THE MAXIMAL RANK CONJECTURE FOR LINEARLY NORMAL CURVES $C \subset \mathbb{P}^r$ with $h^1(C, \mathcal{O}_C(1)) = 1$

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Abstract: Let $C \subset \mathbb{P}^r$ be a general linearly normal curve with prescribed genus and $h^1(C, \mathcal{O}_C(1)) = 1$. Here we prove that $C$ has maximal rank, i.e. that for all integers $t$ the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(C, \mathcal{O}_C(t))$ is either injective or surjective.

AMS Subject Classification: 14H50
Key Words: maximal rank, curves with general moduli, postulation

1. Introduction

Let $C \subset \mathbb{P}^r$ be any projective curve. The curve $C$ is said to have maximal rank if for every integer $x > 0$ the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(x)) \to H^0(C, \mathcal{O}_C(x))$ has maximal rank, i.e. either it is injective or it is surjective.

For any curve $X$ and any spanned $L \in \text{Pic}(X)$ let $h_L : X \to \mathbb{P}^r$, $r := h^0(X, L) − 1$, denote the morphism induced by the complete linear system $|L|$. Here we prove the following result, which improves one of the results in [9].

Theorem 1. Fix integers $g > r \geq 3$ and set $d := g + r − 1$. Fix a general $X \in \mathcal{M}_g$ and a general $L \in W^r_d(X)$. Then $L$ is very ample, $h^0(X, L) = r + 1$ and the curve $h_L(X) \subset \mathbb{P}^r$ has maximal rank.
Take \( g, r, d, X \) as in Theorem 1. We have \( \rho(g, r, d) := (r + 1)g - r(g + r - 1) + r(r + 1) > 0 \). Hence Brill-Noether theory gives \( W_d^r(X) \neq \emptyset \), that \( W_d^r(X) \) is irreducible of dimension \( \rho(g, r, d) \) and that \( W_d^r(X) \neq W_d^{r+1}(X) \), i.e. \( h^i(X, L) = r + 1 \) for a general \( L \in W_d^r(X) \). ([1], Ch. V). Hence \( h^i(X, L) = 1 \) for a general \( L \in W_d^r(X) \). For this range of triples \( (g, r, d) \) it is very easy to prove that a general \( L \in W_d^r(X) \) is very ample (e.g., see the proof of [8], Theorem at pages 26-27). Hence to prove Theorem 1 it is sufficient to prove that \( h_L(X) \) has maximal rank.

2. Proof of Theorem 1

We fix a hyperplane \( H \) of \( \mathbb{P}^r \). For any closed subscheme \( A \subset \mathbb{P}^r \) and any integer \( t \geq 0 \) let \( r_{A,t,x} : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(A, \mathcal{O}_A(t)) \) denote the restriction map. If \( x = r \) we often write \( r_{A,t} \) instead of \( r_{A,t,r} \). If \( x = r - 1 \) and \( \mathbb{P}^{r-1} \) is the hyperplane \( H \) of \( \mathbb{P}^{r-1} \), then we often write \( r_{A,t,H} \) instead of \( r_{A,t,r-1} \).

The following remark is often called Castelnuovo’s lemma or Horace lemma.

**Remark 1.** Fix a closed subscheme \( W \subset \mathbb{P}^r \). Let \( \text{Res}_H(W) \) be the residual scheme of \( W \) with respect to \( H \), i.e. the closed subscheme of \( \mathbb{P}^r \) with \( \mathcal{I}_W : \mathcal{I}_H \) as its ideal sheaf. If \( W \) is reduced, then \( \text{Res}_H(W) \) is the union of the irreducible components of \( W \) not contained in \( H \). For any \( t \in \mathbb{Z} \) we have the following exact sequence of coherent sheaves

\[
0 \to \mathcal{I}_{\text{Res}_H(W)}(t - 1) \to \mathcal{I}_W(t) \to \mathcal{I}_{W \cap H,H}(t) \to 0
\]

(1)

From (1) we get

\[
h^i(\mathcal{I}_W(t)) \leq h^i(\mathcal{I}_{\text{Res}_H(W)}(t - 1)) + h^i(H, \mathcal{I}_{W \cap H,H}(t))
\]

for all \( i \geq 0 \) and all \( t \in \mathbb{Z} \).

