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STABILITY OF AN ADDITIVE FUNCTIONAL INEQUALITY

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Abstract: In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(x)+f(y)+f(z)\| \leq \|f(x+y)+f(z)\|$$

in Banach spaces.

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1. Introduction and Preliminaries

In 1940, S.M. Ulam [1] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows: *Let* $(\mathcal{G}, \circ)$ *be a group and let* $(\mathcal{H}, \star, d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\varepsilon > 0$, *does there exist a* $\delta = \delta(\varepsilon) > 0$ *such that if a mapping* $f : \mathcal{G} \to \mathcal{H}$ *satisfies the inequality*

$$d(f(x \circ y), f(x) \star f(y)) < \delta$$

for all $x, y \in \mathcal{G}$, *then a homomorphism* $F : \mathcal{G} \to \mathcal{H}$ *exits with*

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$$d(f(x), F(x)) < \varepsilon$$

for all $x \in \mathcal{G}$*?*

In the next year, D.H. Hyers [2] gave a first (partial) affirmative answer to the question of Ulam for Banach spaces as follows: *If* $\delta > 0$ *and if* $f : \mathcal{E} \to \mathcal{F}$ *is a mapping between Banach spaces* $\mathcal{E}$ *and* $\mathcal{F}$ *satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in \mathcal{E}$*, then there is a unique additive mapping* $A : \mathcal{E} \to \mathcal{F}$ *such that*

$$\|f(x) - A(x)\| \leq \delta$$

for all $x, y \in \mathcal{E}$*.*

Thereafter, we call that type the Hyers-Ulam stability.

2. Hyers-Ulam Stability in Banach Spaces

Throughout this paper, let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a Banach space. In 2007, C. Park, Y. S. Cho and M.-H. Han [3] proved the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\|$$

in Banach spaces. In 2011, J. R. Lee, C. Park and D. Y. Shin [4] prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(2x) + f(2y) + 2f(z)\| \leq \|2f(x+y+z)\|$$

in Banach spaces. In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y) + f(z)\|$$

in Banach spaces.

Lemma 1. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping. Then it is additive if and only if it satisfies*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y) + f(z)\| \tag{1}$$

for all $x, y, z \in \mathcal{X}$*.*

Proof. If f is additive, then clearly

$$\|f(x)+f(y)+f(z)\| = \|f(x+y)+f(z)\|$$

for all $x, y, z \in \mathcal{X}$.

Assume that f satisfies (1). Letting $x = y = z = 0$ in (1), we gain $\|3f(0)\| \leq \|2f(0)\|$ and so $f(0) = 0$. Putting $z = 0$ and replacing y by $-x$ in (1), we get

$$\|f(x)+f(-x)\| \leq \|2f(0)\| = 0$$

and so $f(-x) = -f(x)$ for all $x \in \mathcal{X}$. Setting $z = -x-y$ in (1), we obtain

$$\|f(x)+f(y)-f(x+y)\| \leq \|f(x+y)-f(x+y)\| = 0$$

for all $x, y \in \mathcal{X}$. Thus we see that

$$f(x+y) = f(x)+f(y)$$

for all $x, y \in \mathcal{X}$. □

Theorem 2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying*

$$\|f(x)+f(y)+f(z)\| \leq \|f(x+y)+f(z)\| + \varphi(x, y, z) \tag{2}$$

and

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^{j+1}x, -2^j y, -2^j z) < \infty \tag{3}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x)-A(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x, x) \tag{4}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x, y, z by $2^{n+1}x, -2^n x, -2^n x$, respectively, and dividing by 2^{n+1} in (2), since f is odd, we gain

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}\right\| \leq \frac{1}{2^{n+1}}\varphi(2^{n+1}x, -2^n x, -2^n x)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. From the above inequality, we get

$$\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right\| \leq \sum_{j=m}^{n-1}\left\|\frac{f(2^{j+1}x)}{2^{j+1}} - \frac{f(2^j x)}{2^j}\right\| \tag{5}$$

$$\leq \sum_{j=m}^{n-1} \frac{1}{2^{j+1}} \varphi(2^{j+1}x, -2^j x, -2^j x)$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, n with $m < n$. By the condition (3), the sequence $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{f(2^n x)}{2^n}\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in \mathcal{X}$. Taking $m = 0$ and letting n tend to ∞ in (5), we have the inequality (4).

