A VARIATIONAL PRINCIPLE AND ITS APPLICATION

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Abstract: Assume that $A$ is a bounded selfadjoint operator in a Hilbert space $H$. Then, the variational principle

$$\max_v \frac{|(Au, v)|^2}{(Av, v)} = (Au, u) \quad (*)$$

holds if and only if $A \geq 0$, that is, if $(Av, v) \geq 0$ for all $v \in H$. We define the left-hand side in (*) to be zero if $(Av, v) = 0$. As an application of this principle it is proved that

$$C = \max_{\sigma \in L^2(S)} \frac{\left| \int_S \sigma(t) dt \right|^2}{\int_S \int_S \frac{\sigma(t)\sigma(s) ds dt}{4\pi |s-t|}}, \quad (***)$$

where $L^2(S)$ is the $L^2$-space of real-valued functions on the connected surface $S$ of a bounded domain $D \in \mathbb{R}^3$, and $C$ is the electrical capacitance of a perfect conductor $D$.

The classical Gauss’ principle for electrical capacitance is an immediate consequence of (*).

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1. Introduction

In many applications a physical quantity of interest can be expressed as a quadratic form. For example, consider electrical charge distributed on the surface of a perfect conductor with density $\sigma(t)$. If the conductor is charged to a
potential \( u = 1 \), then the equation for \( \sigma(t) \) is

\[
A\sigma := \int_S \frac{\sigma(t)dt}{4\pi r_{st}} = 1, \quad s \in S, \quad r_{st} := |s - t|,
\]

where \( dt \) is the element of the surface area, \( S \) is the surface of the conductor \( D \), and \( D \in \mathbb{R}^3 \) is a bounded domain with a connected smooth boundary \( S \). The total charge on \( S \) is \( Q = \int_S \sigma(t)dt \). The physical quantity of interest is electrical capacitance \( C \) of the conductor \( D \). Since \( Q = Cu \) and \( u = 1 \) (see equation (1)), it follows that

\[
C = \int_S \sigma(t)dt = (A\sigma, \sigma),
\]

where \( (f, g) := \int_S f\overline{g}dt \) is the inner product in the Hilbert space \( H = L^2(S) \), and the overbar stands for complex conjugate.

Let us introduce a general theory. Let \( A = A^* \) be a linear selfadjoint bounded operator in a Hilbert space \( H \). Consider an equation \( Au = f \).

We are interested in a quantity \( (Au, u) \) and want to find a variational principle that allows one to calculate and estimate this quantity. Let us write \( A \geq 0 \) if and only if \( (Av, v) \geq 0 \) for all \( v \), and say in this case that \( A \) is non-negative. If \( (Av, v) > 0 \) for all \( v \neq 0 \), we write \( A > 0 \) and say that \( A \) is positive.

The following variational principle is our main abstract result.

**Theorem 1.1.** Let \( A = A^* \) be a linear bounded selfadjoint operator. Formula

\[
(Au, u) = \max_{v \in H} \frac{|(Av, u)|^2}{(Av, v)}
\]

holds if and only if \( A \geq 0 \).

**Remark 1.** We define the right-hand side in (2) to be zero if \( (Av, v) = 0 \).

Theorem 1 can be proved also for unbounded selfadjoint operators \( A \). In this case maximization is taken over \( v \in D(A) \), where \( D(A) \) is the domain of \( A \), a linear dense subset of \( H \).

In Section 2, Theorem 1.1 is proved. Let us illustrate this theorem by an example.

**Example 1.** Let \( A \) be defined in (1). In Section 2, we prove the following lemma.

**Lemma 1.2.** The operator \( A \) in equation (1) is positive in \( H = L^2(S) \).
From Theorem 1.1, Lemma 1.2, and equation (1) it follows that the electrical capacitance $C$ can be calculated by the following variational principle:

$$C = \max_{v \in L^2(S)} \frac{|\int_S v(t) dt|^2}{\int_S \int_S \frac{v(t)v(s)dsdt}{4\pi r_{st}}}.$$  \hspace{1cm} (3)

This variational principle for electrical capacitance is an application of the abstract variational principle formulated in Theorem 1.

Formula (3) can be rewritten as

$$C^{-1} = \min_{v \in L^2(S)} \frac{\int_S \int_S \frac{v(t)v(s)dsdt}{4\pi r_{st}}}{|\int_S v(t) dt|^2}.$$  \hspace{1cm} (4)

In particular, setting $v = 1$ in (3), one gets

$$C \geq \frac{4\pi |S|^2}{J}, \quad J := \int_S \int_S \frac{dsdt}{r_{st}},$$  \hspace{1cm} (5)

where $|S|$ is the surface area of $S$.

In [3] the following approximate formula for the capacitance is derived:

$$C^{(0)} = \frac{4\pi |S|^2}{J}.$$  

This formula is zero-th approximation of an iterative process for finding $\sigma(t)$, the equilibrium charge distribution on the surface $S$ of a perfect conductor charged to the potential $u = 1$.