For all integers \( r \geq 3, g \geq 0 \) and \( d \geq g + r \) let \( Z(d, g, r) \) denote the closure in the Hilbert scheme \( \text{Hilb}(\mathbb{P}^r) \) of \( \mathbb{P}^r \) of the set of all smooth, connected and non-degenerate curves \( C \subset \mathbb{P}^r \) such that \( p_a(C) = g \), \( \deg(C) = d \) and \( h^1(C, \mathcal{O}_C(1)) = 0 \). For all integers \( g > r \geq 3 \) let \( Z'(g, r) \) denote the closure in \( \text{Hilb}(\mathbb{P}^r) \) of the set of all smooth, connected and non-degenerate curves \( C \subset \mathbb{P}^r \) such that \( p_a(C) = g \), \( \deg(C) = g + r - 1 \) and \( h^1(C, \mathcal{O}_C(1)) = 1 \) (since \( C \) is assumed to be non-degenerate the latter condition is equivalent to the linear normality of \( C \)). Obviously \( Z(d, g, r) \) and \( Z'(g, r) \) are irreducible. If \( r \geq 4 \) we write \( Z(d, g, H) \) instead of \( Z(d, g, r - 1) \) to stress that any \( U \in Z(d, g, H) \) is contained in \( H \). If \( 1 \leq d \leq r - 2 \) let \( Z(d, 0, H) \) denote the closure in \( \text{Hilb}(H) \) of
the set of all degree \( d \) smooth rational curves \( T \subset H \) spanning a linear subspace of dimension \( d \).

Fix integers \( r \geq 3 \) and \( k \geq 2 \). Define the integers \( a_{r,k} \) and \( b_{r,k} \) by the relations

\[
k(a_{r,k} + r - 1) + 1 - a_{r,k} + b_{r,k} = \binom{r + k}{r}, \quad 0 \leq b_{r,k} \leq k - 2 \quad (2)
\]

Set \( a_{r,1} := r + 1 \) and \( b_{r,1} = 0 \). As in [5], Definition 2.2, we define the integers \( g(r,k) \) and \( f(r,k) \), \( k \geq 2 \) by the relations

\[
k(g(k,r) + r) + 1 - g(k,r) + f(k,r) = \binom{r + k}{r}, \quad 0 \leq f(k,r) \leq k - 2 \quad (3)
\]

Set \( g(1,r) := 0 \) and \( f(1,r) := 0 \). For any integer \( r \geq 3 \) and \( k \geq 2 \) we have if \( b_{r,k} = g(k,r) + 1 \) and \( b_{r,k} = f(k,r) + 1 \) if \( b_{r,k} \neq 0 \) (or, equivalently, if \( f(k,r) \neq k - 2 \)) and \( a_{r,k} = g(k,r) + 2 \) and \( b_{r,k} = 0 \) if \( b_{r,k} = 0 \) (or, equivalently, if \( f(k,r) = k - 2 \)). These relations allows us to apply the numerical lemmas in [5], §4, loosing at most 2. Using (2) instead of (3) we may extends the proofs in [5], §4, without loosing nothing. Fix an integer \( g > r \). The critical value \( k \) of the pair \((g,r)\) is the minimal integer \( k \) such that \( g \leq a_{r,k} \). Fix any smooth and linearly normal \( C \in Z'(g,r) \) and let \( k \) be the critical value of the pair \((g,r)\). Fix any integer \( t \geq 2 \). Since \( h^1(C,O_C(2)) = 0 \), Riemann-Roch shows that \( h^0(C,O_C(t)) \geq \binom{t + 1}{t} \) if and only if \( t \geq k \). The curve \( C \) has maximal rank if and only if \( h^1(I_C(k)) = 0 \) and \( h^0(C,I_C(k - 1)) = 0 \) (the “only if” part is obvious, the “if” part follows from Castelnuovo-Mumford’s lemma if \( k \geq 3 \), while if \( k \leq 2 \) one need more, e.g. that a canonically embedded curve is projectively normal and then to add \( g - (2r - 2) \) general secant lines). Anyway, the case \( k = 2 \) is obvious by [7].

