FITTING PARETO DISTRIBUTION WITH HYPEREXPONENTIAL TO EVALUATE THE ARL FOR CUSUM CHART

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Abstract: Explicit formulas for the Average Run Length (ARL) of Cumulative Sum (CUSUM) chart are very complicated in regarding the analytical derivation when observations are Long-tailed distributions. The objective of this paper is to fitting Pareto distribution with the hyperexponential distribution to evaluate ARL of CUSUM procedure. The numerical results obtained from analytical solution for the ARL and from numerical approximations are derived and we compared the result with integral equations approach.

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Key Words: cumulative sum chart, average run length, hyperexponential distribution, stopping times

1. Introduction

The Cumulative Sum (CUSUM) chart is a simple but very effective graphical procedure for monitoring the quality in manufacturing industry. CUSUM chart
was first introduced by Page [12] to detected a change in parameter and widely implemented in statistical process control. Mazalov and Zhuravlev [9] implemented CUSUM chart to identified the changing point in a traffic network and Bakhodir [3] employed CUSUM in economic and finance to detected turning point in stock price indices.

The common characteristics of any control chart is the Average Run Lengths (ARL), defined as the expectation of an alarm time $\tau$ taken to trigger a signal about a possible change in parameters distribution. Ideally, an acceptable ARL of an in-control process should be enough large to detect a small change. In the paper we adopt the notation $\text{ARL}_0 = \mathbb{E}_\infty(\tau) = T$ where $\mathbb{E}_\infty(\tau)$ is the expectation corresponding to the target value and is given as large enough. The ARL when the process is out-of-control, is called the Average Delay time ($\text{ARL}_1$), defined as the expectation of delay for true alarm time. This time should minimize the quantity

$$\text{ARL}_1 = \mathbb{E}_\nu(\tau - \nu + 1|\tau \geq \nu)$$

where $\mathbb{E}_\nu(\cdot)$ is the expectation under the assumption that a change-point occurs in time.

In literature several methods for evaluating ARL$\text{ARL}_0$ and ARL$\text{ARL}_1$ for CUSUM procedure have been studied, such as Monte Carlo simulations, Integral Equation (IE) and Markov Chain Approach (MCA). For example, Monte Carlo simulation (MC) is a simple method used to checking the accuracy of analytical results. It is also useful for comparing the performance of different charts but it is very time consuming to run.

Recently, Areepong [1] proposed analytical derivation to find explicit formulas for ARL of EWMA chart when observations are exponential distributed. Mititelu et. al. [11] presented the analytically explicit formulas for determining the ARL of EWMA chart when observations have hyperexponential distributions by Fredholm type integral equations. The technique that we use to derive the formulas when observations are exponential and hyperexponential cannot be used to derive closed form solution for Pareto distribution.

It has been known that every monotone density distribution can be approximated to hyperexponential distributions according to Bernstein’s theorem [see for detail [18]], also called mixture of exponential distributions by different authors [see e.g. [5]].

Consequently, the explicit formulas for the ARL of non-negative CUSUM chart when observations are hyperexponential and fitting Pareto distribution by hyperexponential distribution is investigated.
There are many applications of long-tailed distributions, Kanita et al. [8] fit Weibull distribution to hyperexponential by Feldmann and Whitt (FW) algorithm and evaluate the Average Run Length of CUSUM chart that work well for detecting changes in parameters distribution. The real applications have been studied in network traffic and network intrusions [see e.g. [5, 13].

In this paper, we shall fit Pareto distribution to hyperexponential distributions with Feldmann and Whitt (FW) algorithm in Section 2. In addition, we use it to evaluate ARL of CUSUM chart as shown in Section 3.

2. Fitting Hyperexponential Distribution Using FW Algorithm

2.1. Hyperexponential Distribution

The hyperexponential distribution is characterized by the number of $k$ exponential phases and associate with mean $\alpha_i$ and probability $\lambda_i$ the $i^{th}$ negative exponential distribution (with rate $\alpha_i$ and mean $1/\alpha_i$) is chosen. The phase distribution function is

$$F(x) = 1 - \sum_{i=1}^{k} \lambda_i e^{-\alpha_i x}; x \geq 0.$$  

and the probability function is

$$f(x) = \sum_{i=1}^{k} \lambda_i \alpha_i e^{-\alpha_i x}; x \geq 0.$$  

2.2. Implementation of FW Algorithm

The FW algorithm, it concerns with the long-tailed distribution. We know that any completely monotone distributions can be approximated by hyperexponential distribution [see e.g. [4, 5]]. For example, the Pareto distribution is also completely monotone and can be approximated to hyperexponential distribution.

