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STABILITY OF AN ADDITIVE FUNCTIONAL INEQUALITY WITH THE FIXED POINT ALTERNATIVE

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Abstract: In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\left\|f(x)+2f(y)+2f(z)\right\| \le \left\|f(x)+f(2y+2z)\right\|$$

by using a fixed point method in Banach spaces.

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1. Introduction

In 1940, S.M. Ulam [1] suggested the stability problem of functional equations

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concerning the stability of group homomorphisms as follows: *Let* $(\mathcal{G}, \circ)$ *be a group and let* $(\mathcal{H}, \star, d)$ *be a metric group with the metric* $d(\cdot, \cdot)$. *Given* $\varepsilon > 0$, *does there exist a* $\delta = \delta(\varepsilon) > 0$ *such that if a mapping* $f : \mathcal{G} \to \mathcal{H}$ *satisfies the inequality*

$$d\big(f(x \circ y), f(x) \star f(y)\big) < \delta$$

for all $x, y \in \mathcal{G}$, *then a homomorphism* $F : \mathcal{G} \to \mathcal{H}$ *exits with*

$$d\big(f(x), F(x)\big) < \varepsilon$$

for all $x \in \mathcal{G}$*?*

In the next year, D.H. Hyers [2] gave a first (partial) affirmative answer to the question of Ulam for Banach spaces as follows: *If* $\delta > 0$ *and if* $f : \mathcal{E} \to \mathcal{F}$ *is a mapping between Banach spaces* $\mathcal{E}$ *and* $\mathcal{F}$ *satisfying*

$$\big\|f(x+y) - f(x) - f(y)\big\| \leq \delta$$

for all $x, y \in \mathcal{E}$, *then there is a unique additive mapping* $A : \mathcal{E} \to \mathcal{F}$ *such that*

$$\big\|f(x) - A(x)\big\| \leq \delta$$

for all $x \in \mathcal{E}$.

Thereafter, we call that type the Hyers-Ulam stability.

In 1994, a generalization of the Rassias theorem was obtained by Găvruta [3] as follows.

Suppose $(\mathcal{G}, +)$ is an abelian group, $\mathcal{E}$ is a Banach space, and that the so-called admissible control function $\varphi : \mathcal{G}^2 \to \mathbb{R}$ satisfies

$$\widetilde{\varphi}(x, y) := \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in \mathcal{G}$. If $f : \mathcal{G} \to \mathcal{E}$ is a mapping with

$$\big\|f(x+y) - f(x) - f(y)\big\| \leq \varphi(x, y)$$

for all $x, y \in \mathcal{G}$, then there exists a unique mapping $T : \mathcal{G} \to \mathcal{E}$ such that $T(x+y) = T(x) + T(y)$ and $\big\|f(x) - T(x)\big\| \leq \widetilde{\varphi}(x, x)$ for all $x, y \in \mathcal{G}$.

During the last decades, several stability problems of functional equations have been investigated by a number of mathematicians, see [4, 5, 6] and references therein for more detailed information.

We will recall a fundamental result in fixed point theory for explicit later use.

Theorem 1. (The alternative of fixed point) *[7, 8]*
Suppose we are given a complete generalized metric space $(\mathcal{X}, d)$ and a strictly contractive mapping $\Lambda : \mathcal{X} \to \mathcal{X}$, with the Lipschitz constant L. Then, for each given element $x \in \mathcal{X}$, either

$$d(\Lambda^n x, \Lambda^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(a) $d(\Lambda^n x, \Lambda^{n+1}) < \infty$ *for all* $n \geq n_0$;

(b) *The sequence*$(\Lambda^n x)$ *is convergent to a fixed point* y^* *of* Λ;

(c) y^* *is the unique fixed point of* Λ *in the set* $Y = \{y \in X | d(\Lambda^{n_0}, y) < \infty\}$;

(d) $d(y, y^*) \leq \dfrac{1}{1-L} d(y, \Lambda y)$ *for all* $y \in Y$.

In 1996, Isac and Th. M. Rassias [9] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [10, 11, 12, 13, 14]).

Gilányi [15] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\|, \tag{1}$$

then, it satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y). \tag{2}$$

Moreover, the generalized Hyers-Ulam stability of the functional inequality (1) has been investigated [16, 17, 18].

In this paper, using a fixed point method, we investigate the generalized Hyers-Ulam stability of the functional inequality

$$\big\|f(x) + 2f(y) + 2f(z)\big\| \leq \big\|f(x) + f(2y+2z)\big\| \tag{3}$$

in Banach spaces.

2. Stability of Functional Inequalities (3)

Throughout this section, let $\mathcal{X}$ be a normed vector space and $\mathcal{Y}$ a Banach space.

Lemma 2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. Then it is additive if and only if it satisfies*

$$\big\|f(x) + 2f(y) + 2f(z)\big\| \le \big\|f(x) + f(2y + 2z)\big\| \tag{4}$$

for all $x, y, z \in \mathcal{X}$.

