International Journal of Pure and Applied Mathematics
Volume 77 No. 2 2012, 255-268
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

OPTIMAL PARAMETERS FOR KLEIN-GORDON EQUATION WITH NEUMANN BOUNDARY CONDITION

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Abstract: In this paper we study an identification problem for physical parameters associated with damped Klein-Gordon equation with Neumann boundary conditions. The existence, uniqueness, and continuous dependence of weak solution of Klein-Gordon equations are established. The method of transposition is used to prove the Gâteaux differentiability of the solution map. The Gâteaux differential of the solution map is characterized. The optimal parameters are established.

AMS Subject Classification: 35B30, 49J50
Key Words: identification problem, Neumann boundary conditions, Gâteaux differentiability, optimal parameters

1. Introduction

Let Ω be an open bounded set of R^n with C^1 boundary. Let us consider the following Klein-Gordon equation with Neumann boundary condition.

$$\begin{aligned}
&u_{tt}(t,x) + \alpha u_t(t,x) - \beta \triangle u(t,x) + \delta |u(x,t)|^\gamma = f(x,t);\ (t,x) \in Q \\
&\frac{\partial u}{\partial n}(t,x)|_{x\in\Gamma} = 0,\ t \in\ (0,T) \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega
\end{aligned} \tag{1}$$

where $T > 0$, $Q = (0,T) \times \Omega$, $f \in L^2(Q)$, $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$.

Received: March 13, 2012

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Similarly, α, β, γ, and δ are constants.

Equation (1) is known as one of the nonlinear wave equations frequently appears in relativistic quantum mechanics. For Klein-Gordon model with Dirichlet boundary condition, Ha and Nakagiri estimated the governing parameters. For detail, see [4]. In this paper we estimate the governing parameters of the Klein-Gordon equation for Neumann boundary condition such that the solution of (1) exhibits the desired behavior. More precisely, let

$$\begin{aligned} \mathcal{P} = \{q = (\alpha, \beta, \delta) \in [\alpha_{min}, \alpha_{max}] \\ \times[\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]\}, \end{aligned}$$

where $\beta_{min} > 0$. Define the cost functional $J(q)$ by

$$J(q) = k_1|u(q;T) - z_d^1|^2 + k_2\|u(q;t) - z_d^2\|^2_{L^2(0,T;H)} \tag{2}$$

where $z_d^1 \in H$, $z_d^2 \in L^2(0,T;H)$ and $k_i \geq 0$ for $i = 1,2$ with $k_1 + k_2 > 0$. The data z_d^1 and z_d^2 can be thought of as the targeted behavior of (1). The parameter identification problem for (1) with the objective function (2) is to find $q^* = (\alpha^*, \beta^*, \delta^*) \in \mathcal{P}_{ad}$ satisfying

$$J(q^*) = \inf_{q \in \mathcal{P}_{ad}} J(q). \tag{3}$$

Let $q \to u(q)$ from P into $C([0,T];H)$ be the solution map. The existence, uniqueness, and continuity of solution with respect to data is established in Section 2. Thus the identification problem (3) restricted to a compact subset of P has a solution. Our main focus in this paper is a variational characterization of minimizers q^* of (3). The minimizer is characterized by the inequality.

$$DJ(q^*; q - q^*) = 2k_1((u(q^*;T) - z_d^1), z_1) + 2k_2 \int_0^T (u(q^*;t) - z_d^2), z)dt \geq 0 \tag{4}$$

where $DJ(q^*; q - q^*)$ denotes the Gâteaux derivative of cost functional $J(q)$ at $q = q^*$ in the direction of $q - q^*$. For details, see definition 10. The existence of $DJ(q^*; q - q^*)$ is shown by using the method of transposition. For details, see [8]. Optimal set of parameters is shown in Section 4.