Let $F$ be an arbitrary distribution, we said that and a probability function $f$ is completely monotone if all derivatives of $f$ exist, and $(-1)^n F^n(x) \geq 0$ for all $x > 0$ and $n \geq 1$ [see [18]]. Description of the FM algorithm:

Step 1 Choose number $k$ components and $k$ argument to match $0 < c_k < c_{k-1} < c_1$, assuming the ratio $c_i/c_{i+1}$ is sufficiently large, e.g. $c_i/c_{i+1} \approx 10$ and choose $b$ such that for all $i$ we have $1 < b < c_i/c_{i+1}$. 
**Step 2** In \( k \) steps, the parameter for the phase in hyperexponential is computed. We start with \( k = 1 \) and choose \( \lambda_1 \) and \( p_1 \) to match \( F^c(x) = 1 - F(x) \) the complementary cumulative distribution (ccdf) at arguments and \( c_1 \) and \( bc_1 \).

We assume that \( c_1 \) and \( b \) are known. So, \( F^c(c_1) = p_1 e^{\lambda_1 c_1} \) and \( F^c(bc_1) = p_1 e^{\lambda_1 bc_1} \) and we can find the explicit values for \( \lambda_1 \) and \( p_1 \) as

\[
\lambda_1 = \frac{1}{(b - 1)c_1} \ln \left( \frac{F^c(c_1)}{F^c(bc_1)} \right) \quad \text{and} \quad p_1 = F^c(c_1) e^{-\lambda_1 c_1}.
\] (3)

**Step 3** Now, for any \( 2 \leq i \leq k \), the explicit values for \( \lambda_i \) and \( p_k \) are

\[
\lambda_i = \frac{1}{(b - 1)c_i} \ln \left( \frac{F^c(c_i)}{F^c(bc_i)} \right) \quad \text{and} \quad p_i = F^c(c_i) e^{-\lambda_i c_i}.
\] (4)

**Step 4** Finally, for the last phase \( k \) the explicit values for \( \lambda_k \) and \( p_k \) obtained from \( F^c(c_k) = p_k e^{\lambda_k c_k} \) and \( \sum_{j=1}^{k} p_j = 1 \) are

\[
p_k = 1 - \sum_{j=1}^{k-1} p_j, \quad \text{and} \quad \lambda_k = \frac{1}{c_k} \ln \left( \frac{p_k}{F^c_k(c_k)} \right).
\] (5)

Denote by Pareto(\( \alpha, \beta \)), the Pareto distribution with shape parameter \( \alpha \), scale parameter \( \beta > 0 \) and probability distribution \( F(x) = 1 - (1 + \beta x)^{-\alpha} \). In Table??, we shown some numerical values for the Pareto distribution fitted via an hyperexponential distribution with 14 terms and density \( F(x) = 1 - \sum_{i=1}^{14} \lambda_i e^{-\alpha_i x} \) with \( x > 0 \).

### 3. Explicit Formula for the ARL of CUSUM Chart with Hyperexponential Distribution

The CUSUM chart is often implemented in monitoring and detecting small changes in the parameters of a given distribution. Let \( X_t \) be sequence of independent and identically distribution (i.i.d.) nonnegative random variables as following

\[
Y_t = \max(Y_{t-1} + X_t - a, 0), \quad t = 1, 2, ...
\] (6)

where \( X_t \) are random variables and \( a \) is a non-zero constant. with the corresponding stopping time for Eq.(6) defined as

\[
\tau_b = \inf \{ t > 0; Y_t > b \}
\] (7)
where $b$ is the control limit.

