ON NORMALIZED SEMI PARALLEL

$T'$-VECTOR FIELD IN FINSLER SPACE

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Abstract: The semi-parallel vector field in Riemannian geometry has been introduced by Fulton [3], whereas in Finsler geometry by Singh and Prasad [9], for instance, concurrent vector fields and concircular vector fields are semi parallel. The purpose of the present paper is to introduce Normalized Semi Parallel $T'$-vector field in Finsler space and to study the properties of some special Finsler spaces with this vector field. For instance, there in no such vector field in non-Riemannian C-reducible Finsler space. The notations and terminologies are referred to the monograph [5].

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1. Introduction

Let $M^n$ be an $n(\geq 2)$ dimensional Finsler space endowed with a fundamental function $L = L(x, y)$, where $x = (x^i)$ is a point and $y = (y^i)$ is a supporting element of $M^n$. The metric tensor $g_{ij}$, angular metric tensor $h_{ij}$ and (h)hvtorsion tensor $C_{ijk}$ of $M^n$ are respectively given by
\[ g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad h_{ij} = L \frac{\partial^2 L}{\partial y^i \partial y^j} \text{ and } C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}. \]

So that in terms of normalized supporting element \( l_i = \frac{g_{ij} y^j}{L} \), the angular metric tensor can be written as \( h_{ij} = g_{ij} - l_i l_j \). The T-tensor \( T_{ijkh} \) of \( M^n \) is defined as \[ T_{ijkh} = LC_{ijk}|_h + C_{ijk} l_h + C_{jkh} l_i + C_{khil} l_j + C_{hijl} k. \] (1.1)

where the symbol \(|\) means the v-covariant derivative with respect to Cartan connection \( \Gamma \) of \( M^n \). Transvection of (1.1) by the reciprocal metric tensor \( g^{kh} \) of \( g_{kh} \) gives \[ T_{ij} = LC_{i|j} + C_{il} j + C_{jl} i, \] (1.2)

where \( T_{ij}(= g^{kh} T_{ijkh}) \) and \( C_i (= g^{jk} C_{ijk}) \) are called \( T' \)-tensor and the torsion vector of \( M^n \) respectively. If the \( T' \)-tensor \( T_{ij} \) of \( M^n \) vanishes, then \( M^n \) is called Finsler space with \( T' \)-condition. For instance, a \( C^v \)-reducible Finsler space [7] satisfies \( T' \)-condition as well as \( T' \)-condition. If the \( T' \)-tensor \( T_{il} \) of a Finsler space \( M^n \) is written \( T_{ij} = \alpha h_{ij} \), then \( M^n \) is called a Finsler space with semi-\( T' \)-condition [1] and if the \( T' \)-tensor \( T_{ij} \) of a Finsler space \( M^n \) is written as \( T_{ij} = \alpha' h_{ij} - \left( \frac{\beta}{\partial x^i} \right) C_i C_j \), then \( M^n \) is called a Finsler space with quasi-\( T' \)-condition [2]. For instance, a C-reducible Finsler space satisfies semi-\( T' \)-condition and a semi-C-reducible Finsler space with constant coefficient satisfies quasi-\( T' \)-condition. The semi parallelism of vector fields in Finsler spaces has been introduced by Singh and Prasad [9] as follows.

**Definition.** A normalised vector field \( X_i \) in a Finsler space \( M^n \) is said to be semi parallel if:

(a) \( X_i \) is function of coordinate only,

(b) \( C_{ij} X_i = 0 \), and

(c) \( X_{ij} = \rho (g_{ij} - X_i X_j) \), where the symbol \(|\) means the h-covariant derivative with respect to Cartan connection \( \Gamma \) of \( M^n \).

Further Izumi [10] introduced the h-vector field \( v_i \) which is v-covariant constant with respect to \( \Gamma \) and satisfying \( LC_{ij} X_i = \sigma h_{jk} \). Pandey and Diwedi [8] studied normalized semi parallel Ch-vector field \( (X_i) \) satisfying the condition

(a) \( X_i|_j = 0 \), (b) \( LC_{jk} X_i = \alpha h_{jk} + \beta L^2 C_j C_k \) and (c) \( X_{ij} = \rho (g_{ij} - X_i X_j) \).

