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ON CONVERGENCE OF PARTIAL SUMS OF OPERATORS OF SZÁSZ–MIRAKYAN TYPE

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Abstract: In this paper, the convergence of partial sums of certain operators of Szász–Mirakyan type is examined. The present conditions of convergence differ from given for the Szász–Mirakyan operators.

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1. Introduction

M. Becker presented in [1] approximation properties of the Szász–Mirakyan operators

$$\begin{aligned} S_n(f;x) \equiv S_n(f(t);x) &= e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \qquad (1) \\ &:= \sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right), \end{aligned}$$

$x \in \mathbb{R}_0 = [0,\infty)$, $n \in \mathbb{N} = \{1,2,\ldots\}$, in polynomial weight space C_q, $q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, which is related with the function $w_q(x) = (1+x^q)^{-1}$ if $q \in \mathbb{N}$ and $w_0(x) \equiv 1$, for $x \in \mathbb{R}_0$, i.e. C_q is the set of all $f : \mathbb{R}_0 \longrightarrow \mathbb{R}$ such that $w_q f$ is uniformly continuous and bounded on $\mathbb{R}_0$ and the norm $\|f\|_q = \sup_{x\in\mathbb{R}_0} w_q(x)|f(x)|$.

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Applying similar method, we considered in [6] the strong approximation of $f \in C_q$ by certain positive linear operators. In particular, the presented results imply the following inequality:

$$w_q(x)S_n\left(|f(t)-f(x)|;x\right) = w_q(x)\sum_{k=0}^{\infty} p_k(nx)\left|f\left(\frac{k}{n}\right)-f(x)\right| \leq M_1(q)\omega\left(f;C_q;\sqrt{x/n}\right), \tag{2}$$

for $f \in C_q$, $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$, where $M_1(q) = \text{const.} > 0$ depending only on q, and $\omega(f;C_q)$ is the modulus of continuity of f, i.e.
$\omega(f;C_q;t) = \sup_{0\leq h\leq t} \|\Delta_h f(\cdot)\|_q$ for $t \in \mathbb{R}_0$, and $\Delta_h f(x) = f(x+h) - f(x)$.

For application and calculation, several authors (e.g. [2], [4], [5] and [8]) considered the partial sums of $S_n(f)$:

$$S_{n,N}(f;x) \equiv S_{n,N}\left(f(t);x\right) = \sum_{k=0}^{N} p_k(nx) f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0, n \in \mathbb{N}, \tag{3}$$

where $N = N(n,x)$ is a positive integer.

The results obtained in [4] and [5] say that if, for a fixed $x_0 \in \mathbb{R}_0$, is $N > nx_0$ and

$$\lim_{n\to\infty} \frac{N - nx_0}{\sqrt{n}} = \infty, \tag{4}$$

then

$$\lim_{n\to\infty} S_{n,N}(f;x_0) = f(x_0) \tag{5}$$

holds for every $f \in C_q$, $q \in \mathbb{N}_0$.

As is known ([3]), the Borel method B_r, $r \in \mathbb{N}$, of summability of series is connected with the expansion

$$A_r(t) := \sum_{k=0}^{\infty} \frac{t^{rk}}{(rk)!} \quad \text{for } t \in \mathbb{R}. \tag{6}$$

Obviously $A_1(t) = e^t$ and $A_2(t) = \cosh t$ for $t \in \mathbb{R}_0$. In [7] we proved that

$$A_{2m}(t) = \frac{1}{m}\left[\cosh t + \sum_{k=1}^{m-1} \exp\left(t\cos\frac{k\pi}{m}\right)\cos\left(t\sin\frac{k\pi}{m}\right)\right] \tag{7}$$

for $2 \leq m \in \mathbb{N}$, and

$$A_{2m+1}(t) = \frac{1}{2m+1}\left[e^t + \right. \tag{8}$$

$$+ 2\sum_{k=1}^{m} \exp\left(t\cos\frac{2k\pi}{2m+1}\right)\cos\left(t\sin\frac{2k\pi}{2m+1}\right)\Bigg]$$

for $m \in \mathbb{N}$ and $t \in \mathbb{R}_0$.

From the above we see that $A_3(t) = \frac{1}{3}\left[e^t + 2e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right]$ and $A_4(t) = \frac{1}{2}(\cosh t + \cos t)$ for $t \in \mathbb{R}_0$.

Applying the expansion (6), we examined in [7] approximation properties of the following operators of Szász-Mirakyan type:

$$L_{n,r}(f;x) \equiv L_{n,r}(f(t);x) = \tag{9}$$
$$= \frac{1}{A_r(nx)}\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!}f\left(\frac{rk}{n}\right) := \sum_{k=0}^{\infty} a_{k,r}(nx)f\left(\frac{rk}{n}\right)$$

for $f \in C_q$, $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. Clearly $L_{n,1}(f;x) \equiv S_n(f;x)$.

