LOWER AGAINST NUMBER IN GRAPHS

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Abstract: A negative function of a graph $G = (V, E)$ is a function $f : V \to \{-1, +1\}$ such that for every vertex $v$, the sum of the values of $f$ over the closed neighborhood of $v$ is at most 1. A negative function $f$ is maximal if there does not exist a negative function $g, f \neq g$, for which $g(v) \geq f(v)$ for every $v \in V$. The weight of a negative function is $w(f) = \sum_{v \in V(G)} f(v)$. The lower against number $\beta_N^*(G)$ of $G$ is the minimum weight of a maximal negative function on $G$. In this paper we establish a sharp lower bound on $\beta_N^*(G)$ for general graphs. Our result generalizes previous results for regular graphs and nearly regular graphs with minimum degree being even.

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1. Introduction

Let $G = (V, E)$ be a finite, simple and undirected graph with vertex set $V$ and edge set $E$. Terminology and notation not defined here will in general conform to that in [1]. The open neighborhood of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N_G(v)$. The degree of $v$ in $G$, denoted by $d_G(v)$, is defined as the cardinality of $N_G(v)$, and the minimum degree and maximum degree of $G$ is denoted by $\delta(G)$ and $\Delta(G)$, respectively. If the graph $G$ is clear from context, we will omit the subscript $G$. If $d(v) = k$ for all $v \in V$, then we call $G$ a $k$-regular graph. If $d(v) = k - 1$ or $k$ for all $v \in V$, then $G$ is said to be a nearly $k$-regular graph. For $S \subseteq V$, we
define $d_S(v) = |S \cap N(v)|$. The subgraph of $G$ induced by $S$ is denoted by $G[S]$. For two disjoint subsets $X, Y \subseteq V(G)$, we use $e(X, Y)$ to denote the number of edges between $X$ and $Y$.

Let $f : V \to \{-1, +1\}$ be a function which assigns to each vertex of $G$ an element of the set $\{-1, +1\}$. The weight of $f$ is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we denote $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. For a vertex $v \in V$, we will replace $f(N[v])$ by $f[v]$ for notational convenience. The function $f$ is defined in [2] as a signed dominating function (SDF) of $G$ if $f[v] \geq 1$ for every $v \in V$. The signed domination number of $G$, denoted by $\gamma_s(G)$, is the minimum weight of a SDF of $G$. This parameter has been investigated in [2,3,4,5] and surveyed in [6,8].

In [11], the function $f$ is called a negative function (NF) of $G$ if $f[v] \leq 1$ for every $v \in V$. The maximum weight of a NF on $G$ is called against number of $G$, denoted by $\beta_N(G)$. A negative function $f$ of $G$ is maximal if there does not exist a NF $g : V \to \{-1, +1\}$, $f \neq g$, for which $g(v) \geq f(v)$ for every vertex $v \in V$. The lower against number, denoted $\beta_N^*(G)$, is the minimum weight of a maximal NF of $G$. Noting that the against number is called signed 2-independence number in [7,9], and is a certain dual to the signed domination number of a graph. For the applications of negative number in a social network, the reader is referred to [11]. Wang [10] define the negative decision number by simply changing “$f[v] \leq 1$” in the definition of negative function to “$f(N(v)) \leq 1$”. In [12], negative decision number is also called signed total 2-independence number.


**Theorem 1.** (Wang [11]) If $G = (V, E)$ is an $r$-regular graph of order $n$, then

$$
\beta_N^*(G) \geq \begin{cases} 
\left( \frac{r+2-r^2}{r+2+r^2} \right) n & \text{for } r \text{ even}, \\
\left( \frac{1-r}{1+r} \right) n & \text{for } r \text{ odd}.
\end{cases}
$$

This bound is best possible.

**Theorem 2.** (Wang [11]) If $G$ is a nearly $r$-regular graph of order $n$, then

$$
\beta_N^*(G) \geq \left( \frac{1-r}{1+r} \right) n.
$$

For $r \geq 3$ even, the equality holds if and only if $G \in \mathcal{F}_0'(r)$. For $r \geq 3$ odd, the equality holds if and only if $G \in \mathcal{F}_1'(r)$. 
Furthermore, Wang [11] posed an open problem: What is a sharp lower bound on $\beta^*_N(G)$ for a general graph $G$?

In this paper, we establish a sharp lower bounds on $\beta^*_N(G)$ for a general graph $G$ in terms of minimum degree, maximum degree and order.