2. Existence, Uniqueness, and Continuity of Weak Solution

Let $H = L^2(\Omega)$ be a Hilbert space with following inner product and norm

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x)dx, \qquad |\phi| = (\phi, \phi)^{\frac{1}{2}} \tag{5}$$

for all ϕ , $\psi \in L^2(\Omega)$. Let $V = H^1(\Omega)$ be a Hilbert space with following inner product and norm

$$((\phi, \psi)) = (\phi, \psi) + (\nabla\phi, \nabla\psi), \qquad \|\phi\| = ((\phi, \phi))^{\frac{1}{2}} \tag{6}$$

for all ϕ , $\psi \in H^1(\Omega)$. The dual H' is identified with H leading to $V \subset H \subset V'$ with compact, continuous, and dense injections [9]. Hence there exists a constant $K_1 = K_1(\Omega)$ such that

$$|w| \leq K_1\|w\| \quad \text{for any} \quad w \in V. \tag{7}$$

Let $< u, v >_{V, V'}$ denote the duality pairing between V and V'. To use the variational formulation let us define the following bilinear form on $V \times V$

$$a_\beta(u, v) = \int_\Omega u\ v dx + \beta \int_\Omega \nabla u \nabla v dx \tag{8}$$

for any $u, v \in H^1(\Omega)$ and diffusion coefficient β. For $\beta > 0$, $a_\beta(u, v)$ is bounded and coercive in V. Define a linear operator $A_\beta : D(A_\beta) = \{u : u \in V, A_\beta u \in H\}$ into H by $a_\beta(u, v) = (A_\beta u, v)$ for all $u \in D(A_\beta)$ and for all $v \in V$. Let the norm on $D(A_\beta)$ be $\|u\|_\beta^2 = \int_\Omega |u|^2 dx + \beta \int_\Omega |\nabla u|^2 dx$ then A_β is an isomorphism between $D(A_\beta)$ and H. The operator $A_\beta :\ D(A_\beta) \subset H$ into H is a self-adjoint and bounded so A_β^{-1} exists. In addition, A_β^{-1} is bounded,self-adjoint compact operator. Let $\{\lambda_k\}_{k=1}^{\infty}\}$ and $\{w_k\}_{k=1}^{\infty}\}$ respectively are the nonzero eigenvalues and eigenfunctions for the operator $-\Delta + I$ defined in V such that $\{w_K\}_{k=1}^{\infty}$ forms an orthonormal basis in H. Then functions $\{\frac{w_k}{\sqrt{\mu_k}}\}_{K=1}^{\infty}$ form an orthonormal basis in V. We recall the following Sobolev embeddings.

$$\begin{cases} H_0^1(\Omega) \hookrightarrow L^q(\Omega), \forall q < \infty \ if\ n = 1, 2, q = 6\ if\ n = 3 \\ H^1(\Omega) \hookrightarrow C(\bar{\Omega}), if\ n = 1, \\ H^2(\Omega) \hookrightarrow C(\bar{\Omega}), if\ n = 2, 3 \end{cases}$$

To handle nonlinear term in (1), we suppose a nonlinear real valued function $h(u) = u|u|^\gamma$ and γ is assumed to satisfy

$$\begin{cases} 0 \leq \gamma < \infty, & if\ n = 1, 2 \\ 0 \leq \gamma \leq 2, & if\ n = 3, \\ 0 & if\ n \geq 4 \end{cases} \tag{9}$$

then we have $h \in C^1(R)$ and $h'(u) = (\gamma + 1)|u|^\gamma$. From now on the dependency on x is suppressed, and $'$ and $''$ stand for the time derivatives. Let

$$W(0, T) = \{u\ :\ u \in L^2(0, T; V), u' \in L^2(0, T; H),$$

$$u'' \in L^2(0,T;V')\}. \tag{10}$$

The weak solution of (1) is a function $u \in W(0,T)$ satisfying

$$\begin{aligned} &\langle u'', w_j\rangle + \alpha(u', w_j) + a_\beta(u, w_j) + \delta(h(u), w_j) = (f, w_j) + (u, w_j),\ \forall j \in \mathbb{N},\\ &u(0) = u_0 \in V, \quad u'(0) = u_1 \in H, \end{aligned} \tag{11}$$

where the equations in t are satisfied in the distributional sense. Since the span $\{w_1, w_2, w_3, ...\}$ is dense in V, (11) is satisfied for any $v \in V$

$$\begin{aligned} &\langle u'' + \alpha u' + A_\beta u + \delta h(u), v\rangle = \langle f + u, v\rangle,\\ &u(0) = u_0 \in V, \quad u'(0) = u_1 \in H. \end{aligned} \tag{12}$$

Thus

$$\begin{aligned} &u'' + \alpha u' + A_\beta u + \delta h(u) = f + u,\\ &u(0) = u_0 \in V, \quad u'(0) = u_1 \in H \end{aligned} \tag{13}$$

which is understood in the sense of distributions on $(0,T)$ with the values in V'. For more details see [5]. The following lemma is of critical importance.