Formula (4) yields a well-known Gauss’ principle (see [2]), which says that if the total charge $Q = \int_S v(t) dt$ is distributed on the surface $S$ of a perfect conductor with a density $v(t)$ and $u(s)$ is the corresponding distribution of the potential on $S$, then the minimal value of the functional

$$Q^{-2} \int_S \int_S \frac{v(t)v(s)dsdt}{4\pi r_{st}} = \min$$  \hspace{1cm} (6)

is equal to $C^{-1}$, where $C$ is the electrical capacitance of the conductor, and this minimal value is attained at $v(t) = \sigma(t)$, where $\sigma(t)$ solves equation (1).
2. Proofs

Proof of Theorem 1.1. The sufficiency of the condition $A \geq 0$ for the validity of (2) is clear: if $A = A^* \geq 0$, then the quadratic form $\langle u, u \rangle := (Au, u)$ is non-negative and the standard argument yields the Cauchy inequality

$$|(Au, v)|^2 \leq (Au, u)(Av, v).$$

The equality sign in (7) is attained if and only if $u$ and $v$ are linearly dependent. Dividing (7) by $(Av, v)$, one obtains (2), and the maximum in (2) is attained if $v = \lambda u$, $\lambda = \text{const}$.

Let us prove the necessity of the condition $A \geq 0$ for (2) to hold. Let us assume that there exist $z$ and $w$ such that $\langle Az, z \rangle > 0$ and $\langle Aw, w \rangle < 0$, and prove that then (2) cannot hold.

Note that if $\langle Av, v \rangle \leq 0$ for all $v$, then (2) cannot hold. Indeed, if $\langle Av, v \rangle \leq 0$ for all $v$, then (2) implies $|(Bu, v)|^2 \geq (Bu, u)(Bv, v)$, where $B = -A \geq 0$. This is a contradiction to the Cauchy inequality. This contradiction proves that $\langle Av, v \rangle \leq 0$ for all $v$ cannot hold if (2) holds.

Let us continue the proof of necessity. Take $v = \lambda z + w$, where $\lambda$ is an arbitrary real number. Then, (2) yields

$$\frac{|(Au, \lambda z + w)|^2}{q(\lambda)} \leq (Au, u),$$

where

$$q(\lambda) := a\lambda^2 + 2b\lambda + c, \quad a := \langle Az, z \rangle > 0, \quad c = \langle Aw, w \rangle < 0,$$

and $b := \text{Re}(Az, w)$. The polynomial $q(\lambda)$ has two real roots $\lambda_1 < 0$ and $\lambda_2 > 0$, $q^{-1}(\lambda) \to +\infty$ if $\lambda \to \lambda_1 - 0$ or if $\lambda \to \lambda_2 + 0$. The quadratic polynomial $p(\lambda) := |(Au, \lambda z + w)|^2$ has also two roots, and by (2), the ratio $\frac{p(\lambda)}{q(\lambda)}$ is bounded when $\lambda \to \lambda_1 - 0$ and $\lambda \to \lambda_2 + 0$. Therefore, one concludes that $p(\lambda)$ has the same roots as $q(\lambda)$, that is, $\lambda_1$ and $\lambda_2$ are roots of $p(\lambda)$.

Since $\lambda_1 \lambda_2 < 0$ and

$$p(\lambda) = |(Au, z)|^2 \lambda^2 + 2\lambda \text{Re}(Au, z)(Au, w) + |(Au, w)|^2,$$

it follows that

$$\frac{|(Au, w)|^2}{|(Au, z)|^2} < 0.$$

This is a contradiction which proves that there are no elements $z$ and $w$ such that $\langle Az, z \rangle > 0$ and $\langle Aw, w \rangle < 0$. 


Theorem 1.1 is proved. □

Proof of Lemma 1.2. It is known that

\[
F\left(\frac{1}{|x|}\right) := \int_{\mathbb{R}^3} \frac{e^{-i\zeta \cdot x}}{|x|} dx = \frac{4\pi}{|\zeta|^2} > 0,
\]

where the Fourier transform \(F\) is understood in the sense of distributions (see, e.g., [1]). Therefore,

\[
(A\sigma, \sigma) = \int_S \int_S \frac{\sigma(t)\sigma(s)}{4\pi|s-t|} ds dt = \int_{\mathbb{R}^3} \frac{|F\sigma(\zeta)|^2}{|\zeta|^2} d\zeta \geq 0,
\]

which proves Lemma 1.2. □

In (12), \(F\sigma(\zeta)\) is the Fourier transform of the distribution \(\sigma(t)\) with support on the surface \(S\). There are many results about the rate of decay of the Fourier transform of a function (measure) supported on a surface. For example, if the Gaussian curvature of the surface \(S\) is strictly positive, then (see [4])

\[
F\sigma(\zeta) := \int_S \sigma(t)e^{-i\zeta \cdot t} dt = O\left(\frac{1}{|\zeta|}\right), \quad |\zeta| \to \infty, \quad \zeta \in \mathbb{R}^3,
\]

provided that \(\sigma(t)\) is sufficiently smooth.

References


