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SOME ASPECTS OF ATANASSOV'S INTUITIONISTIC FUZZY SUBMODULE

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Abstract: In this paper, we obtain several properties of intuitionistic fuzzy submodules of a module and established necessary and sufficient conditions for intuitionistic fuzzy submodules. Some special types of intuitionistic fuzzy submodules are also introduced.

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1. Introduction

After the introduction of fuzzy sets by Zadeh [12], a number of generalizations of this fundamental concept have come up. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] (also see [2], [3]) is one among them. Algebraic structures play a vital role in Mathematics and numerous applications of these structures are seen in many disciplines such as computer sciences, information sciences, theoretical physics, control engineering and so on. This inspires researchers to study and carry out research in various concepts of abstract algebra in fuzzy setting. Biswas [4] applied the concept of intuitionistic fuzzy sets to the

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theory of groups and studied intuitionistic fuzzy subgroups of a group. Fuzzy submodules of a module M over a ring R were first introduced by Naegoita and Ralescu [9]. Since then different types of fuzzy submodules were investigated in the last two decades. Davvaz et. al. 2006 [5] have defined intuitionistic fuzzy submodules of a module M over a ring R. In this paper we present it by simply dropping one condition which is later shown that it follows from the other five conditions. We obtain various properties of intuitionistic fuzzy submodules and establish necessary and sufficient conditions for intuitionistic fuzzy submodules. Some special types of intuitionistic fuzzy submodules are introduced. We also investigate the algebraic nature of such type of IF submodules under homomorphism.

2. Basic Definitions and Notations

By R we mean a commutative ring with unity 1 and M denotes an R-module. The zero elements of R and M are 0 and θ respectively. By a fuzzy set of a module M we mean any mapping μ from M to $[0,1]$. By $[0,1]^M$ we will denote the set of all fuzzy subsets of M. For any mapping f from X to Y, we can define in X a new fuzzy set μ^f by putting $\mu^f(x) = \mu(f(x))$ for all $x \in X$, where $\mu \in [0,1]^Y$. It is clear that if for $x_1, x_2 \in f^{-1}(x)$, then $f(x_1) = f(x_2) = x$ from which we get, $\mu^f(x_1) = \mu^f(x_2)$ for $x_1, x_2 \in f^{-1}(x)$.

For each fuzzy set μ in X and any $\alpha \in [0,1]$ we define two sets

$$U(\mu,\alpha)=\{x \in X | \mu(x) \geq \alpha\} \text{ and } L(\mu,\alpha) = \{x \in X | \mu(x) \leq \alpha\},$$

which are called an upper level cut and lower level cut set of μ respectively and can be used to the characterization of μ. The complement of μ, denoted by μ^c, is the fuzzy set on X defined by $\mu^c(x) = 1 - \mu(x)$.

Definition 1. (see [8]) If $A \subseteq X$ and $\alpha \in (0,1]$, then α_A is defined as:

$$\alpha_A(x) = \begin{cases} \alpha, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

If $A=\{x\}$, then $\alpha_{\{x\}}$ becomes a fuzzy point and it is denoted by x_α. When $\alpha = 1$ then 1_A, is known as the characteristic function of A.

If $\mu, \sigma \in [0,1]^X$, then:

(1) $\mu \subseteq \sigma$ if and only if $\mu(x) \leq \sigma(x)$;

(2) $(\mu \cup \sigma)(x) = max\{\mu(x), \sigma(x)\} = \mu(x) \vee \sigma(x)$;

(3) $(\mu\cap\sigma)(x)= min\{\,\mu(x),\sigma(x)\,\}=\mu(x)\wedge\sigma(x)$;

For any family $\{\mu_i|i\in I\}$ of fuzzy subsets of X, where I is any non-empty index set,

(4) $(\cup_{i\in I}\mu_i)(x)=sup_{i\in I}\mu_i(x)=\vee_{i\in I}\mu_i(x)$;

(5) $(\cap_{i\in I}\mu_i)(x)=inf_{i\in I}\mu_i(x)=\wedge_{i\in I}\mu_i(x)$ for all $x\in X$.

Let $r\in R$, $\mu\in[0,1]^M$. We define $r\mu\in[0,1]^M$ and $r\circ\mu\in[0,1]^M$ as follows:

$$r\mu(x)=\vee\{\mu(y)|\,y\in M,\, ry=x\}$$

and

$$r\circ\mu(x)=\wedge\{\mu(y)|y\in M,\, ry=x\}$$

for all $x\in M$.

Definition 2. (see [8]) Let X and Y be any two non-empty sets, and $f:X\to Y$ be a mapping. Let $\mu\in[0,1]^X$ and $\sigma\in[0,1]^Y$ then the *image* $f(\,\mu\,)\in[0,1]^Y$ and the *inverse image* $f^{-1}(\,\sigma\,)\in[0,1]^X$ are defined as follows: For all $y\in Y$

$$f(\mu)(y)=\begin{cases}\vee\{\mu(x)|x\in X, f(x)=y\}, & \text{if } f^{-1}(y)\neq\phi;\\ 0, & \text{otherwise.}\end{cases}$$

and $f^{-1}(\sigma)(x)=\sigma(f(x))$ for all $x\in X$.