We recall the following Assertion \( H(k) \), \( k \geq 1 \), proved in [5], Lemma 3.1:

\( H(k) \), \( k \geq 1 \): The map \( r_{A,k} \) is bijective for a general \( A \in Z(g(k,r) + r, g(k,r) - f(k,r)) \).

Of course, before proving \( H(k) \) we proved that \( Z(g(k,r) + r, g(k,r) - f(k,r)) \) is well-defined, i.e. \( g(k,r) \geq f(k,r) \).

**Proof of Theorem 1.** Since the case \( r = 3 \) is true by [4], Theorem 1, we may assume \( r \geq 4 \). To apply the numerical lemmas in [5] we also assume \( r \geq 5 \) (anyway, the case \( r = 4 \) is known by [2]). Fix an integer \( g > r \) and a general \( C \in Z'(g,r) \). Let \( k \) be the critical value of the pair \((g,r)\). If \( k = 1 \), then \( C \) is a canonically embedded smooth curve; hence it is projectively normal; hence it
has maximal rank. If \( k = 2 \), then \( C \) is projectively normal ([7]). From now on we assume \( k \geq 3 \). The assumption that \( k \) is the critical value of the pair \((g, r)\) is equivalent to the following two inequalities:

\[
k(g + r - 1) + 1 - g \leq \binom{r + k}{r} \tag{4}
\]

\[
(k - 1)(g + r - 1) + 1 - g > \binom{r + k - 1}{r} \tag{5}
\]

From (4) and (5) we get

\[
g + r - 2 \leq \binom{r + k - 1}{r - 1} \tag{6}
\]

Since \( Z'(g, r) \) is irreducible, it is sufficient to find \( A, B \in Z'(g, r) \) such that 
\[
h^1(\mathcal{I}_A(k)) = 0 \quad \text{and} \quad h^1(\mathcal{I}_B(k - 1)) = 0.
\]
In steps (a) and (b) we prove the existence of \( A \), while in steps (c) and (d) we prove the existence of \( B \). Take a general \( E \in Z(g(k - 1, r) + r, g(k - 1, r) - f(k - 1, r)) \). Since \( H(k - 1) \) is true ([5], Lemma 3.1) the linear map \( r_{E, k-1} \) is bijective. For a general \( E \) we may assume that \( E \) is transversal to \( H \) and that any subset of \( E \cap H \) is in linearly general position in \( H \), i.e. \( H \) is spanned by any \( S' \subset E \cap H \) such that \( \sharp(S') = r \).

To prove the existence of \( A \) and of \( B \) we may use induction on the critical value \( k \).

(a) Here we prove the existence of the curve \( A \) under the assumption \( g \geq g(k - 1, r) + r \). Take a general \( A_2 \in Z(g - g(k - 1, r) + r - 1, g - g(k - 1, r), H) \). By [3] (case \( r - 1 = 3 \)) or [5] (case \( r - 1 \geq 4 \), \( A_2 \) has maximal rank. Taking the difference between the equation in (3) and the same equation for the integer \( k' := k - 1 \) we get the following equation ([5], eq. (4)):

\[
(k - 2)(g(k, r) - g(k - 1, r)) = \binom{r - 1 + k}{r - 1} - g(k, r) + f(k - 1, r) - f(k, r) \tag{7}
\]