Recently, [10] used the integral equation method to derive explicit formulas for the $\text{ARL} = j(x) = \mathbb{E}_x \tau_b$ of CUSUM procedure in the case when observations are i.i.d. and hyperexponentially distributed the d.f.

$$F(x) = 1 - \sum_{i=1}^{n} \lambda_i e^{-\alpha_i x}, \text{ for } \forall \lambda_i \in \mathbb{R}, \sum_{i=i}^{n} \lambda_i = 1, \quad (8)$$

**Theorem 1.** The solutions of the integral equation

$$j(x) = 1 + \int_0^b j(y) \sum_{i=1}^{n} \lambda_i \alpha_i e^{\alpha_i(x-a-y)} dy + (1 - \sum_{i=1}^{n} \lambda_i e^{-\alpha_i(a-x)})) j(0) \text{ if } x \in [0, a]$$

is

$$j(x) = 1 + j(0) + \sum_{i=1}^{n} [d_i - \lambda_i j(0)] e^{\alpha_i(x-a)} , \text{ for } x \in [0, a] \text{ with } a > b \quad (9)$$

<table>
<thead>
<tr>
<th>Number of terms</th>
<th>Pareto(1.2, 5) (c_1 = 10^7, c_k = 0.0264)</th>
<th>Pareto(1.8, 2.5), (c_1 = 10^7, c_k = 0.009)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$\lambda_i \times 10^{-9}$</td>
<td>$\alpha_i$</td>
</tr>
<tr>
<td>1</td>
<td>1.32578</td>
<td>8.31777 $\times 10^{-8}$</td>
</tr>
<tr>
<td>2</td>
<td>9.60654 $\times 10^{-9}$</td>
<td>6.20966 $\times 10^{-7}$</td>
</tr>
<tr>
<td>3</td>
<td>6.05603 $\times 10^{-8}$</td>
<td>3.10062 $\times 10^{-6}$</td>
</tr>
<tr>
<td>4</td>
<td>3.76242 $\times 10^{-7}$</td>
<td>0.0000144274</td>
</tr>
<tr>
<td>5</td>
<td>2.33135 $\times 10^{-6}$</td>
<td>0.0000661781</td>
</tr>
<tr>
<td>6</td>
<td>0.0000144374</td>
<td>0.00302646</td>
</tr>
<tr>
<td>7</td>
<td>0.0000893744</td>
<td>0.00138313</td>
</tr>
<tr>
<td>8</td>
<td>0.000552607</td>
<td>0.00631899</td>
</tr>
<tr>
<td>9</td>
<td>0.00339858</td>
<td>0.0288428</td>
</tr>
<tr>
<td>10</td>
<td>0.0204019</td>
<td>0.131128</td>
</tr>
<tr>
<td>11</td>
<td>0.110135</td>
<td>0.585927</td>
</tr>
<tr>
<td>12</td>
<td>0.393489</td>
<td>2.45634</td>
</tr>
<tr>
<td>13</td>
<td>0.440172</td>
<td>9.02115</td>
</tr>
<tr>
<td>14</td>
<td>0.0317445</td>
<td>33.0862</td>
</tr>
</tbody>
</table>

Table 1: Fitting parameters for the Pareto distribution
where
\[ j(0) = \frac{1 + \sum_{i=1}^{n} d_i e^{-\alpha_i a}}{\sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}}, \]
and the coefficients \(d_i\) are the solution of the \(n \times n\) linear system of equations:
\[
\begin{cases}
    d_1(D - M_{1,n}e^{-\alpha_1 a} - \lambda_1 \alpha_1 be^{-\alpha_1 a} D) + d_2(-M_{1,n}e^{-\alpha_2 a} - \lambda_1 \alpha_1 be^{-\alpha_2 a} A_{2,1} D)
    
    + \ldots + d_n(-M_{1,n}e^{-\alpha_n a} - \lambda_1 \alpha_1 be^{-\alpha_n a} A_{n,1} D) = \lambda_1(1 - e^{-b\alpha_1})D + M_{1,n}
    
    d_1(-M_{2,n}e^{-\alpha_1 a} - \lambda_2 \alpha_2 be^{-\alpha_1 a} A_{1,2} D) + d_2(D - M_{2,n}e^{-\alpha_2 a} - \lambda_2 \alpha_2 be^{-\alpha_2 a} D)
    