Lemma 1. *(Gronwall's Lemma) Let $\xi(t)$ be a nonnegative, summable function on [0,T] which satisfies the integral inequality*

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2 \ \ for\ constants\ C_1\ , C_2 \geq 0 \tag{14}$$

almost everywhere $t \in$ [0,T]. Then

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t})\ a.e.\ on\ 0 \leq t \leq T. \tag{15}$$

In particular, if

$$\begin{aligned} &\xi(t) \leq C_1 \int_0^t \xi(s)ds\ \ a.e.\ on\ 0 \leq t \leq T,\ then\\ &\xi(t)\ =\ 0\ \ a.e.\ on\ [0,\ T] \end{aligned} \tag{16}$$

For proof see [12].

Theorem 2. *Let $q = (\alpha, \beta, \delta) \in \mathcal{P}$, $u_0 \in V$, $u_1 \in H$, γ described be as in (9) and $f \in L^2(0,T;H)$. Then*
(i). There exists a unique weak solution $u(t;q)$ of (13). This solution satisfies $u \in C([0,T];V) \cap W(0,T)$, $u' \in C([0,T];H)$, and if $u_0 \in D(A_\beta), u_1 \in V$ and

$f \in L^2(0,T;H)$ then $u \in C([0,T];D(A_\beta), u' \in C([0,T];V)$. In addition, for all $t \in [0,T]$ there exist a constant $C(q,\gamma)$ such that

$$\|u(t)\|^2 + |u'(t)|^2) \leq C(\|u_0\|^2 + |u_1|^2 + \|u_0\|^{\gamma+2}_{L^{\gamma+2}}(\Omega) \\ + \int_0^t |u'(s)|^2 ds + \|f\|^2_{L^2(0,T;H)}) \tag{17}$$

(ii). The solution $u(t,x)$ is bounded. That is there exists a constant $M(q,\Omega) > 0$ such that

$$\max_{(x,t)\in Q} |u(x,t)| \leq M \tag{18}$$

For proof see [5]

Now we show continuity of solution with respect to parameters. Let us consider the following theorem

Theorem 3. *The map $q \to u(q)$ from $\mathcal{P} \to W(0,T)$ is weakly continuous.*

Proof. Let $q_m = (\alpha_m, \beta_m, \delta_m) \to q = (\alpha, \beta, \delta)$ in R^3. We show $u(q_m) \to u(q)$ weakly in $W(0,T)$. Let $u_m(q_m)$ be weak solution of (1). Thus we have

$$u''_m + \alpha_m u'_m + \beta_m A u_m + \delta_m h(u_m) = f(t) \\ u_m(0) = u_0, \;\; u'_m(0) = u_1 \tag{19}$$

Using (20) and lemma 1 we have

$$\max_{0\leq t\leq T}(\|u(t)\|^2 + |u'(t)|^2) \leq C(\|u_0\|^2 \\ + |u_1|^2 + \|u_0\|^{\gamma+2}_{L^{\gamma+2}}(\Omega) + \|f\|^2_{L^2(0,T;H)}) \tag{20}$$

for all $t \in [0,T]$. Thus the sequence $\{u_m\}_{m=1}^\infty$ is bounded in $L^2(0,T;V)$, $\{u'_m\}_{m=1}^\infty$ is bounded in $L^2(0,T;H)$, and $\{u''_m\}_{m=1}^\infty$ is bounded in $L^2(0,T;V')$, where V' is the dual space of V. Since $L^2(0,T;V)$, $L^2(0,T;H)$, and $L^2(0,T;V')$ are reflexive spaces, there exist a subsequence $\{u_{m_k}\}_{k=1}^\infty \subset \{u_m\}_{k=1}^\infty$ and $z \in L^2(0,T;V)$, $d^1 \in L^2(0,T;H)$, $d^2 \in L^2(0,T;V')$ such that