Definition 3. (see [9]) A fuzzy submodule of a module M is a fuzzy subset $\mu\in[0,1]^M$ such that:

(1) $\mu(\theta)=1$,

(2) $\mu(x+y)\geq\mu(x)\wedge\mu(y)$ for all $x,\,y\in M$, and

(3) $\mu(rx)\geq\mu(x)$, for all $r\in R$, $x\in M$.

By $F(M)$ we will indicate the set of all fuzzy submodules of M. Since $(-1)x=-x$ for all $x\in M$. So $\mu(-x)\geq(x)$.

An intuitionistic fuzzy set (abbreviated as IFS) introduced by Atanassov in [1] was defined as follows:

An intuitionistic fuzzy set in a non-empty set X, is an object of the form,

$$A=\{(x,\mu_A(x),\nu_A(x))\mid x\in X\}$$

where μ_A and ν_A are fuzzy sets in X and denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. By IFS(X) we denote the set of all IFSs of X.

Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs of X. Then:

(1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$;

(2) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x)), \nu_A(x) \vee \nu_B(x)) \mid x \in X\}$;

(3) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)) \mid x \in X\}$;

(4) $\Box A = \{(x, \mu_A(x), \mu_A^c(x))|x \in M\}$;

(5) $\diamond A = \{(x, \nu_A^c(x), \nu_A(x))|x \in M\}$.

For our convenience we shall use the notation $A(x) \geq B(x)$, when $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ for all $x \in X$.

Definition 4. Let $A = (\mu_A, \lambda_A) = \{(x, \mu_A(x), \lambda_A(x))| x \in M\}$, and $B = (\mu_B, \lambda_B) = \{(x, \mu_B(x), \lambda_B(x))|x \in M\}$ be IFSs of M and let r, $s \in [0, 1]$. Then:

(1) $A + B = (\mu_A \widetilde{+} \mu_B, \lambda_A \widehat{+} \lambda_B)$, where $\mu_A \widetilde{+} \mu_B$ and $\lambda_A \widehat{+} \lambda_B$ are defined by

$$(\mu_A \widetilde{+} \mu_B)(x) = \vee\{\mu_A(y) \wedge \mu_B(z)|y,\, z \in M,\, y + z = x\}$$

and

$$(\lambda_A \widehat{+} \lambda_B)(x) = \wedge\{\lambda_A(y) \vee \lambda_B(z)|y,\, z \in M,\, y + z = x\}$$

(2) $rA = (r\mu_A, r \circ \mu_A)$.

3. Preliminaries

In this section, we present some results that are needed in the sequel.

Theorem 5. (see [8]) *If $r \in R$, then $r1_{\{\theta\}} = 1_{\{\theta\}}$.*

Theorem 6. (see [8]) *Let M and N be modules over the same ring R and f be a homomorphism from M into N. Let $r, s \in R$ and μ, $\nu \in [0, 1]^M$. Then:*

(1) $f(\mu \widetilde{+} \nu) = f(\mu) \widetilde{+} f(\nu)$;

(2) $f(r\mu) = rf(\mu)$;

(3) $f(r\mu \widetilde{+} s\nu) = rf(\mu) \widetilde{+} sf(\nu)$.

Lemma 7. (see [8]) *Let $\mu, \nu \in F(M)$. Then $\mu \widetilde{+} \nu \in F(M)$.*

Lemma 8. (see [8]) *Let* $\mu_i \in F(M)$, *for each* $i \in I$ *where* $|I| > 1$. *Then* $\sum_{i\in I} \mu_i \in F(M)$ *and* $< \cup_{i\in I}\mu_i >= \sum_{i\in I} \mu_i$.

Lemma 9. (see [10]) *Let* $\mu \in [0,1]^M$. *Then the level subset* $\mu_t = \{x \in M : \mu(x) \geq t\}$, $t \in Im(\mu)$ *is a submodule of* M *if and only if* μ *is a fuzzy submodule of* M.

Theorem 10. (see [7]) *Let* r, $s \in R$ *and* $A = (\mu_A, \lambda_A)$, $B = (\mu_B, \lambda_B) \in IFS(M)$. *Then the following assertions hold:*

(1) $1A = A$, $(-1)A = -A$;

(2) $A \subseteq B \Rightarrow rA \subseteq rB$;

(3) $r(sA) = (rs)A$;

(4) $r(A + B) = rA + rB$;

(5) $(rA)(rx) \geq A(x)$;

(6) $B(rx) \geq A(x)$ *for all* $x \in M \Leftrightarrow rA \subseteq B$;

(7) $(r\mu_A \widetilde{+} s\mu_B)(rx+sy) \geq \mu_A(x) \wedge \mu_B(y)$ *and* $(r\lambda_A \widehat{+} s\lambda_B)(rx+sy) \leq \lambda_A(x) \vee \lambda_B(y)$;

(8) *Let* $C = (\mu_C, \lambda_C) \in IFS(M)$. *Then* $\mu_C(rx + sy) \geq \mu_A(x) \wedge \mu_B(y)$ *and* $\lambda_C(rx + sy) \leq \lambda_A(x) \vee \lambda_B(y)$ *for all* x, $y \in M$ *if and only if* $rA + sB \subseteq C$.