2. Lower Against Number in General Graphs

In this section, we present a sharp lower bound on $\beta^*_N$ for a general graph in terms of its minimum degree, maximum degree and order. To this aim, we shall need the following lemma due to Wang [11].

**Lemma 3.** (Wang [11]) A negative function of a graph $G = (V, E)$ is maximal if and only if for every vertex $v \in V$ with $f(v) = -1$, there exists a vertex $u \in N[v]$ such that $f[u] = 0$ or $1$.

**Theorem 4.** If $G$ is a graph of order $n$ with minimum degree $\delta$ and maximum degree $\Delta$, then

$$
\beta^*_N(G) \geq \begin{cases} 
\frac{(\delta + 2 + \delta \Delta - 2\Delta^2)}{(\delta + 2 - \delta \Delta + 2\Delta^2)} n & \text{for } \delta \text{ even}, \\
\frac{(\delta + 1 - \Delta + \delta \Delta - 2\Delta^2)}{(\delta + 1 + \Delta - \delta \Delta + 2\Delta^2)} n & \text{for } \delta \text{ odd}.
\end{cases}
$$

Furthermore, these bounds are sharp.

**Proof.** Let $f$ be a maximal negative function on $G$ satisfying $\beta^*_N(G) = f(V)$, and let $P$ and $M$ denote the sets of those vertices in $G$ that are assigned under $f$ the value $+1$ and $-1$, respectively. For notational convenience, we set $\lfloor \delta/2 \rfloor = k$ and $\lfloor (\Delta + 2)/2 \rfloor = l$.

By Lemma 3, we can claim that $P \neq \emptyset$. Let $|P| = p$ and $|M| = m$. Thus, $w(f) = |P| - |M| = 2p - n$.

For any vertex $v \in M$, $f[v] = 2d_P(v) - d(v) - 1$, and so $d_P(v) \leq \lfloor (d(v) + 2)/2 \rfloor$ as $f[v] \leq 1$. Hence we can partition $M$ into $l + 1$ sets by defining $M_i = \{v \in M \mid d_P(v) = i\}$ and letting $|M_i| = m_i$ for $i = 0, 1, \ldots, l$. Then we have

$$
n = m + p = m + \sum_{i=0}^{l} m_i.
$$

(1)

Since each vertex in $P$ is adjacent to at most $\Delta$ vertices of $M$, we have

$$
\sum_{i=1}^{l} i m_i = e(M, P) \leq \Delta p.
$$

(2)
Suppose that \( M_0 = \emptyset \). If \( \Delta = \delta = r \), then desired result follows by Theorem 1. Hence, \( \Delta \geq \delta + 1 \). Then, by (1) and (2), we have

\[
    n = p + \sum_{i=1}^{l} m_i \leq p + \sum_{i=1}^{l} im_i \leq (\Delta + 1)p.
\]

This implies that \( p \geq n/(\Delta + 1) \), and so \( \beta_N^*(G) = 2p - n \geq (1 - \Delta)n/(1 + \Delta) \).

Observing that

\[
    \left( \frac{1 - \Delta}{1 + \Delta} \right) n \geq \max \left\{ \left( \frac{\delta + 2 + \delta \Delta - 2\Delta^2}{\delta + 2 - \delta \Delta + 2\Delta^2} \right) n, \left( \frac{\delta + 1 - \Delta + \delta \Delta - 2\Delta^2}{\delta + 1 + \Delta - \delta \Delta + 2\Delta^2} \right) n \right\},
\]

and so the Theorem 4 holds. Hence in what follows we may assume that \( M_0 \neq \emptyset \).

According to our partition for \( M_1 \), we have \( f[v] \leq -1 \) for every \( v \in (\bigcup_{i=0}^{k} M_i) \).

For any vertex \( v \in M_0 \), since \( f \) is maximal, by Lemma 3, \( v \) has at least one neighbor \( u \notin M_0 \) such that \( f[u] \in \{0, 1\} \). Let \( W = \{u \in N(M_0) \mid f[u] = 0 \text{ or } 1\} \). Then \( W \subseteq \bigcup_{i=k+1}^{l} M_i \). So

\[
m_0 = |M_0| \leq e(M_0, W) \leq e(M_0, \bigcup_{i=k+1}^{l} M_i) \leq \sum_{i=k+1}^{l} (\Delta - i)m_i \leq (\Delta - k - 1) \sum_{i=k+1}^{l} m_i. \tag{3}
\]

We now distinguish two cases depending on the parity of \( \delta \).