$$\begin{aligned} u_{m_k} &\rightharpoonup z, \quad \text{in} \quad L^2(0,T;V), \\ u'_{m_k} &\rightharpoonup d^1, \quad \text{in} \quad L^2(0,T;H), \\ u''_{m_k} &\rightharpoonup d^2, \quad \text{in} \quad L^2(0,T;V'), \end{aligned} \tag{21}$$

where $\rightharpoonup$ indicates the weak convergence. Since the convergence in $W(0,T)$ is the distributional convergence, we have

$$u'_{m_k} \rightharpoonup z', \quad \text{in} \quad L^2(0,T;H),$$

$$u''_{m_k} \rightharpoonup z'' \quad \text{in} \quad L^2(0,T;V') \quad \text{as} \quad k \to \infty. \tag{22}$$

But the weak limit is unique when it exists. So $d^1 = z'$ and $d^2 = z''$. Energy estimate (20) also implies that $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^{\infty}(0,T;V)$ and the sequence $\{u'_m\}_{m=1}^{\infty}$ is bounded in $L^{\infty}(0,T;H)$. By the Alaoglu Theorem, [16] we can find subsequences $\{u_{m_k}\}_{m=1}^{\infty}$ and $\{u'_{m_k}\}_{m=1}^{\infty}$ of $\{u_m\}_{m=1}^{\infty}$ and $\{u'_m\}_{m=1}^{\infty}$ respectively such that

$$\begin{aligned} &u_{m_k} \rightharpoonup z \quad \text{weak star in} \quad L^{\infty}(0,T;V), \\ &u'_{m_k} \rightharpoonup z' \quad \text{weak star in} \quad L^{\infty}(0,T;H) \\ &z(0) = u_0 \;\; z'(0) = u_1 \end{aligned} \tag{23}$$

Now it remains to show that

$$\int_0^T (h(u_m(t)) - h(z(t)), \phi(t))dt \to 0, \;\; \forall\, \phi \in L^2(0,T;V). \tag{24}$$

Since V is compactly embedded in H, then by the classical compactness theorem [5] $u_{m_k} \to z$ in $L^2(0,T;H)$. By mean value theorem we have,

$$\begin{aligned} &(h(u_m(t)) - h(z(t)), \phi(t)) = \int_{\Omega} G_m(t,x)\phi(t,x)(u_m(t,x) - z(t,x))dx, \\ &G_m(t,x) = (\gamma+1)\int_0^1 |\theta u_m(t,x) + (1-\theta)z(t,x)|^{\gamma} d\theta \end{aligned} \tag{25}$$

Integrating (25) over $[0,T]$ and using Holder's inequality we have,

$$\begin{aligned} &\|(|u_m| + |z|^{\gamma})\phi\|_{L^{\mathbf{2}}(0,T;H)} \le c(\|u_m\|_{C([0,T];V)} + \|z\|_{L^{\infty}((0,T;V)})^{\gamma} \\ &\|\phi\|_{L^{\mathbf{2}}(0,T;V)} \end{aligned} \tag{26}$$

Using (26) and (20) we get,

$$\int_0^T (h(u_m(t)) - h(z(t)), \phi(t))dt \to 0, \;\; \forall\, \phi \in L^2(0,T;V). \tag{27}$$

We multiply (3) by $\phi \in C([0,T];V)$ with $\phi(0) = \phi(T) = 0$ and integrate over $[0,T]$ to get

$$\begin{aligned} &\int_0^T (u'_m, \phi')dt + \int_0^T (\alpha_m u'_m + \beta_m A u_m + \delta_m h(u_m), \phi)dt \\ &= \int_0^T (f, \phi)dt \end{aligned} \tag{28}$$

Using (21), (22), (23), (27), and (28), we can pass the limit to have

$$\begin{aligned} &z'' + \alpha z' + \beta Az + \delta h(z) = f \;\; in \; (0,T) \\ &z(0) = u_0, \;\; z'(0) = u_1 \end{aligned} \tag{29}$$

By uniqueness of weak solution, we show $u(q_m)) \rightharpoonup u(q)$ in $W(0,T)$. □

The following theorem is immediate by Theorem 3 and lower semi-continuity of norms.