4. Main Results

Definition 11. Let M be a module over a ring R. An IFS $A = (\mu_A, \lambda_A)$ of M is called an intuitionistic fuzzy submodule of M if for all x, $y \in M$, and for all $r \in R$, the following five properties hold:

(1) $\mu_A(\theta) = 1$;

(2) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$;

(3) $\mu_A(rx) \geq \mu_A(x)$;

(4) $\lambda_A(x + y) \leq \lambda_A(x) \vee \lambda_A(y)$; and

(5) $\lambda_A(rx) \leq \lambda_A(x)$.

The set of all intuitionistic fuzzy submodules of M will be denoted by $IF(M)$. It can be easily verified that $1_{\widetilde{\{\theta\}}} = (1_{\{\theta\}}, 1_{\{\theta\}^c})$ satisfies all the above five properties for intuitionistic fuzzy submodule of M, and so $1_{\widetilde{\{\theta\}}} \in IF(M)$. Also $\lambda_A(\theta) = 0$, because $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in R$ and $\mu_A(\theta) = 1$.

Definition 12. Let M and N be modules over the same ring R and let f be a mapping from M to N. Let $A = (\mu_A, \lambda_A) \in IFS(M)$ and $B = (\mu_B, \lambda_B) \in IFS(N)$. Then $f(A) = (f(\mu_A), \lambda_A(f)) \in IFS(N)$ can be defined as follows:

For all $y \in N$

$$f(\mu_A)(y) = \begin{cases} \bigvee\{\mu_A(x)|x \in M,\ f(x) = y\}, & \text{if } f^{-1}(y) \neq \phi; \\ 0, & \text{otherwise.} \end{cases}$$

$$\lambda_A(f)(y) = \begin{cases} \bigwedge\{\lambda_A(x)|x \in M,\ f(x) = y\}, & \text{if } f^{-1}(y) \neq \phi; \\ 1, & \text{otherwise.} \end{cases}$$

and for all $x \in M$, $f^{-1}(B) = B(f(x))$ are called, respectively , the image of A under f and the pre-image(or inverse image) of B under f.

Theorem 13. *If $r \in R$, then $r1_{\widetilde{\theta}} = 1_{\widetilde{\theta}}$.*

Proof. From Theorem 5 we have $r1_{\{\theta\}} = 1_{\{\theta\}}$. Now, for $x \in M$

$$r \circ 1_{\{\theta\}^c}(x) = \bigwedge\{1_{\{\theta\}^c}(y)|y \in M,\ 1y = x\} = \begin{cases} 0, & \text{for } x = \theta; \\ 1, & \text{if } x \neq \theta. \end{cases}$$

Therefore, we have $r \circ 1_{\{\theta\}^c} = 1_{\{\theta\}^c}$, and hence $r1_{\widetilde{\theta}} = 1_{\widetilde{\theta}}$. □

Theorem 14. *Let $r, s \in R$ and $A = (\mu_A, \lambda_A) \in IFS(M)$. Then:*

(1) $rA \subseteq A \Leftrightarrow A(rx) \geq A(x)$ for all $x \in M$

(2) $rA + sA \subseteq A \Leftrightarrow \mu_A(rx + sy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\lambda_A(rx + sy) \leq \lambda_A(x) \vee \lambda_A(y)$ for all $x, y \in M$.

Proof. The proof follows from (6) and (8) of Theorem 10. □

Theorem 15. *Let M and N be modules over same the ring R and f be a homomorphism from M into N. Let $r, s \in R$ and $A = (\mu_A, \lambda_A), B = (\mu_B, \lambda_B)$ be in IFS(M). Then:*

(1) $f(A + B) = f(A) + f(B)$,

(2) $f(rA) = rf(A)$,

(3) $f(rA + sB) = rf(A) + sf(B)$.

Proof. (1) It is enough to show $f(\mu_A \widetilde{+} \mu_B) = f(\mu_A) \widetilde{+} f(\mu_B)$ and $(\lambda_A \widehat{+} \lambda_B)(f) = \lambda_A(f) \widehat{+} \lambda_B(f)$. First equality follows from (1) of Theorem 6. For second equality, let $y \in N$. Then

$$\begin{aligned}(\lambda_A \widehat{+} \lambda_B)(f)(y) &= \bigwedge\{(\lambda_A \widehat{+} \lambda_B)(x) | x \in M, f(x) = y\},\\ &= \bigwedge\{(\wedge\{\lambda_A(u) \vee \lambda_B(v) | u, v \in M, u + v = x\}) | x \in M,\\ &\quad f(x) = y\},\\ &= \bigwedge\{\lambda_A(u) \vee \lambda_B(v) | u, v \in M,\, p, q \in N,\, f(u) = p,\\ &\quad f(v) = q, p + q = y\},\\ &= \bigwedge\{(\wedge\{\lambda_A(u) | u \in M, f(u) = p\}) \vee (\wedge\{\lambda_B(v) | v \in M,\\ &\quad f(v) = q\}) | p, q \in N,\, p + q = y\},\\ &= \bigwedge\{(\lambda_A(f))(p) \vee (\lambda_B(f))(q) | p, q \in N, p + q = y\},\\ &= (\lambda_A(f) \widehat{+} \lambda_B(f))(y).\end{aligned}$$

Therefore we have $(\lambda_A \widehat{+} \lambda_B)(f) = \lambda_A(f) \widehat{+} \lambda_B(f)$.