Proof. If f is additive, then clearly

$$\|f(x) + 2f(y) + 2f(z)\| = \|f(x) + f(2y + 2z)\|$$

for all $x, y, z \in \mathcal{X}$.

Assume that f satisfies (4). Letting $x = y = 0$ in (4), we gain $\|5f(0)\| \le \|2f(0)\|$ and so $f(0) = 0$. Putting $x = 0$ and $z = -y$ in (4), we get

$$\|2f(y) + 2f(-y)\| \le \|f(0)\| = 0$$

and so $f(-y) = -f(y)$ for all $y \in \mathcal{X}$. Replacing x by $2x$ and setting $y = -x$ and $z = 0$ in (4), we obtain

$$\|f(2x) - 2f(x)\| \le \|f(2x) - f(2x)\| = 0$$

and so $f(2x) = 2f(x)$ for all $x \in \mathcal{X}$. Taking $x = -2y - 2z$ in (4), we see that

$$\| - f(2y + 2z) + 2f(y) + 2f(z)\| \le \| - f(2y + 2z) + f(2y + 2z)\| = 0$$

for all $y, z \in \mathcal{X}$. Thus we see that

$$f(y + z) = f(y) + f(z)$$

for all $y, z \in \mathcal{X}$. □

Now, using fixed point methods, we investigate the generalized Hyers-Ulam stability of the functional inequality (3) in Banach spaces.

Theorem 3. *Suppose that an odd mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|f(x) + 2f(y) + 2f(z)\| \le \|f(x) + f(2y + 2z)\| + \phi(x, y, z) \tag{5}$$

for all $x, y, z \in \mathcal{X}$*, where* $\phi : \mathcal{X}^3 \to [0, \infty)$ *is a given function. If there exists* $L < 1$ *such that*

$$\phi(x, y, z) \leq \frac{1}{2} L\phi(2x, 2y, 2z) \tag{6}$$

for all $x, y, z \in \mathcal{X}$*. Then there exists a unique Cauchy additive mapping* $A : \mathcal{X} \to \mathcal{Y}$ *satisfying*

$$\|f(x) - A(x)\| \leq \frac{L}{2 - 2L}\phi(2x, -x, 0) \tag{7}$$

for all $x \in \mathcal{X}$.

Proof. Consider a set $S := \{g \,|\, g : \mathcal{X} \to \mathcal{Y}\}$ and introduce a generalized metric d on S as follows:

$$d(g, h) = d_\phi(g, h) := \inf S_\phi(g, h),$$

where

$$\begin{aligned} &S_\phi(g, h) \\ &\quad := \{C \in (0, \infty) : \|g(x) - h(x)\| \leq C\phi(2x, -x, 0) \text{ for all } x \in \mathcal{X}\} \end{aligned}$$

for all $g, h \in S$. Now we show that (S, d) is complete. Let $\{h_n\}$ be a Cauchy sequence in (S, d). Then, for any $\varepsilon > 0$ there exists an integer $N_\varepsilon > 0$ such that $d(h_m, h_n) < \varepsilon$ for all $m, n \geq N_\varepsilon$. Since $d(h_m, h_n) = \inf S_\phi(h_m, h_n) < \varepsilon$ for all $m, n \geq N_\varepsilon$, there exists $C \in (0, \varepsilon)$ such that

$$\|h_m(x) - h_n(x)\| \leq C\phi(2x, -x, 0) \leq \varepsilon\phi(2x, -x, 0) \tag{8}$$

for all $m, n \geq N_\varepsilon$ and all $x \in \mathcal{X}$. So $\{h_n(x)\}$ is a Cauchy sequence in $\mathcal{Y}$ for each $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, $\{h_n(x)\}$ converges for each $x \in \mathcal{X}$. Thus a mapping $h : \mathcal{X} \to \mathcal{Y}$ can be defined by

$$h(x) := \lim_{n\to\infty} h_n(x) \tag{9}$$

for all $x \in \mathcal{X}$. Letting $n \to \infty$ in (8), we have

$$\begin{aligned} m \geq N_\varepsilon &\Rightarrow \|h_m(x) - h(x)\| \leq \varepsilon\phi(2x, -x, 0) \\ &\Rightarrow \varepsilon \in S_\phi(h_m, h) \\ &\Rightarrow d(h_m, h) = \inf S_\phi(h_m, h) \leq \varepsilon \end{aligned}$$

for all $x \in \mathcal{X}$. This means that the Cauchy sequence $\{h_n\}$ converges to h in (S, d). Hence (S, d) is complete.