The purpose of this paper is to give conditions of convergence of partial sums of $L_{n,r}(f)$:

$$L_{n,r;N}(f;x) \equiv L_{n,r;N}(f(t);x) := \sum_{k=0}^{N} a_{k,r}(nx)f\left(\frac{rk}{n}\right), \tag{10}$$

for $f \in C_q$, $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and every fixed $r \in \mathbb{N}$, where $N = N(n,r,x)$ is a positive integer. In particular we have $L_{n,1;N}(f;x) \equiv S_{n,N}(f;x)$.

2. Theorems and Corollaries

By (9) and (10) it follows that for every fixed $r \in \mathbb{N}$:

$$\sum_{k=0}^{\infty} a_{k,r}(t) = 1 \quad \text{for } t \in \mathbb{R}_0, \tag{11}$$

and

$$L_{n,r}(f(t);0) = f(0) = L_{n,r;N}(f(t);0) \tag{12}$$

for $f \in C_q$, $n \in \mathbb{N}$ and $N = N(n,r,x)$.
By elementary calculation we can derive the following lemma from (6)–(8).

Lemma 1. *Let A_r, $r \in \mathbb{N}$, be defined by (6). Then*

$$\lim_{t\to\infty}\frac{A_r(t)}{e^t} = \frac{1}{r}$$

and

$$r^{-1} < e^{-t}A_r(t) \le 1 \quad \textit{for } t \in \mathbb{R}_0.$$

Consequently, for every fixed $r \in \mathbb{N}$ *we have*

$$1 \le \frac{e^t}{A_r(t)} < r \quad \textit{for } t \in \mathbb{R}_0. \tag{13}$$

First we shall estimate the approximation order of f by $L_{n,r}(f)$.

Theorem 1. *Let* $q \in \mathbb{N}_0$ *and* $r \in \mathbb{N}$ *be fixed. Then there exists* $M_2(q,r) = \text{const.} > 0$ *(depending only on* q *and* r*) such that*

$$\begin{aligned} w_q(x)\,|L_{n,r}(f;x) - f(x)| &\le w_q(x)L_{n,r}\left(|f(t)-f(x)|;x\right) \\ &\le M_2(q,r)\omega\left(f;C_q;\sqrt{x/n}\right) \end{aligned} \tag{14}$$

holds for every $f \in C_q$, $x \in \mathbb{R}_0$ *and* $n \in \mathbb{N}$.

Proof. The formulas (9), (11), (13) and (1) imply that

$$\begin{aligned} &|L_{n,r}(f(t);x) - f(x)| = |L_{n,r}(f(t)-f(x);x)| \\ &\quad \le L_{n,r}\left(|f(t)-f(x)|;x\right) \le re^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!}\left|f\left(\frac{rk}{n}\right) - f(x)\right| \\ &\quad \le rS_n\left(|f(t)-f(x)|;x\right), \end{aligned} \tag{15}$$

for $f \in C_q$, $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. Using now (2), we immediately obtain (14) from (15).

Corollary 1. *For every* $f \in C_q$ *and* $r \in \mathbb{N}$ *we have*

$$\lim_{n\to\infty} L_{n,r}\left(|f(t)-f(x)|;x\right) = 0 \quad \textit{at every } x \in \mathbb{R}_0,$$

which implies that

$$\lim_{n\to\infty} L_{n,r}(f;x) = f(x) \quad \textit{at every } x \in \mathbb{R}_0.$$

This convergence is uniform in every interval $[x_1, x_2]$, $0 \le x_1, x_2 < \infty$.

Now we shall prove the main theorem.

Theorem 2. *Let* $q \in \mathbb{N}_0$, $r \in \mathbb{N}$, $x \in \mathbb{R}_0$ *and* $n \in \mathbb{N}$ *be fixed and let* $N = N(n,r,x)$ *be an integer such that* $rN > nx$. *If* $f \in C_q$, *then*

$$\begin{aligned} &w_q(x)\,|L_{n,r;N}(f;x) - f(x)| \le \\ &\quad \le M_2(q,r)\omega\left(f;C_q;\sqrt{x/n}\right) + \|f\|_q \frac{1}{\sqrt{2\pi rN}}\frac{nx}{rN - nx + 1}, \end{aligned} \tag{16}$$

where $M_2(q,r)$ *is the same positive constant as in (14).*

Proof. By (12) we consider only $x > 0$. For given $f \in C_q$, $x > 0$, $n \in \mathbb{N}$, $r \in \mathbb{N}$ and $N = N(n,r,x)$ such that $rN > nx$ we can write by (10) and (11):

$$L_{n,r,N}(f(t);x) - f(x) = L_{n,r;N}(f(t) - f(x);x) - f(x) \sum_{k=N+1}^{\infty} a_{k,r}(nx) \tag{17}$$

and next, by (9) and (14),

$$w_q(x)\left|L_{n,r;N}(f(t) - f(x);x)\right| \le w_q(x) L_{n,r;N}\left(|f(t) - f(x)|;x\right) \le w_q(x) L_{n,r}\left(|f(t) - f(x)|;x\right) \le M_2(q,r)\omega\left(f; C_q; \sqrt{x/n}\right). \tag{18}$$