(2) In view of (2) of Theorem 6, it is enough to show $(r \circ \lambda_A)(f) = r \circ (\lambda_A(f)$. Let $y \in N$. Then

$$\begin{aligned}((r \circ \lambda_A)(f))(y) &= \bigwedge\{(r \circ \lambda_A)(x) | x \in M, f(x) = y\},\\ &= \bigwedge\{(\wedge\{\lambda_A(u) | u \in M, ru = x\}) | x \in M, f(x) = y\},\\ &= \bigwedge\{\lambda_A(u) | u \in M, f(ru) = y\},\\ &= \bigwedge\{\lambda_A(u) | u \in M, rf(u) = y\},\\ &= r \circ (\lambda_A(f))(y).\end{aligned}$$

Thus we have $(r \circ \lambda_A)(f) = r \circ (\lambda_A(f)$.

(3) Now (1) and (2) together implies (3). □

Theorem 16. [7] *Let $A = (\mu_A, \lambda_A) \in IFS(M)$. Then $A = (\mu_A, \lambda_A) \in IF(M)$ if and only if it satisfies the following properties:*

(i) $\mu_A(\theta) = 1$,

(ii) $\mu_A(rx + sy) \geq \mu_A(x) \wedge \mu_A(y)$, and

(iii) $\lambda_A(rx + sy) \leq \lambda_A(x) \vee \lambda_A(y)$, for all $r, s \in R$ and for all $x, y \in M$.

The following Theorem can be proved by verifying the corresponding conditions.

Theorem 17. *Let A be a non-empty subset of M. Then $1_{\widetilde{A}} = (1_A, 1_{A^c})$ is an IF submodule of M if and only if A is a submodule of M.*

Theorem 18. *A fuzzy set μ_A is a fuzzy submodule of M if and only if $A = (\mu_A, \mu_A^c)$ is an IF submodule of M.*

Proof. If $A = (\mu_A, \mu_A^c)$ is an IF submodule of M, then clearly, $\mu_A \in F(M)$. Conversely if $\mu_A \in F(M)$, then $\mu_A^c(x+y) = 1 - \mu_A(x+y) \leq 1 - \mu_A(x) \wedge \mu_A(y) = (1 - \mu_A(x)) \vee (1 - \mu_A(y)) = \mu_A^c(x) \vee \mu_A^c(y)$ and $\mu_A^c(rx) = 1 - \mu_A(rx) \leq 1 - \mu_A(x) = \mu_A^c(x)$. Other conditions follow from the fact that $\mu_A \in F(M)$. □

The following Theorem can be proved by verifying the corresponding conditions.

Theorem 19. *Let $A = (\mu_A, \lambda_A) \in IFS(M)$. Then $A \in IF(M)$ if and only if (i) $\mu_A \in F(M)$ and (ii) $\lambda_A^C \in F(M)$.*

Theorem 20. *Let $A = (\mu_A, \lambda_A)$, $B = (\mu_B, \lambda_B) \in IF(M)$. Then $A \cap B \in IF(M)$.*

Proof. Since $\mu_A(\theta) = 1$ and $\mu_B(\theta) = 1$, so we have $\mu_A \cap \mu_B(\theta) = 1$. Now, let for all $r, s \in R$ and $x, y \in M$. Then

$$\begin{aligned}(\mu_A \cap \mu_B)(rx + sy) &= \mu_A(rx + sy) \wedge \mu_B(rx + sy),\\ &\geq (\mu_A(x) \wedge \mu_A(y)) \wedge (\mu_B(x) \wedge \mu_B(y)),\\ &\text{(see Theorem 16)}\\ &= (\mu_A(x) \wedge \mu_B(x)) \wedge (\mu_A(y) \wedge \mu_B(y)),\\ &= ((\mu_A \cap \mu_B)(x)) \wedge ((\mu_A \cap \mu_B)(y)).\end{aligned}$$

Thus $(\mu_A \cap \mu_B)(rx + sy) \geq ((\mu_A \cap \mu_B)(x)) \wedge ((\mu_A \cap \mu_B)(y))$.
Similarly, we have

$$(\lambda_A \cup \lambda_B)(rx + sy) \leq ((\lambda_A \cup \lambda_B)(x)) \vee ((\lambda_A \cup \lambda_B)(y))$$