Theorem 4. *If $P_{ad} \subset P$ is compact, then there exists at least one optimal parameter $\in q^*$ for the cost functional* (3).

3. Weak Gâteaux Differentiability of the Solution Map

Let

$$\mathcal{H} = \left\{ G = \begin{pmatrix} \xi \\ g \end{pmatrix} : \; \xi \in H \quad \text{and} \quad g \in L^2(0,T;H) \right\}. \tag{30}$$

Then H is a Hilbert space with the following inner product and the norm

$$(G_1, G_2)_{\mathcal{H}} = (\xi_1, \xi_2)_H + (g_1, g_2)_{L^2(0,T;H)}, \quad \|G\|_{\mathcal{H}} = (G,G)_{\mathcal{H}}^{\frac{1}{2}}, \tag{31}$$

where $G_1 = \begin{pmatrix} \xi_1 \\ g_1 \end{pmatrix} \in \mathcal{H}$ and $G_2 = \begin{pmatrix} \xi_2 \\ g_2 \end{pmatrix} \in \mathcal{H}$.
To show the weak Gâteaux differentiability of $J(q)$ at $q^* \in \mathcal{P}$ we have to estimate the quotient

$$z_\lambda = \frac{u(q_\lambda) - u(q^*)}{\lambda}, \tag{32}$$

where $q_\lambda = q^* + \lambda(q - q^*), \quad \lambda \in (0,1]$. Generally it is desirable to estimate z_λ in the solution space $W(0,T)$. Since the second order evolution equations for z_λ have the forcing term containing a diffusion operator, it is not easy or impossible to solve equation (45) by standard variational manner as in [7]. Hence we will restrict ourselves to an estimate of $\begin{pmatrix} z_\lambda(T) \\ z_\lambda(t) \end{pmatrix} \in H \times L^2(0,T;H)$ as $\lambda \to 0$ based on the method of transposition presented in [8].
Now we show the Gâteaux differentiability of the solution map $q \to \begin{pmatrix} u(q;T) \\ u(q;t) \end{pmatrix}$ of $\mathcal{P}$ into $H \times L^2(0,T;H)$ via the method of transposition and characterize its

Gâteaux derivative.
Fix $q = (\alpha, \beta, \delta) \in \mathcal{P}$ and $h'(u) \in \mathbb{R}$. Let $G = \begin{pmatrix} \xi \\ g \end{pmatrix} \in \mathcal{H}$.
Let us consider the following linear terminal value problem

$$\begin{aligned} &\phi'' - \alpha\phi' + A_\beta\phi + (\delta h'(u) - 1)\phi = g \quad \text{in} \quad (0, T) \\ &\phi(T) = 0, \ \ \phi'(T) = \xi. \end{aligned} \tag{33}$$

Let $\phi(T - s, x) = w(s, x)$ for any $x \in (0, 1)$. Then we have $\phi_t(T - s, x) = -w_s(s, x)$ and $\phi_{tt}(T - s, x) = w_{ss}(s, x)$. Then (33) can be written as

$$\begin{aligned} &w'' + \alpha w' + A_\beta w + (\delta h'(u) - 1)w = g \quad \text{in} \quad (0, T) \\ &w(0) = 0, \ \ w'(0) = -\xi. \end{aligned} \tag{34}$$

Arguing as in Section 2, we can conclude that (34) has a unique weak solution. Hence (33) has a unique weak solution $\phi = \phi(\xi, g) \in W(0, T)$ that satisfies the energy estimate

$$|\phi'(t)|^2 + \|\phi(t)\|^2 \leq c(|\xi|^2 + \|g\|^2_{L^{\mathbf{2}}(0,T;H)}), \quad t \in [0, T]. \tag{35}$$

Definition 5. Solution map: Given $G \in \mathcal{H}$ define the solution map from $\mathcal{H}$ into $W(0, T)$ by $\tau(G) = \phi$, where ϕ is the weak solution of (33).