Moreover, by (11) and $rN > nx$ we deduce that

$$\sum_{k=N+1}^{\infty} a_{k,r}(nx) = \sum_{k=1}^{\infty} a_{N+k,r}(nx) = \frac{1}{A_r(nx)} \sum_{k=1}^{\infty} \frac{(nx)^{r(N+k)}}{(rN+rk)!}$$
$$= a_{N,r}(nx) \sum_{k=1}^{\infty} \frac{(nx)^{rk}}{(rN+1)(rN+2)\dots(rN+rk)}$$
$$\le a_{N,r}(nx) \sum_{k=1}^{\infty} \left(\frac{nx}{rN+1}\right)^{rk}$$
$$= a_{N,r}(nx) \frac{(nx)^r}{(rN+1)^r - (nx)^r} \le a_{N,r}(nx) \frac{nx}{r(rN - nx + 1)},$$

and, by (13),

$$a_{N,r}(t) = \frac{1}{A_r(t)} \frac{t^{rN}}{(rN)!} < \frac{r}{(rN)!} e^{-t} t^{rN} \quad \text{for } t > 0.$$

Since the function $g(t) = e^{-t}t^{rN}$ is increasing for $0 < t < rN$, so we can write

$$a_{N,r}(nx) < \frac{r}{(rN)!} e^{-rN} (rN)^{rN}$$

and next

$$a_{N,r}(nx) < \frac{r}{\sqrt{2\pi rN}},$$

by the Stirling formula. Consequently,

$$w_q(x) \left| f(x) \sum_{k=N+1}^{\infty} a_{k,r}(nx) \right| \le \|f\|_q \frac{1}{\sqrt{2\pi rN}} \frac{nx}{rN - nx + 1}. \tag{19}$$

Combining (17)–(19), we obtain the desired estimation (16).

Corollary 2. *Let $r \in \mathbb{N}$ and $x_0 > 0$ be fixed and let $N = N(n, r, x_0)$ satisfies the conditions:*

$$rN > nx_0 \quad \text{for } n \in \mathbb{N} \tag{i}$$

and

$$\lim_{n\to\infty} \frac{nx_0}{\sqrt{N}\,(rN - nx_0 + 1)} = 0. \tag{ii}$$

Then

$$\lim_{n\to\infty} L_{n,r;N}(f; x_0) = f(x_0) \tag{20}$$

holds for every $f \in C_q$, $q \in \mathbb{N}_0$. In particular, if (i) and

$$\lim_{n\to\infty} \frac{rN - nx_0}{\sqrt{n}} = \infty \tag{21}$$

hold for $x_0 > 0$ and $r \in \mathbb{N}$, then (20) holds for every $f \in C_q$ at the point x_0. Clearly, if $r = 1$, then the conditions (21) and (4) are identical and the statements (20) and (5) are identical also.

Corollary 3. *Let $x > 0$ and $r \in \mathbb{N}$ be fixed, and let $N = [nx + 1]$ for $n \in \mathbb{N}$, ($[y]$ denotes the integral part of $y \in \mathbb{R}$). Then $rnx < rN \le r(nx + 1)$ and the condition (21) is not satisfied for $r = 1$, but (21) holds for $2 \le r \in \mathbb{N}$ at this x. Moreover, for $N = [nx + 1]$ and $2 \le r \in \mathbb{N}$, we have by (16):*

$$w_q(x)\,|L_{n,r;N}(f; x) - f(x)| \le M_3 \left[\omega\left(f; C_q; \sqrt{x/n}\right) + \|f\|_q \frac{1}{\sqrt{nx}}\right] \tag{22}$$

for every $f \in C_q$, where $M_3 = M_3(q, r) = \text{const.} > 0$.

Corollary 4. *Let $x > 0$ and $r \in \mathbb{N}$ be fixed, and let $(b_n)_1^\infty$ be a sequence of the numbers $b_n > r$ such that $\lim_{n\to\infty} \frac{b_n}{\sqrt{n}} = \infty$. If $N = \left[\frac{nx+b_n}{r}\right]$, then the conditions (i) and (21) hold, and consequently (20) holds for every $f \in C_q$ at the point x.*

Remark. We can verify that the condition (4) at a point $x_0 > 0$ implies (21) for every $2 \le r \in \mathbb{N}$, but not conversely. The example is $N = [n_0 x + 1]$. Hence the conditions (i) and (21) with $2 \le r \in N$ and $x_0 > 0$ ensure the convergence (20) at the point x_0, but no (5).

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