Therefore from Theorem 16 we have $A \cap B \in IF(M)$. □

Theorem 21. *Let $A = (\mu_A, \lambda_A) \in IFS(M)$. Then $A \in IF(M)$ if and only if (i) $\Box A \in IF(M)$ and (ii) $\diamond A \in IF(M)$.*

Proof. Suppose $A \in IF(M)$. To show (i) $\Box A \in IF(M)$ and (ii) $\diamond A \in IF(M)$. Now, $A \in IF(M)$ implies $\mu_A, \lambda_A^C \in F(M)$ (see Theorem 19). Also, Theorem 16 implies $\Box A = (\mu_A, \mu_A^c) \in IF(M)$ and $\diamond A = (\lambda_A^C, \lambda_A) \in IF(M)$. Conversely, we assume (i) $\Box A \in IF(M)$ and (ii) $\diamond A \in IF(M)$, then $\mu_A, \lambda_A^C \in F(M)$. Therefore from Theorem 19, we have $A \in IF(M)$. □

Theorem 22. *An IFS $A = (\mu_A, \lambda_A)$ is an IF submodule of M if and only if $\mu_A(\theta) = 1$ and $M_A^{(\alpha,\beta)} = \{x \in M | \mu_A(x) \geq \alpha, \lambda_A(x) \leq \beta\}$ is a submodule of M for every $(\alpha, \beta) \in Im(\mu_A) \times Im(\lambda_A)$ such that $\alpha + \beta \leq 1$, i.e., if and only if all non-empty level subsets $U(\mu_A, \alpha)$ and $L(\lambda_A, \beta)$ are submodules of M.*

Proof. Suppose $A = (\mu_A, \lambda_A) \in IF(M)$. To show that $M_A^{(\alpha,\beta)}$ is a submodule of M for every $(\alpha, \beta) \in Im(\mu_A) \times Im(\lambda_A)$ such that $\alpha + \beta \leq 1$. Let $x, y \in M_A^{(\alpha,\beta)}$, then $\mu_A(x) \geq \alpha, \mu_A(y) \geq \alpha$ and $\lambda_A(x) \leq \beta$, $\lambda_A(y) \leq \beta$. Now,

$\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha$ and $\lambda_A(x+y) \leq \lambda_A(x) \vee \lambda_A(y) \leq \beta$. Therefore we have $x+y \in M_A^{(\alpha,\beta)}$. Let $r \in R$. Then $\mu_A(rx) \geq \mu_A(x) \geq \alpha$ and $\lambda_A(rx) \leq \lambda_A(x) \leq \beta$. Therefore we have $rx \in M_A^{(\alpha,\beta)}$. Hence $M_A^{(\alpha,\beta)}$ is a submodule M.

Conversely, we assume $M_A^{(\alpha,\beta)}$ is a submodule M and $\mu_A(\theta) = 1$. To show $A = (\mu_A, \lambda_A)$ is an IF submodule of M. Since $M_A^{(\alpha,\beta)}$ is a Submodule of M for every $(\alpha, \beta) \in Im(\mu_A) \times Im(\lambda_A)$ such that $\alpha+\beta \leq 1$. Therefore, all non-empty level subsets $U(\mu_A, \alpha)$ and $L(\lambda_A, \beta)$ are submodule of M.

Let $x, y \in M$. Take $\alpha = \mu_A(x) \wedge \mu_A(y)$, then $\mu_A(x) \geq \alpha$, $\mu_A(y) \geq \alpha$. That is $x, y \in U(\mu_A, \alpha)$. Since $U(\mu_A, \alpha)$ is a submodule of M, therefore we have $x + y \in U(\mu_A, \alpha)$, and so we have $\mu_A(x + y) \geq \alpha = \mu_A(x) \wedge \mu_A(y)$. Thus we have $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$.

Let $x, y \in M$. Take $\beta = \lambda_A(x) \vee \lambda_A(y)$, then $\lambda_A(x) \leq \beta$, $\lambda_A(y) \leq \beta$. That is $x, y \in L(\lambda_A, \beta)$. Since $L(\lambda_A, \beta)$ is a submodule of M, therefore we have $x + y \in L(\lambda_A, \beta)$, and so $\lambda_A(x + y) \leq \beta = \lambda_A(x) \vee \lambda_A(y)$. Thus we have $\lambda_A(x + y) \leq \lambda_A(x) \vee \lambda_A(y)$.

Let $r \in R$ and $x \in M$. Take $\alpha = \mu_A(x)$, then $x \in U(\mu_A, \alpha)$. Since $U(\mu_A, \alpha)$ is a submodule of M, therefore we have $rx \in U(\mu_A, \alpha)$, and so we have $\mu_A(rx) \geq \alpha = \mu_A(x)$. Thus we have $\mu_A(rx) \geq \mu_A(x)$.

