, x_{t-d_t})| \\
&\geq \frac{M+N}{2} - |b|M - \sum_{s=T}^{\infty}\sum_{t=s}^{\infty} Q_t \\
&> \frac{M+N}{2} - |b|M - \left(\frac{1}{2} - |b|\right)M + N = \frac{3N}{2} > N,
\end{aligned}
$$

which yield (2.7). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

Next we construct three examples to explain Theorems 2.1-2.3.

Example 2.1. Consider the second order neutral delay difference equation

$$\Delta\left(x_n + \frac{8}{9}x_{n-\tau}\right) + \frac{x^2_{n-5(-1)^n}\sin^3(3x_{n-(-1)^n})}{n^6 + |x_{n-(-1)^n}|} = 0, \quad n \geq n_0 = 1, \tag{2.13}$$

where $\tau \in \mathbb{N}$ is fixed. Let M and N be two positive constants with $M > N$ and

$$b = \frac{8}{9}, \quad c_n = 5(-1)^n, \quad d_n = (-1)^n, \quad f(n,u,v) = \frac{u^2\sin^3(3v)}{n^6 + |v|},$$

$$P_n = \frac{M(2+9M)n^6 + 3M^2(1+3M)}{(n^6+N)^2}, \quad Q_n = \frac{M^2}{n^6+N}, \quad n \geq 1,\ u,v \in \mathbb{R}.$$

It is easy to see that the conditions (2.1)-(2.3) are satisfied. Thus Theorem 2.1 ensures that Eq.(2.13) has a bounded positive solution in $A(N,M)$.

Example 2.2. Consider the second order neutral delay difference equation

$$\Delta\left(x_n - \frac{6}{7}x_{n-\tau}\right) + \frac{(x_{n-3})^2}{n^3(n+1) + (x_{n-(-1)^{n+1}})^2} = 0, \quad n \geq n_0 = 4, \tag{2.14}$$

where $\tau \in \mathbb{N}$ is fixed. Let M and N be two positive constants with $M > N$ and

$$b = -\frac{6}{7}, \quad c_n = 3, \quad d_n = (-1)^{n+1}, \quad f(n,u,v) = \frac{u^2}{n^3(n+1) + v^2},$$

$$P_n = \frac{2Mn^3(n+1) + 4M^3}{\left(n^3(n+1) + N^2\right)^2}, \quad Q_n = \frac{M^2}{n^3(n+1) + N^2}, \quad n \geq 4,\ u \in \mathbb{R}.$$

It is easy to verify that the conditions (2.1)-(2.3) hold. Therefore Theorem 2.2 guarantees that Eq.(2.14) has a bounded positive solution in $A(N,M)$.

Example 2.3. Consider the second order neutral delay difference equation

$$\Delta\left(x_n + \frac{2}{5}x_{n-\tau}\right) + \frac{n(x_{n-(-1)^n})^2 - \cos(x_{n-7})}{n^8 + 3n^3 - 2n^2 + 1} = 0, \quad n \geq n_0 = 3, \tag{2.15}$$

where $\tau \in \mathbb{N}$ is fixed. Let M and N be two positive constants with $M > 10N$ and

$$b = \frac{2}{5}, \quad c_n = (-1)^n, \quad d_n = 7, \quad f(n,u,v) = \frac{nu^2 - \cos v}{n^8 + 3n^3 - 2n^2 + 1},$$

$$P_n = \frac{2nM + 1}{n^8 + 3n^3 - 2n^2 + 1}, \quad Q_n = \frac{nM^2 + 1}{n^8 + 3n^3 - 2n^2 + 1}, \quad n \geq 3,\ u \in \mathbb{R}.$$

Clearly, the conditions (2.1)-(2.3) are satisfied. Thus Theorem 2.3 means that Eq.(2.15) has a bounded positive solution in $A(N,M)$.

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