    + \ldots + d_n(-M_{2,n}e^{-\alpha_n a} - \lambda_2 \alpha_2 be^{-\alpha_n a} A_{n,2} D) = \lambda_2(1 - e^{-b\alpha_2})D + M_{2,n}
    
    \vdots
    
    d_1(-M_{n,n}e^{-\alpha_1 a} - \lambda_n \alpha_n be^{-\alpha_1 a} A_{1,n} D) + d_2(-M_{n,n}e^{-\alpha_2 a} - \lambda_n \alpha_n be^{-\alpha_2 a} A_{2,n} D)
    
    + \ldots + d_n(D - M_{n,n}e^{-\alpha_n a} - \lambda_n \alpha_n be^{-\alpha_n a} D) = \lambda_n(1 - e^{-b\alpha_1})D + M_{n,n}
\end{cases}
\]
\[
(10)
\]
with
\[ D = \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}, \quad M_{k,n} = (1 - e^{-b\alpha_k})\lambda_k - \lambda_k \alpha_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} A_{i,k}, \quad (11) \]
and where
\[ A_{i,k} = \begin{cases} 
\frac{e^{(\alpha_i - \alpha_k)b} - 1}{b(\alpha_i - \alpha_k)}, & i \neq k \\
1, & i = k.
\end{cases} \]
\[ (12) \]

Proof. For a full proof (see [10]). \(\Box\)

4. Numerical Solution for the ARL Integral Equation

In this section, we present a method to evaluate numerical solutions of Fredholm-type integral equation (IE). It can be shown that the ARL of CUSUM chart, \(j(x) = \mathbb{E}_x \tau_b\) is a solution of the integral equation
\[ j(x) = 1 + \mathbb{E}_x \{I(0 < X_1 < b)j(X_1)\} + \mathbb{P}_x \{X_1 = 0\} j(0), \quad (13) \]

Let \(X_n\) be i.i.d. random variables with a given a distribution function \(F(x)\) and density \(f(x) = \frac{dF(x)}{dx}\) then we can rewrite Eq.(13) as a Fredholm type of integral equation of the form
\[ j(x) = 1 + j(0)F(a - x) + \int_0^b j(y)f(y + a - x)dy. \quad (14) \]
Using Elementary quadrature rule, we can approximate the integral \( \int_{0}^{b} f(x) \, dx \) by a sum of areas of rectangles with bases \( \frac{b}{m} \) with heights chosen as the value of \( f \) at the midpoints of intervals of length \( \frac{b}{m} \) beginning at zero, i.e. on the interval \([0, b]\) with the division points \( 0 \leq a_1 \leq a_2 \leq \ldots \leq a_m < b \) \( w_k = \frac{b}{m} \geq 0 \)
we can writing as follow

\[
\int_{0}^{b} f(y) \, dy \approx \sum_{k=1}^{m} w_k f(a_k), \quad \text{with} \quad a_k = \frac{b}{m} \left( k - \frac{1}{2} \right) \quad ; k = 1, 2, \ldots, m , \quad (15)
\]

If \( j^*(x) \) denotes the the approximate solution of \( j(x) \) then the last term in Eq.(16) can be express as

\[
\sum_{k=1}^{m} w_k j^*(a_k + a - a_i) \quad ; i = 1, 2, \ldots, m . \quad (16)
\]

Then, the integral in Eq.(16) becomes a system of \( m \) linear equations in the \( m \) unknowns \( j^*(a_1), j^*(a_2), \ldots, j^*(a_m) \)

\[
\begin{align*}
   j^*(a_1) &= 1 + j^*(a_1) [F(a - a_1) + w_1 f(a)] \\
      &\quad + \sum_{k=2}^{m} w_k j^*(a_k) f(a_k + a - a_1) \\
   j^*(a_2) &= 1 + j^*(a_1) [F(a - a_2) + w_1 f(a_1 + a - a_2)] \\
      &\quad + \sum_{k=2}^{m} w_k j^*(a_k) f(a_k + a - a_2) \\
   \vdots \\
   j^*(a_m) &= 1 + j^*(a_1) [F(a - a_m) + w_1 f(a_1 + a - a_m)] \\
      &\quad + \sum_{k=2}^{m} w_k j^*(a_k) f(a_k + a - a_m) \\
\end{align*}
\]