Replacing x, y, z by $2^n x, 2^n x, 2^n x$, respectively, and dividing by 2^n in (2), we obtain

$$\begin{aligned} &\left\| \frac{f(2^n x)}{2^n} + \frac{f(2^n y)}{2^n} + \frac{f(2^n z)}{2^n} \right\| \\ &\qquad \leq \left\| \frac{f(2^n x + 2^n y)}{2^n} + \frac{f(2^n z)}{2^n} \right\| + \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) \end{aligned}$$

for all $x, y, z \in \mathcal{X}$ and all nonnegative integers n. Since (3) gives that

$$\lim_{n\to\inf} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n y) = 0$$

for all $x, y, z \in \mathcal{X}$, letting n tend to ∞ in the above inequality, we see that A satisfies the inequality (1) and so it is additive by Lemma 1.

Let $B : \mathcal{X} \to \mathcal{Y}$ be another mapping satisfying (4). Since both A and B are additive, we have

$$\begin{aligned} \|A(x) - B(x)\| &= \frac{1}{2^n} \|A(2^n x) - B(2^n x)\| \\ &\leq \frac{1}{2^n} \big(\|A(2^n x) - f(2^n x)\| + \|f(2^n x) - B(2^n x)\| \big) \\ &\leq \frac{1}{2^n} \widetilde{\varphi}(2^n x, 2^n x, 2^n x) = \sum_{j=n}^{\infty} \frac{1}{2^j} \varphi(2^{j+1}x, -2^j x, -2^j x) \end{aligned}$$

which goes to zero as $n \to \infty$ for all $x \in \mathcal{X}$ by (3). Hence A is a unique additive mapping satisfying (4), as desired. □

Corollary 3. *Let $\theta \in [0, \infty)$ and $p \in [0, 1)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y) + f(z)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \theta \frac{2 + 2^p}{2 - 2^p} \|x\|^p$$

for all $x \in \mathcal{X}$.

Proof. In Theorem 2, taking

$$\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p),$$

we have

$$\widetilde{\varphi}(x, y, z) = \frac{2\theta}{2 - 2^p}(2^p\|x\|^p + \|y\|^p + \|z\|^p) < \infty$$

for all $x, y, z \in \mathcal{X}$. And we obtain the desired result. □

Theorem 4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping. If there is a function $\psi : X^3 \to [0, \infty)$ satisfying*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y) + f(z)\| + \psi(x, y, z) \tag{6}$$

and

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} 2^j \psi\left(\frac{1}{2^{j+1}}x, \frac{1}{2^{j+1}}y, -\frac{1}{2^j}z\right) < \infty$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \widetilde{\psi}(x, x, x)$$

for all $x \in \mathcal{X}$. In fact, the additive mapping $A : \mathcal{X} \to \mathcal{Y}$ is given by

$$A(x) := 2^n f\left(\frac{1}{2^n}x\right)$$

for all $x \in \mathcal{X}$.

Proof. Replacing x, y, z by $\frac{1}{2^{n+1}}x, \frac{1}{2^{n+1}}x, -\frac{1}{2^n}x$, respectively, and multiplying by 2^n in (6), since f is odd, we gain

$$\left\|2^{n+1} f\left(\frac{1}{2^{n+1}}x\right) - 2^n f\left(\frac{1}{2^n}x\right)\right\| \leq 2^n \psi\left(\frac{1}{2^{n+1}}x, \frac{1}{2^{n+1}}x, -\frac{1}{2^n}x\right)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n.

The rest of the proof is similar to the corresponding part of the proof of Theorem 2. □

Corollary 5. *Let $\theta \in [0,\infty)$ and $p \in (1,\infty)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y) + f(z)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \frac{2^p + 2}{2^p - 2}\theta\|x\|^p$$

for all $x \in \mathcal{X}$.

Proof. In Theorem 4, taking

$$\psi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p),$$

we have

$$\widetilde{\psi}(x, y, z) = \theta\frac{\|x\|^p + \|y\|^p + 2^p\|z\|^p}{2^p - 2} < \infty$$

for all $x, y, z \in \mathcal{X}$. And we obtain the desired result. □

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