Since \( a_{r,k-1} < g \leq a_{r,k}, a_{r,k} \in \{g(k, r), g(k, r) + 1\}, a_{r,k-1} \in \{g(k - 1, r), g(k - 1, r) + 1\} \), \( f(k, r) \leq k - 2 \) and \( a_{r,k-1} \geq 2r + k \), we have \( k(g - g(k - 1, r) + r - 1) + 1 - (g - g(k - 1, r)) \leq \binom{r + k - 1}{r - 1} \). Since \( A_2 \) has maximal rank, the map \( r_{A_2, k,H} \) is surjective. Fix \( S \subset H \) such that \( \sharp(S) = r + 2 \) and \( S \) is in linearly general position in \( H \). Since any two such sets \( S \) are projectively equivalent, we may assume \( S \subset A_2 \) and \( S \subset E \cap H \). For a general \( A_2 \) we may also assume \( S = (E \cap H) \cap A_2 \). Hence \( S = E \cap A_2 \) and \( E \cup A_2 \) is a nodal curve of degree \( g + r - 1 \) with arithmetic genus \( g \). By [6], Lemma 2.7, we
have $E \cup A_2 \in Z'(g,r)$. Hence by semicontinuity to prove the existence of $A$ it is sufficient to prove $h^1(\mathcal{I}_{A_2 \cup E}(k)) = 0$. Since $r_{A_2,k,H}$ is surjective we have $h^0(H, \mathcal{I}_{A_2}(k)) = \binom{r+k-1}{r-1} - k(g - g(k-1,r) + r - 1) + (g - g(k-1,r) - 1).$ Taking the difference between (4) and the case $k' = k - 1$ we get $\sharp(E \cap H) - \sharp(S) \leq h^0(H, \mathcal{I}_{A_2}(k)) = 0$. Since $h^1(\mathcal{I}_E(k-1)) = 0$, Remark 1 shows that to prove $h^1(\mathcal{I}_{A_2 \cup E}(k)) = 0$ it is sufficient to prove $h^1(H, \mathcal{I}_{A_2 \cup (E \cap H)}(k)) = 0$, i.e. it is sufficient to prove $h^0(H, \mathcal{I}_{(E \cap H) \cup A_2}(k)) = h^0(H, \mathcal{I}_{A_2}(k)) - \sharp(A_2 \cap H) - \sharp(E \cap A_2).$

Apply the proof of part (b) of [5], Lemma 1.6.

(b) Here we prove the existence of the curve $A$ under the assumption $g \leq g(k-1,r) + r - 1$. By (3) for $k' := k - 1$ we get $k(g + r - 1) + 1 \leq \binom{r+k-1}{r-1} - k + 1$. Let $E'$ be a general element of $Z'(a_{k-1}, r)$. Since the pair $(a_{k-1},r)$ has critical value $k - 1$, the inductive assumption gives $h^1(\mathcal{I}_{E'}(k)) = 0$. We may also assume that $E'$ is transversal to $H$ and that each subsets of $E' \cap H$ are in linearly general position in $H$. Take $S' \subset E' \cap H$ such that $\sharp(S') = \min\{r, g - a_{k-1}, r + 1\}$. Set $\epsilon := \max\{g - g(k-1,r) - r + 1, 0\}$. Take a general $A_3 \in Z(g - a_{k-1}, \epsilon, r)$ containing $S'$ (it exists, because either $\epsilon = 0$ and $\sharp(S') = g - a_{k-1} - 1$ or $\epsilon = r$ and $g - a_{k-1} \geq r - 1$). Since any two subsets of $H$ with cardinality $\sharp(S')$ and in linearly general position are projectively equivalent, we may assume $S' = E' \cap A_3$. Repeat the proof of step (a) taking $E'$, $A_3$ and $S'$ instead of $E$, $A_2$ and $S$, respectively.