(17)

For numerical implementation is preferable to writing the linear system in Eq.(17) in matrix form as follow

\[
\mathbf{J}_{m \times 1} = \mathbf{1}_{m \times 1} + \mathbf{R}_{m \times m} \mathbf{J}_{m \times 1} , \quad \text{or} \quad (\mathbf{I}_m - \mathbf{R}_{m \times m}) \mathbf{J}_{m \times 1} = \mathbf{1}_{m \times 1} \quad (18)
\]

where

\[
\mathbf{J}_{m \times 1} = \begin{pmatrix} j^*(a_1) & j^*(a_2) & \ldots & j^*(a_m) \end{pmatrix}^T
\]

and

\[
\mathbf{1}_{m \times 1} = \begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix}^T
\]
\[ R_{m \times m} = \begin{pmatrix}
F(a - a_1) + w_1 f(a) & w_2 f(a_2 + a - a_1) & \ldots & w_m f(a_2 + a - a_1) \\
F(a - a_1) + w_1 f(a_1 + a - a_2) & w_2 f(a) & \ldots & w_m f(a_2 + a - a_2) \\
\vdots & \vdots & \ddots & \vdots \\
F(a - a_m) + w_1 f(a_1 + a - a_m) & w_2 f(a_2 + a - a_m) & \ldots & w_m f(a) 
\end{pmatrix}, \]

and \( I_m = \text{diag}(1, 1, \ldots, 1) \) is the unit matrix order \( m \). If \( (I_m - R_{m \times m})^{-1} \) exists, then the solution of Eq.(18) is

\[ J_{m \times 1} = (I_m - R_{m \times m})^{-1}1_{m \times 1}. \tag{21} \]

Solving the set of equations (21) for approximate values of \( j^*(a_1), j^*(a_2), \ldots, j^*(a_m) \)

we may approximate the function \( j(x) \) as

\[ j(x) \approx 1 + j^*(a_1)F(a - x) + \sum_{k=1}^{m} w_k j^*(a_k) f(a_k - a - x), \tag{22} \]

with

\[ w = \frac{b}{m} \quad \text{and} \quad a_k = \frac{b}{m} \left( k - \frac{1}{2} \right). \tag{23} \]

5. Numerical Results

In this section, we give some example to show how the hyperexponential distribution can be fit to some long-tailed distributions by FW algorithm see Table?? in Section2. Furthermore, we show numerical result from explicit formula for ARL CUSUM chart with hyperexponential distribution.

Now, we apply these results to evaluate the ARL for Pareto distribution \( \text{Pareto}(\alpha, \beta) \), with shape parameter \( \alpha \), scale parameter \( \beta > 0 \) and probability distribution \( F(x) = 1 - (1 + \beta x)^{-\alpha} \).

Let the \( \text{ARL}_{(\text{IE})} \), denotes the value of ARL for Pareto using numerical integral equation and \( \text{ARL}_{(\text{hyper})} \), the value of ARL for Pareto when is approximated by a hyperexponential distribution with 14 exponentials, and

\[ \varepsilon_r(\%) = 100 \times \frac{|\text{ARL}_{\text{numerical}} - \text{ARL}_{\text{hyper}}|}{\text{ARL}_{\text{numerical}}} \tag{24} \]
be the relative error in percentage as show in Table 2.

We found the relative errors around 0.3% by approximating Pareto distribution with 14 exponentials for a typical run of 800 iterations, and a computational time of 54-55 minutes.

Explicit formula for hyperexponential distribution takes less than 35 sec. computational time. Furthermore, we extend to change shape parameter of Pareto distribution to compare $\text{ARL}_0$ and $\text{ARL}_1$. The results are shown in Table 2.

Table 2 shown a comparison of the numerical result for hyperexponential CUSUM chart obtain in Eq.(9) with the results of integral equation for the number of division point $m = 800$. The explicit formula are an useful approach when it is not possible to obtain explicit expressions for the $\text{ARL}_0$ and $\text{ARL}_1$. 