Definition 6. Fix $q = (\alpha, \beta, \delta) \in \mathcal{P}$ and $h'(u) \in \mathbb{R}$. Let the solution space $\mathcal{X}(q; h'(u)) = \tau(\mathcal{H})$ be defined by

$$\mathcal{X}(q, h'(u)) = \{\phi : \phi \text{ is solution of (33) for each G} \in \mathcal{H}\}.$$

Let the linear operator $\mathcal{L}(q; h)$ from $\mathcal{X}(q; h)$ into $\mathcal{H}$ be defined by

$$\mathcal{L}(q; h'(u))\phi = \begin{pmatrix} \phi'(T) \\ \phi'' - \alpha\phi' + A_\beta\phi + (\delta h'(u) - 1)\phi. \end{pmatrix} = \begin{pmatrix} \phi'(T) \\ g \end{pmatrix}. \tag{36}$$

Let the inner product $(.\ ,.)$ in $\mathcal{X}(q; h)$ be defined by

$$(\phi, \psi)_{\mathcal{X}(q;h'(u))} = \left(\mathcal{L}(q; h'(u))\phi, \mathcal{L}(q; h'(u))\psi\right)_{\mathcal{H}}. \tag{37}$$

In terms of the operator $\mathcal{L}(q; h'(u))$ the energy estimate (35) can be written as

$$|\phi'(t)|^2 + \|\phi(t)\|^2 \leq c(\|\mathcal{L}(q; h'(u))\phi\|^2_{\mathcal{H}}) = c\|\phi\|^2_{\mathcal{X}(q,h'(u))}. \tag{38}$$

Definition 7. Given $q \in \mathcal{P}$, $h \in L^2(0,T;H)$, and $f \in L^2(0,T;V')$, the element $\bar{z} = \begin{pmatrix} z_1 \\ z \end{pmatrix} \in \mathcal{H}$, $z_1 \in H$, $z \in L^2(0,T;H)$ is called a weakened solution of the problem

$$\begin{aligned} & z''(t) + \alpha z'(t) + A_\beta z(t) + (\delta h'(u) - 1)z(t) = f(t) \\ & z(0) = 0 \ , \ z'(0) = 0, \quad t \in (0,T), \end{aligned} \tag{39}$$

if

$$(\bar{z}, \mathcal{L}(q; h'(u))\phi)_{\mathcal{H}} = \int_0^T \langle f(t), \phi(t) \rangle dt \tag{40}$$

for any $\phi \in \mathcal{X}(q; h'(u))$. That is,

$$(z_1, \xi)_H + \int_0^T (z(t), g(t))dt = \int_0^T \langle f(t), \phi(t) \rangle dt \tag{41}$$

for all $\phi \in \mathcal{X}(q, h'(u))$.

Remark 8. If $f \in L^2(0,T;H)$ and $z(t)$ is the weak solution (in the sense of Section 2) of the problem (39), then the integration by parts shows that $\bar{z} = \begin{pmatrix} z'(T) \\ z(t) \end{pmatrix}$ also is its weakened solution.

Lemma 9. *If $f \in L^2(0,T;V')$, then there exists a unique weakened solution of the problem* (39).

Proof. By the method of transposition of Lions for details see [8], if F is a bounded linear functional on $\mathcal{X}(q; h'(u))$, then there exists a unique $\bar{\xi} \in \mathcal{H}$ such that

$$F(\phi) = (\bar{\xi}(t), \mathcal{L}(q; h'(u))(\phi)(t))_{\mathcal{H}} \quad \text{for} \quad \text{any} \quad \phi \in \mathcal{X}(q; h'(u)). \tag{42}$$

Let

$$F(\phi) = \int_0^T \langle f(t), \phi(t) \rangle dt, \ \ \phi \in \mathcal{X}(q, h'(u)).$$