**Case 1.** \( \delta \) is even. If \( \Delta = \delta \), then the assertion is trivial by Theorem 1. Thus we have \( \Delta \geq \delta + 1 \). Noting that \( k = \delta/2 \). Then, by Eqs. (1), (2) and (3), we obtain

\[
n \leq p + (\Delta - k - 1) \sum_{i=k+1}^{l} m_i + \sum_{i=1}^{l} m_i
\]

\[
= p + \sum_{i=1}^{k} m_i + (\Delta - k) \sum_{i=k+1}^{l} m_i
\]

\[
\leq p + \sum_{i=1}^{k} m_i + \frac{\Delta - k}{k + 1} \sum_{i=k+1}^{l} im_i
\]
\[
\begin{align*}
&\leq p + \frac{2\Delta - \delta}{\delta + 2} \sum_{i=1}^{l} im_i \\
&\leq p + \frac{2\Delta - \delta}{\delta + 2} \Delta p
\end{align*}
\]

Thus, we have \( p \geq (\delta + 2)n/(\delta + 2 - \delta\Delta + 2\Delta^2) \). Consequently,

\[
\beta_N^*(G) = 2p - n \geq \left(\frac{\delta + 2 + \delta\Delta - 2\Delta^2}{\delta + 2 - \delta\Delta + 2\Delta^2}\right) n.
\]

**Case 2.** \( \delta \) is odd. Then \( k = (\delta - 1)/2 \). If \( \delta = 1 \), then by Eqs. (1), (2) and (3) again, we have

\[
\begin{align*}
n &\leq p + (\Delta - 1) \sum_{i=1}^{l} m_i + \sum_{i=1}^{l} m_i \\
&\leq p + \Delta \sum_{i=1}^{l} im_i \\
&\leq p + \Delta^2 p,
\end{align*}
\]

which means that \( p \geq n/(1 + \Delta^2) \). So

\[
\beta_N^*(G) = 2p - n \geq \left(\frac{1 - \Delta^2}{1 + \Delta^2}\right) n = \left(\frac{\delta + 1 - \Delta + \delta\Delta - 2\Delta^2}{\delta + 1 + \Delta - \delta\Delta + 2\Delta^2}\right) n.
\]

Hence we may assume that \( \delta > 1 \). According to Eqs. (1), (2) and (3), it follows that

\[
\begin{align*}
n &\leq p + (\Delta - k - 1) \sum_{i=k+1}^{l} m_i + \sum_{i=1}^{l} m_i \\
&= p + \sum_{i=1}^{k} m_i + (\Delta - k) \sum_{i=k+1}^{l} m_i \\
&\leq p + \sum_{i=1}^{k} m_i + \frac{\Delta - k}{k + 1} \sum_{i=k+1}^{l} im_i \\
&\leq p + \frac{2\Delta - \delta + 1}{\delta + 1} \sum_{i=1}^{l} im_i
\end{align*}
\]
\[ p + \frac{2\Delta - \delta + 1}{\delta + 1}\Delta p, \]

which implies that

\[ p \geq \frac{(\delta + 1)n}{(\delta + 1 + \Delta - \delta \Delta + 2\Delta^2)}. \]

Thus,

\[ \beta^*_N(G) = 2p - n \geq \left( \frac{\delta + 1 - \Delta + \delta \Delta - 2\Delta^2}{\delta + 1 + \Delta - \delta \Delta + 2\Delta^2} \right)n. \]

The bounds in Theorem 4 are sharp in the case when \( \delta = \Delta \), and the extremal graphs attaining the bounds are characterized in [11].

Obviously, if \( \delta = \Delta = r \), then

\[ \frac{(\delta + 2 + \delta \Delta - 2\Delta^2)n}{(\delta + 2 - \delta \Delta + 2\Delta^2)} = \frac{(r + 2 - r^2)n}{(r + 2 + r^2)} \]

for \( r \) being even, and

\[ \frac{(\delta + 1 - \Delta + \delta \Delta - 2\Delta^2)n}{(\delta + 1 + \Delta - \delta \Delta + 2\Delta^2)} = \frac{(1 - r)n}{(1 + r)} \]

for \( r \) being odd. If \( \delta = r - 1 \) and \( \Delta = r \), then

\[ \frac{(\delta + 2 + \delta \Delta - 2\Delta^2)n}{(\delta + 2 - \delta \Delta + 2\Delta^2)} = \frac{(1 - r)n}{(1 + r)} \]

for \( r \) being odd. Hence, Theorem 4 generalizes Theorem 1 and Theorem 2 for the case that minimum degree being even.

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References