Let $r \in R$, $x \in M$. Take $\beta = \lambda_A(x)$, then $x \in L(\lambda_A, \beta)$. Since $L(\lambda_A, \beta)$ is a submodule of M, therefore we have $rx \in L(\lambda_A, \beta)$, and so $\lambda_A(rx) \leq \beta = \lambda_A(x)$. Thus we have $\lambda_A(rx) \leq \lambda_A(x)$. Hence A is a IF submodule of M. □

Theorem 23. *Let $A = (\mu_A, \lambda_A) \in IFS(M)$. Then $A \in IF(M)$ if and only if A satisfies* (1) $1_{\widetilde{\theta}} \subseteq A$, (2) $rA \subseteq A$,*for all* $r \in R$, *and* (3) $A + A \subseteq A$.

Proof. Suppose $A \in IF(M)$. Then $\mu_A(\theta) = 1$ and $\lambda_A(\theta) = 0$ imply (1) $1_{\widetilde{\theta}} \subseteq A$. And (2) and (3) follow from Theorem 14 and the fact $A \in IF(M)$. Conversely (1) imply $\mu_A(\theta) = 1$ and the other conditions follows as a combination of Theorem 14 and (2), (3). □

The following Theorem is a consequence of the above Theorem.

Theorem 24. *Let $A = (\mu_A, \lambda_A) \in IFS(M)$. Then $A \in IF(M)$ if and only if A satisfies* (1) $1_{\widetilde{\theta}} \subseteq A$ *and* (2) $rA + sA \subseteq A$, *for all* $r, s \in R$.

Theorem 25. *Let $A = (\mu_A, \lambda_A)$, $B = (\mu_B, \lambda_B) \in IF(M)$. Then $A + B \in IF(M)$.*

Proof. Now,

$$\begin{aligned}(\mu_A \widetilde{+} \mu_B)(\theta) &= \vee\{\mu_A(y) \wedge \mu_B(z) | y,\, z \in M,\, y+z=\theta\},\\ &\geq \mu_A(\theta) \wedge \mu_B(\theta) = 1.\end{aligned}$$

Thus we have $(\mu_A \widetilde{+} \mu_B)(\theta) = 1$, and so $1_\theta \subseteq \mu_A \widetilde{+} \mu_B$. And

$$\begin{aligned}(\lambda_A \widehat{+} \lambda_B)(\theta) &= \wedge\{\lambda_A(y) \vee \lambda_B(z) | y,\, z \in M,\, y+z=\theta\},\\ &\leq \lambda_A(\theta) \vee \lambda_B(\theta) = 0 \vee 0 = 0.\end{aligned}$$

Thus we have $(\lambda_A \widehat{+} \lambda_B)(\theta) = 0$, and so $1_{\{\theta\}^c} \supseteq \lambda_A \widehat{+} \lambda_B$.

(I) Therefore we have $1_{\widetilde{\theta}} \subseteq A + B$.

Since $A,\, B \in IF(M)$, so we have (i) $rA \subseteq A$ and $rB \subseteq B$ and (ii) $A+A \subseteq A$ and $B + B \subseteq B$ (see Theorem 23). Now (ii) implies

$$\mu_A \widetilde{+} \mu_A \subseteq \mu_A,\ \ \mu_B \widetilde{+} \mu_B \subseteq \mu_B$$

and

$$\lambda_A \widehat{+} \lambda_A \supseteq \lambda_A,\ \ \lambda_B \widehat{+} \lambda_B \supseteq \lambda_B$$

In view of (i) and (4) of Theorem 10 we have

(II) $r(A + B) = rA + rB \subseteq A + B$. Also, $(\mu_A \widetilde{+} \mu_B) \widetilde{+} (\mu_A \widetilde{+} \mu_B) \subseteq (\mu_A \widetilde{+} \mu_B)$, because $\mu_A \widetilde{+} \mu_B \in F(M)$. And

$$\begin{aligned}(\lambda_A \widehat{+} \lambda_B) \widehat{+} (\lambda_A \widehat{+} \lambda_B) &= \lambda_A \widehat{+} (\lambda_B \widehat{+} \lambda_A) \widehat{+} \lambda_B,\\ &= \lambda_A \widehat{+} (\lambda_A \widehat{+} \lambda_B) \widehat{+} \lambda_B,\\ &= (\lambda_A \widehat{+} \lambda_A) \widehat{+} (\lambda_B \widehat{+} \lambda_B),\\ &\supseteq \lambda_A \widehat{+} \lambda_A\\ &\text{(because } \lambda_A \widehat{+} \lambda_A \supseteq \lambda_A,\ \lambda_B \widehat{+} \lambda_B \supseteq \lambda_B)\end{aligned}$$

(III) Therefore we have $(A + B) + (A + B) \subseteq A + B$.