Using the energy estimate (38) we get

$$\begin{aligned} |F(\phi)| &\leq \|f\|_{L^2(0,T:V')} \|\phi\|_{L^2(0,T;V)} = \|f\|_{L^2(0,T:V')} \sqrt{\int_0^T \|\phi(t)\|_V^2 dt} \\ &\leq \|f\|_{L^2(0,T:V')} \sqrt{c} \int_0^T \|\phi(t)\|^2_{\mathcal{X}(q,h'(u))} dt \end{aligned}$$

$$\leq \sqrt{cT}\|f\|_{L^2(0,T:V')}\ \|\phi(t)\|_{\mathcal{X}}(q,h'(u)) \tag{43}$$

and the result follows. $\square$

Definition 10. Let $q, q^* \in \mathcal{P}$. Let $q_\lambda = q^* + \lambda(q - q^*)$ for $\lambda \in (0,1]$. The solution map $q \to \bar{u}(q) = \begin{pmatrix} u'(T;q) \\ u(t;q) \end{pmatrix}$ of $\mathcal{P}$ into $\mathcal{H}$ is said to be weakly Gateaux differentiable at q^* in the direction $q - q^*$ if there exist $\bar{z} \in \mathcal{H}$ such that

$$\lim_{\lambda \to 0+} \frac{1}{\lambda}(\bar{u}(q_\lambda) - \bar{u}(q^*), \bar{v})_{\mathcal{H}} = (\bar{z}, \bar{v})_{\mathcal{H}} \tag{44}$$

for any $\bar{v} \in \mathcal{H}$.

Theorem 11. *Let $q = (\alpha, \beta, \delta), q^* = (\alpha^*, \beta^*, \delta^*) \in \mathcal{P}$. Then the weak Gâteaux derivative $\bar{z} \in \mathcal{H}$ at $q^* \in \mathcal{P}$ in the direction $q - q^*$ is the unique weakened solution of the problem*

$$\begin{aligned} &z''(t) + \alpha^* z'(t) + A_{\beta^*} z(t) + (\delta^* h'(u(t;q^*)) - 1)z(t) = f_0(t), \\ &z(0) = 0,\ z'(0) = 0,\ t \in (0,T), \end{aligned} \tag{45}$$

where $f_0(t) = (\alpha^* - \alpha)u'(t;q^*) + (A_{\beta^*} - A_\beta)u(t;q^*) + (\delta^* - \delta)h(u(t;q^*))$.

Remark 12. For $\mathcal{X}$ and $\mathcal{L}$ defined by (37) and (36) respectively with q^* and $h = h'(u(q^*))$ the solution $\bar{z} = \begin{pmatrix} z(T) \\ z(t) \end{pmatrix}$ satisfies

$$(\bar{z}(t), \mathcal{L}(q^*; h'(u(t;q^*))\phi(t))_{\mathcal{H}} = \int_0^T \langle f_0(t), \phi(t)\rangle dt \tag{46}$$

for any $\phi \in \mathcal{X}(q^*; h'((u(q^*))))$.

For proof see [9]

4. Optimal Parameters

From Theorem 11 the map $q \to \bar{u}(q)$ is weakly Gâteaux differentiable at $q = q^* \in \mathcal{P}$ in any direction of $q - q^*$, and its weak Gâteaux derivative $\bar{z}(t,x) = D\bar{u}(q^*; q - q^*)(t,x)$ can be described by (46).
Let us consider the functional

$$J(q) = k_1|u(q;T) - z_d^1|^2 + k_2\|u(q;t) - z_d^2\|^2_{L^2(0,T;H)} \tag{47}$$

where $z_d^1 \in H$, $z_d^2 \in L^2(0,T;H)$ and $k_i \geq 0$ for $i = 1, 2$ with $k_1 + k_2 > 0$.

Lemma 13. *$J(q)$ is Gâteaux differentiable, and its Gâteaux derivative is given by*

$$DJ(q^*; q - q^*) = 2k_1((u(q^*; T) - z_d^1), z_1) + 2k_2 \int_0^T (u(q^*; t) - z_d^2), z)dt, \quad (48)$$

where $\bar{z}$ is the solution of integral equation (46).