Define a mapping $\Lambda : S \to S$ by

$$\Lambda h(x) := 2h\Big(\frac{x}{2}\Big) \tag{10}$$

for all $x \in \mathcal{X}$. We claim that Λ is strictly contractive on S. For any given $g, h \in S$, let $C_{gh} \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le C_{gh}$. Then

$$\begin{aligned} d(g,h) &\le C_{gh} \\ &\Rightarrow \|g(x) - h(x)\| \le C_{gh}\phi(2x, -x, 0) \quad \text{for all } x \in \mathcal{X} \\ &\Rightarrow \Big\|2g\Big(\frac{x}{2}\Big) - 2h\Big(\frac{x}{2}\Big)\Big\| \le 2C_{gh}\phi\Big(x, -\frac{x}{2}, 0\Big) \quad \text{for all } x \in \mathcal{X} \\ &\Rightarrow \Big\|2g\Big(\frac{x}{2}\Big) - 2h\Big(\frac{x}{2}\Big)\Big\| \le LC_{gh}\phi(2x, -x, 0) \quad \text{for all } x \in \mathcal{X}, \end{aligned}$$

that is, $d(\Lambda g, \Lambda h) \le LC_{gh}$. Hence we see that $d(\Lambda g, \Lambda h) \le Ld(g, h)$ for any $g, h \in S$. Therefore Λ is strictly contractive mapping on S with the Lipschitz constant $0 < L < 1$. Replacing x by $2x$ and putting $y = -x$ and $z = 0$ in (5), we have

$$\|f(2x) - 2f(x)\| \le \phi(2x, -x, 0) \tag{11}$$

for all $x \in \mathcal{X}$. It follows from (11) that

$$\Big\|f(x) - 2f\Big(\frac{x}{2}\Big)\Big\| \le \phi\Big(x, \frac{x}{2}, 0\Big) \le \frac{L}{2}\phi(2x, -x, 0) \tag{12}$$

for all $x \in \mathcal{X}$. Thus $d(f, \Lambda f) \le \frac{L}{2}$. Therefore, it follows from Theorem 1 that the sequence $\{\Lambda^n f\}$ converges to a fixed point A of Λ, i.e.,

$$A : \mathcal{X} \to \mathcal{Y}, \quad A(x) = \lim_{n\to\infty} (\Lambda f)(x) = \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big)$$

and $A(2x) = 2A(x)$ for all $x \in \mathcal{X}$. Also A is the unique fixed point of Λ in the set $S^* = \{g \in S \,|\, d(f, g) < \infty\}$ and

$$d(A, f) \le \frac{1}{1-L} d(\Lambda f, f) \le \frac{L}{2 - 2L},$$

i.e., the inequality (7) holds for all $x \in \mathcal{X}$. It follows from the definition of A and (5) that

$$\|A(x) + 2A(y) + 2A(z)\| \le \|A(x) + A(2y + 2z)\|$$

for all $x, y, z \in \mathcal{X}$. By Lemma 2, the mapping $A : \mathcal{X} \to \mathcal{Y}$ is a Cauchy additive mapping. Therefore, there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying (7). □

Corollary 4. *Let $p > 1$ and θ be non-negative real numbers and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\|f(x) + 2f(y) + 2f(z)\| \leq \|f(x) + f(2y + 2z)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \quad (13)$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \frac{2^p + 1}{2^p - 2}\theta\|x\|^p \quad (14)$$

for all $x \in \mathcal{X}$.

Proof. In Theorem 3, take $\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in \mathcal{X}$. Then, we can choose $L = 2^{1-p}$ and we have the desired result. □

Theorem 5. *Suppose that an odd mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|f(x) + 2f(y) + 2f(z)\| \leq \|f(x) + f(2y + 2z)\| + \phi(x, y, z) \quad (15)$$

for all $x, y, z \in \mathcal{X}$, where $\phi : \mathcal{X}^3 \to [0, \infty)$ is given function. If there exists $L < 1$ such that

$$\phi(x, y, z) \leq 2L\phi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big) \quad (16)$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x) - A(x)\| \leq \frac{1}{2 - 2L}\phi(2x, -x, 0) \quad (17)$$

for all $x \in \mathcal{X}$.

Proof. Consider the complete generalized metric space (S, d) given in the proof of Theorem 3. Now we consider the linear mapping $\Lambda : S \to S$ given by

$$\Lambda h(x) = \frac{1}{2}h(2x)$$

for all $x \in \mathcal{X}$. For any given $g, h \in S$, let $C_{gh} \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq C_{gh}$. Hence we obtain

$$d(\Lambda g, \Lambda h) \leq Ld(g, h)$$

for all $g, h \in S$. It follows from (11) that $d(f, \Lambda f) \leq \frac{1}{2}$. The rest of the proof is similar to the corresponding part of the proof of Theorem 3. □

Corollary 6. *Let $\theta \in [0, \infty)$ and $p \in [0, 1)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\|f(x) + 2f(y) + 2f(z)\| \leq \|f(x) + f(2y + 2z)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \quad (18)$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \frac{1 + 2^p}{2 - 2^p}\theta\|x\|^p \quad (19)$$

for all $x \in \mathcal{X}$.

Proof. In Theorem 5, take $\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in \mathcal{X}$. Then we can choose $L = 2^{p-1}$ and we have the desired result. □

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