Hence (I), (II),(III) together imply $A + B \in IF(M)$ (see Theorem 23). □

Theorem 26. *Let M and N be modules over the same ring R and f be a homomorphism from M into N. If $A = (\mu_A, \lambda_A)$ is an IF submodule of N, then $A^f = (\mu_A^f, \lambda_A^f)$ is an IF submodule of M.*

Proof. (1)Let θ_M and θ_N be the additive identities of M and N respectively. f being homomorphism we have $f(\theta_M) = \theta_N$. Now,

$$\mu_A^f(\theta_M) = \mu_A(f(\theta_M)) = \mu_A(\theta_N) = 1$$

Thus we have $\mu_A^f(\theta_M) = 1$. Let $x, y \in M$ and $r \in R$. Then

(2)

$$\begin{aligned}\mu_A^f(x+y) &= \mu_A(f(x+y)),\\ &= \mu_A(f(x)+f(y)),\\ &\geq \mu_A(f(x)) \wedge \mu_A(f(y)) = \mu_A^f(x) \wedge \mu_A^f(y).\end{aligned}$$

(3)

$$\begin{aligned}\lambda_A^f(x+y) &= \lambda_A(f(x+y)),\\ &= \lambda_A(f(x)+f(y)),\\ &\leq \lambda_A(f(x)) \vee \lambda_A(f(y)) = \lambda_A^f(x) \vee \lambda_A^f(y).\end{aligned}$$

(4)

$$\begin{aligned}\mu_A^f(rx) &= \mu_A(f(rx)),\\ &= \mu_A(rf(x)),\\ &\geq \mu_A(f(x)) = \mu_A^f(x).\end{aligned}$$

(5)

$$\begin{aligned}\lambda_A^f(rx) &= \lambda_A(f(rx)),\\ &= \lambda_A(rf(x)),\\ &\leq \lambda_A(f(x)) = \lambda_A^f(x).\end{aligned}$$

Therefore $A^f = (\mu_A^f, \lambda_A^f)$ is an IF submodule of M. □

Theorem 27. *Let M and N be modules over the same ring R and f be a homomorphism from M into N. If $A^f = (\mu_A^f, \lambda_A^f)$ is an IF submodule of M, then $A = (\mu_A, \lambda_A)$ is an IF submodule of N.*

Proof. (1) Let θ_M and θ_N be the additive identities of M and N respectively and since f is a homomorphism, so we have $f(\theta_M) = \theta_N$. Now,

$$\mu_A(\theta_N) = \mu_A(f(\theta_M)) = \mu_A^f(\theta_M) = 1$$

Thus we have $\mu_A(\theta_N) = 1$. Let $x, y \in N$, then since f is surjective mapping, so there exists $x_1, y_1 \in M$ such that $f(x_1) = x$ and $f(y_1) = y$. Now

(2)

$$\begin{aligned}\mu_A(x+y) &= \mu_A(f(x_1)+f(y_1)),\\ &= \mu_A(f(x_1+y_1)),\\ &= \mu_A^f(x_1+y_1),\\ &\geq \mu_A^f(x_1) \wedge \mu_A^f(y_1),\\ &= \mu_A(f(x_1)) \wedge \mu_A(f(y_1)) = \mu_A(x) \wedge \mu_A(y).\end{aligned}$$

(3)

$$\begin{aligned}\lambda_A(x+y) &= \lambda_A(f(x_1)+f(y_1)),\\ &= \lambda_A(f(x_1+y_1)),\\ &= \lambda_A^f(x_1+y_1),\\ &\le \lambda_A^f(x_1)\vee\lambda_A^f(y_1),\\ &= \lambda_A(f(x_1))\vee\lambda_A(f(y_1)) = \lambda_A(x)\vee\lambda_A(y).\end{aligned}$$

(4)

$$\begin{aligned}\mu_A(rx) &= \mu_A(rf(x_1)),\\ &= \mu_A(f(rx_1)),\\ &= \mu_A^f(rx_1),\\ &\ge \mu_A^f(x_1),\\ &= \mu_A(f(x_1)) = \mu_A(x).\end{aligned}$$

(5)

$$\begin{aligned}\lambda_A(rx) &= \lambda_A(rf(x_1)),\\ &= \lambda_A(f(rx_1)),\\ &= \lambda_A^f(rx_1),\\ &\le \lambda_A^f(x_1),\\ &= \lambda_A(f(x_1)) = \lambda_A(x).\end{aligned}$$

Therefore $A=(\mu_A,\lambda_A)$ is an IF submodule of N. □

The following theorem is a consequence of the above two theorems.

Theorem 28. *Let $f: M \longrightarrow N$ be an epimorphism of module. Then $A^f=(\mu_A^f,\lambda_A^f)$ is an IF submodule of M if and only if $A=(\mu_A,\lambda_A)$ is an IF submodule of N.*

Theorem 29. *Let A be a non-empty subset of a module M. Then an IFS $A=(\mu_A,\lambda_A)$ defined by*

$$\mu_A(x)=\begin{cases}1, & \text{if } x\in A;\\ \alpha, & \text{otherwise.}\end{cases}$$

and

$$\lambda_A(x)=\begin{cases}0, & \text{if } x\in A;\\ \beta, & \text{otherwise.}\end{cases}$$

where $0\le\alpha<1$, $0<\beta\le 1$ and $\alpha+\beta\le 1$ is an IF submodule of M if and only if A is a submodule of M.

Proof. Let $A = (\mu_A, \lambda_A)$ is an IF submodule of M. Let $a, b \in A$ and $r \in R$. Since $0 \leq \alpha < 1$, so we have $\mu_A(x) \leq 1$ for all $x \in M$. This implies $\mu_A(a+b) \leq 1$, $\mu_A(ra) \leq 1$ and $\mu_A(ar) \leq 1$, and so $a+b, ar, ra \in A$. Therefore, A is a submodule of M.