For proof see [9].
Since $\mathcal{P} = \{q = (\alpha, \beta, \delta) \in [\alpha_{min}, \alpha_{max}] \times [\beta_{min}, \beta_{max}] \times [\delta_{min}, \delta_{max}]\}$ is a closed and convex subset of $\mathbb{R}^3$, then we have the following optimality condition

$$2k_1((u(q^*; T) - z_d^1), z_1) + 2k_2 \int_0^T (u(q^*; t) - z_d^2), z)dt \geq 0 \quad \text{for} \quad q \in \mathcal{P}, \quad (49)$$

where $\begin{pmatrix} z_1 \\ z \end{pmatrix}$ is a solution of the integral equation (46).

Let us introduce the adjoint state p defined to be the weak solution of the following adjoint system

$$\begin{aligned} & p'' - \alpha^* p' + A_\beta^* p + (\delta^*(h'(u(q^*)) - 1))p = k_2(u(q*; t) - z_d^2) \\ & p(T) = 0 \quad p'(T) = k_1(u(q^*; T) - z_d^1). \end{aligned} \quad (50)$$

System (50) can be written as

$$\begin{aligned} & \mathcal{L}(q^*; (h'(u(q^*))))p(q^*) = \begin{pmatrix} k_1 u(q^*; T) - z_d^1 \\ k_2 u(q^*; t) - z_d^2 \end{pmatrix} \in \mathcal{H} \\ & p(T) = 0, \quad p'(T) = k_1(u(q^*; T) - z_d^1). \end{aligned} \quad (51)$$

Since $k_2(u(q^*; t) - z_d^2) \in L^2(0, T; H)$, as shown in Section 2 problem in (50) has a unique weak solution. Using $p(q^*)$ in place of ϕ in (46) equation (48) can be written as

$$\begin{aligned} & DJ(q^*; q - q^*) = 2 \int_0^T \langle (\alpha^* - \alpha)u'(t; q^*) + (A_{\beta^*} - A_\beta)u(t; q^*) \\ & +(\delta^* - \delta)h(u(t; q^*)), p(q^*)\rangle dt. \end{aligned} \quad (52)$$

Thus we obtain the following result.

Theorem 14. *The Gâteaux derivative of the objective function $J(q)$ has the following representation*

$$DJ(q^*; q - q^*) = (\alpha^* - \alpha)a(q^*) + (\beta^* - \beta)b(q^*) + (\delta^* - \delta)c(q^*)dt, \quad (53)$$

where

$$a = -\frac{\partial J}{\partial \alpha} = -2\int_0^T (u_t(t,x;q^*), p(t,x;q^*))dt, \tag{54}$$

$$c = -\frac{\partial J}{\partial \delta} = -2\int_0^T (h(u(t,x;q^*)), p(t,x;q^*))dt, \tag{55}$$

and

$$b = -\frac{\partial J}{\partial \beta} = -2\int_0^T (\nabla u(t,x), \nabla p(t,x))dt, \tag{56}$$

The optimality condition $DJ(q^*; q - q^*) \geq 0$ for any $q \in \mathcal{P}$ is

$$(\alpha^* - \alpha)a(q^*) + (\beta^* - \beta)b(q^*) + (\delta^* - \delta)c(q^*) \geq 0 \tag{57}$$

for any $(\alpha, \beta, \delta) \in \mathcal{P}$.

In addition, the optimal coefficient $q^* \in \mathcal{P}$ for nonzero (a, b, c) can be compactly written as

$$\alpha^* = \frac{1}{2}\{\text{sign}(a) + 1\}\alpha_{max} - \frac{1}{2}\{\text{sign}(a) - 1\}\alpha_{min}, \tag{58}$$

$$\beta^* = \frac{1}{2}\{\text{sign}(b) + 1\}\beta_{max} - \frac{1}{2}\{\text{sign}(b) - 1\}\beta_{min}, \tag{59}$$

and

$$\delta^* = \frac{1}{2}\{\text{sign}(c) + 1\}\delta_{max} - \frac{1}{2}\{\text{sign}(c) - 1\}\delta_{min}. \tag{60}$$

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