<table>
<thead>
<tr>
<th>Start point</th>
<th>Pareto(1.2, 5), $a = 5.0, b = 4.15$</th>
<th>Pareto(1.8, 2.5), $a = 5.5, b = 3.624$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$\text{ARL}_{\text{hyper}}$ (time : min)</td>
<td>$\text{ARL}_{\text{IE}}$ (time : min)</td>
</tr>
<tr>
<td>0</td>
<td>100.71 (0.003)</td>
<td>100.52 (54.83)</td>
</tr>
<tr>
<td>0.5</td>
<td>100.646 (0.003)</td>
<td>100.452 (54.83)</td>
</tr>
<tr>
<td>1.0</td>
<td>100.572 (0.003)</td>
<td>100.374 (54.83)</td>
</tr>
<tr>
<td>1.5</td>
<td>100.488 (0.003)</td>
<td>100.285 (54.83)</td>
</tr>
<tr>
<td>2.0</td>
<td>100.389 (0.003)</td>
<td>100.183 (54.83)</td>
</tr>
<tr>
<td>2.5</td>
<td>100.271 (0.003)</td>
<td>100.065 (54.83)</td>
</tr>
<tr>
<td>3.0</td>
<td>100.13 (0.003)</td>
<td>99.9248 (54.83)</td>
</tr>
</tbody>
</table>

Table 2: Comparison by hyperexponential of 14 terms exponentials fit Pareto($\alpha, \beta$) with IE.
### Table 3: Comparison by hyperexponential of 14 terms exponentials fit $\text{Pareto}(\alpha, \beta)$ with IE

<table>
<thead>
<tr>
<th>$(\alpha, \beta)$</th>
<th>$\text{ARL}_{\text{hyper}}$</th>
<th>$\text{ARL}_{\text{IE}}$</th>
<th>$\epsilon_r(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 100$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(1.2, 5.0)$</td>
<td>100.71</td>
<td>100.52</td>
<td>0.189</td>
</tr>
<tr>
<td>1.2, 4.5</td>
<td>88.94</td>
<td>88.80</td>
<td>0.158</td>
</tr>
<tr>
<td>1.2, 4.0</td>
<td>77.43</td>
<td>77.33</td>
<td>0.129</td>
</tr>
<tr>
<td>1.2, 3.5</td>
<td>64.94</td>
<td>66.14</td>
<td>1.814</td>
</tr>
<tr>
<td>1.2, 3.0</td>
<td>55.46</td>
<td>55.26</td>
<td>0.362</td>
</tr>
<tr>
<td>1.2, 2.5</td>
<td>44.71</td>
<td>44.72</td>
<td>0.022</td>
</tr>
<tr>
<td>$T = 300$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.8, 2.5</td>
<td>300.136</td>
<td>300.054</td>
<td>0.023</td>
</tr>
<tr>
<td>1.8, 2.4</td>
<td>279.747</td>
<td>279.642</td>
<td>0.006</td>
</tr>
<tr>
<td>1.8, 2.3</td>
<td>259.998</td>
<td>259.870</td>
<td>0.037</td>
</tr>
<tr>
<td>1.8, 2.2</td>
<td>240.895</td>
<td>240.745</td>
<td>0.071</td>
</tr>
<tr>
<td>1.8, 2.1</td>
<td>222.718</td>
<td>222.271</td>
<td>0.108</td>
</tr>
<tr>
<td>$T = 500$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>2.5, 1.7</td>
<td>500.274</td>
<td>500.079</td>
<td>0.0003</td>
</tr>
<tr>
<td>2.5, 1.6</td>
<td>435.350</td>
<td>435.146</td>
<td>0.0035</td>
</tr>
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<td>2.5, 1.5</td>
<td>375.760</td>
<td>375.540</td>
<td>0.0061</td>
</tr>
<tr>
<td>2.5, 1.4</td>
<td>321.340</td>
<td>321.110</td>
<td>0.0080</td>
</tr>
<tr>
<td>2.5, 1.3</td>
<td>271.940</td>
<td>271.696</td>
<td>0.0094</td>
</tr>
</tbody>
</table>

6. Conclusions

Fitting Pareto distribution with the hyperexponential to evaluate ARL of CUSUM procedure is effective to detect change when the observed random variable have change in parameters. The numerical results can be found that they perform good agreement, and obviously our formula takes computational time less than IE.

References