Conversely, we assume A is a submodule of M. Since $\theta \in A$, so we have $\mu_A(\theta) = 1$. Let $x, y \in M$, and $r \in R$.

If $x + y \notin A$, then either $x \notin A$ or $y \notin A$. This implies either $\mu_A(x) = \alpha$ or $\mu_A(y) = \alpha$. In this case we have $\mu_A(x+y) = \mu_A(x) \wedge \mu_A(y) = \alpha$. If $x + y \in A$, then $\mu_A(x+y) = 1 \geq \mu_A(x) \wedge \mu_A(y)$. Thus we have $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$.

If $x \in A$, then $rx \in A$, and so $\mu_A(rx) = \mu_A(x) = 1$. If $x \notin A$, then $\mu_A(rx) \geq \mu_A(x)$.

If either $x \notin A$ or $y \notin A$, then $\lambda_A(x) \vee \lambda_A(y) = \beta \geq \lambda_A(x+y)$. If $x, y \in A$, then $x + y \in A$, and so in this case $\lambda_A(x) \vee \lambda_A(y) = 0 = \lambda_A(x+y)$. Thus we have $\lambda_A(x+y) \leq \lambda_A(x) \vee \lambda_A(y)$.

If $x \notin A$, then $\lambda_A(x) = \beta \geq \lambda_A(rx)$. If $x \in A$, then $rx \in A$, and so $\lambda_A(x) = 0 = \lambda_A(rx)$. Thus we have $\lambda_A(rx) \leq \lambda_A(x)$.

Hence $A = (\mu_A, \lambda_A)$ is a IF submodule M. $\square$

Corollary 30. *A non-empty subset A of M is a submodule of M if and only if* $1_{\widetilde{A}} = (1_A, 1_{A^c})$ *is an IF submodule of M.*

Theorem 31. *Let* $A = (\mu_A, \lambda_A)$ *be in IFS(M) and let* $\mu_A^+(x) = \mu_A(x) + 1 - \mu_A(\theta)$, $\lambda_A^+(x) = \lambda_A(x) - \lambda_A(\theta)$. *If* (1) $\mu_A^+(x) + \lambda_A^+(x) \leq 1$, (2) $\mu_A(rx+sy) \geq \mu_A(x) \wedge \mu_A(y)$ *and* (3) $\lambda_A(rx+sy) \leq \lambda_A(x) \vee \lambda_A(y)$, *for all* $x, y \in M$, $r, s \in R$, *then* $A^+ = (\mu_A^+, \lambda_A^+)$ *is an IF submodule of M containing A.*

Proof. Clearly, $\mu_A^+(\theta) = 1$, and $\lambda_A^+(\theta) = 0$. Since $\mu_A(\theta) \leq 1$ and $\lambda_A(x) \geq \lambda_A(\theta)$, for all $x \in M$. Therefore we have $\mu_A^+(x) \geq 0$ and $\lambda_A^+(x) \geq 0$. Moreover, $\mu_A^+(x) + \lambda_A^+(x) \leq 1$ implies $0 \leq \mu_A^+(x) \leq 1$ and $0 \leq \lambda_A^+(x) \leq 1$ for all $x \in M$. Thus $A^+ = (\mu_A^+, \lambda_A^+)$ is an IFS of M. Let $x, y \in M$ and $r, s \in R$. Then

$$\begin{aligned} \mu_A^+(rx+sy) &= \mu_A(rx+sy) + 1 - \mu_A(\theta), \\ &\geq \mu_A(x) \wedge \mu_A(y) + 1 - \mu_A(\theta), \\ &= (\mu_A(x) + 1 - \mu_A(\theta)) \wedge (\mu_A(y) + 1 - \mu_A(\theta)), \\ &= \mu_A^+(x) \wedge \mu_A^+(y). \end{aligned}$$

$$\begin{aligned} \lambda_A^+(rx+sy) &= \lambda_A(rx+sy) - \lambda_A(\theta), \\ &\leq \lambda_A(x) \vee \lambda_A(y) - \lambda_A(\theta), \\ &= (\lambda_A(x) - \lambda_A(\theta)) \vee (\lambda_A(y) - \lambda_A(\theta)), \\ &= \lambda_A^+(x) \wedge \lambda_A^+(y). \end{aligned}$$

Therefore $A^+ = (\mu_A^+, \lambda_A^+)$ is an IF submodule of M containing A. $\square$

Corollary 32. *If $A \in IF(M)$, then $A^+ = A$.*

5. Conclusion

In this paper we have presented the basic results on intuitionistic fuzzy submodules of a module. Also, we have investigated various properties of intuitionistic fuzzy submodules. In our opinion this is an opening for investigations of different types of IF submodules of a module. This will lead us to the study of modules with finiteness conditions on IF submodules. Using these concepts one may expect to explore the Goldie like structures in fuzzy setting.

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