(c) Here we prove the existence of the curve $B$ under the assumption $g \geq g(k-1,r) + r + 1$. There is $A_1 \in Z(g - 1 - g(k-1,r), g - 1 - g(k-1,r) - r)$ such that $\sharp(A_1 \cap E) = r + 2$ and $E \cup A_1$ is nodal. By [6], Lemma 2.7, we have $E \cup A_1 \in Z'(g,r)$. Hence by the semicontinuity theorem for cohomology to prove the existence of $B$ it is sufficient to prove the injectivity of $r_{E \cup A_1,k-1}$. Since $r_{E,k-1}$ is injective, $r_{E \cup A_1,k-1}$ is injective.

(d) Here we prove the existence of the curve $B$ under the assumption $g \leq g(k-1,r) + r$. Fix a general $F \in Z(g(k-2), r, g(k-2) - f(k-2, r))$. Since $H(k-2)$ is true ([5], Lemma 3.1) the linear map $r_{F,k-2}$ is bijective (here we use that $k \geq 3$). For a general $F$ we may assume that $F$ is transversal to $H$ and that any subset of $F \cap H$ is in linearly general position in $H$. Fix $S_1 \subset F \cap H$ such that $\sharp(S_1) = r + 1$ (it exists, because $k \geq 3$ and $\deg(F) = a_{r,k-2} + r - 1 \geq a_{r,1} - r - 1 = 2r$). Since $g > a_{r,k-1}$ and $g(k-1,r) - g(k-2,r)$, we have $g - a_{r,k-2} - 1 \geq r - 1$. Take a general $A_3 \in Z(g - a_{r,k-2} - 1, g - a_{r,k-2} - r, H)$ containing $S_1$ (it exists, because $g - a_{r,k-2} - 1 \geq r - 1$). We may also assume $S_1 = F \cap A_3$. Hence $F \cup A_3$ is a nodal curve of degree $g + r - 1$ and arithmetic genus $g$ spanning $\mathbb{P}^r$. By [6], Lemma 2.7, we have $F \cup A_3 \in Z'(g,r)$. By semicontinuity to prove the existence of the curve $B$ it is sufficient to prove $h^0(\mathcal{I}_{A_3 \cup F}(k-1)) = 0$. 

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Since $A_3$ is a general non-special curve of its degree and genus, it has maximal rank ([3],[5]). Hence either $h^1(H, \mathcal{I}_{A_3}(k - 1)) = 0$ or $h^0(H, \mathcal{I}_{A_3}(k - 1)) = 0$. First assume $h^0(H, \mathcal{I}_{A_3}(k - 1)) = 0$. Since $h^0(\mathcal{I}_F(k - 1)) = 0$, Remark 1 implies $h^0(\mathcal{I}_{A_3 \cup F}(k - 1)) = 0$. Now assume $h^1(H, \mathcal{I}_{A_3}(k - 1)) = 0$, i.e. $h^0(H, \mathcal{I}_{A_3}(k - 1)) = (k-r-2) - (k - 1)(g - a_{r,k-2} - 1) - 1 + (g - a_{r,k-2} - r)$. Taking the difference of the equation in (3) for the integer $k' := k - 1$ with the same equation for the integer $k' := k - 2$ we get

$$a_{r,k-2} + (k - 1)(a_{r,k-1} - a_{r,k-2}) + b_{r,k-1} - b_{r,k-1} = \binom{r + k - 2}{r - 1} \quad (8)$$

Since $g > a_{r,k-1}$ and $b_{r,k-1} \leq k - 3$, we get $\sharp(F \cap H) - \sharp(F \cap A_3) \geq h^0(H, \mathcal{I}_{A_3}(k - 1))$. Hence we may apply [5], Lemma 1.4, and get $h^0(H, \mathcal{I}_{A_3 \cup (F \cap H)}(k - 1)) = 0$. Apply Remark 1. \square

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References