The purpose of the present paper is to introduce normalized semi parallel \( T' \)-vector field and to study the properties of some special Finsler spaces admitting this field.
Definition 1.1. A normalized vector field $X_i$ in a Finsler space is said to be semi parallel $T'$-vector field if:

(a) $X_i|_j = 0$,
(b) $LC^i_{jk}X_i = T_{jk}$, and
(c) $X_{ij} = \rho(x)(g_{ij} - X_iX_j)$.

Throughout the paper the vector field $X_i$ is assumed to be positively homogeneous of degree zero in $y^i$.

Remark. A normalized semi parallel $T'$-vector field $X_i$, in a Finsler space with $T$ or $T'$-condition, is a normalized semi parallel vector field. A normalized semi parallel $T'$-vector field $X_i$, in a Finsler space with semi-$T'$-condition and quasi-$T'$-condition, is normalized semi parallel $h$-vector field and $Ch$-vector field respectively.

Proposition 1.1. There is no normalized semi parallel $T'$-vector field parallel or perpendicular to line element $l^i$.

Proof. If possible let $X_i = \lambda(x, y)y^i$, where $\lambda$ is homogenous of degree (-1) with respect to $y$. Differentiating v-covariantly with respect to $y^j$ and using condition (a) of definition (1.1), we have $y^i\frac{\partial \lambda}{\partial y^j} + \lambda \delta^i_j = 0$. Summing with respect to $i$ and $j$ using homogenity of $\lambda$, we have $(n - 1)\lambda = 0$, i.e., $\lambda = 0$, which is a contradiction, because $X^i \neq 0$.

Further if we assume $X_iy^i = 0$, then differentiating v-covariantly with respect to $y^j$ and using condition (a) of definition (1.1), we have $X_i = 0$, which is again a contradiction.

Proposition 1.2. The scalar $\rho$ in definition (1.1) is function of position only.

Proof. If possible let $\rho = \rho(x, y)$. Consider the second Ricci identity [5], for normalized semi parallel $T'$-vector field $X_i$:

$$X_{i|j|k} - X_i|_{k|j} = -X_hP^h_{ijk} - X_i|_hC^h_{jk} + X_i|_hF^h_{jk}$$

(1.3)

Using (a) and (c) of definition (1.1) we have

$$\frac{\partial \rho}{\partial y^k}(g_{ij} - X_iX_j) = -X_hP^h_{ijk} - \rho(C_{ijk} - X_iL^{-1}T_{jk})$$

(1.4)

Contracting by $y^j$ and using $P^h_{ijk}y^j = 0$, $T_{jk}y^j = 0$ and $C_{ijk}y^j = 0$, we have $\frac{\partial \rho}{\partial y^k}(y_i - X_0X_i) = 0$. Since $(y_i - X_0X_i) = 0$, contradicts the proposition (1.1), we have $\frac{\partial \rho}{\partial y^k} = 0$, that is, $\rho$ is function of position only.
**Theorem 1.1.** If \( X_i \) be a normalized semi parallel \( T' \)-vector field in a Finsler space \( F^n \) then:

(a) \( X_h S_{ijk}^h = 0 \),

(b) \( X_h P_{ijk}^h = -\rho (C_{ijk} - X_i L^{-1} T_{jk}) \), and

(c) \( X_h R_{ijk}^h = -g_{ij} (\rho_k + \rho^2 X_k) + g_{ik} (\rho_j + \rho^2 X_j) + X_i (\rho_k X_j - \rho_j X_k) \), where \( \rho_k \) stands for \( \rho|_k \).

**Proof.** Consider the third Ricci identity [5]

\[
X_i |j|k - X_i |k|j = -X_h S_{ijk}^h
\]

Using condition (a) of definition (1.1), we have \( X_h S_{ijk}^h = 0 \). Using the fact \( \rho \) is function of position only in equation (1.4), we have

\[
X_h P_{ijk}^h = -\rho (C_{ijk} - X_i L^{-1} T_{jk}).
\]

Finally consider the Ricci first identity

\[
X_i |j|k - X_i |k|j = -X_h R_{ijk}^h - X_i |h R_{jk}^h.
\]

Using conditions (a) and (c) of definition (1.1), we have

\[
X_h R_{ijk}^h = -g_{ij} (\rho_k + \rho^2 X_k) + g_{ik} (\rho_j + \rho^2 X_j) + X_i (\rho_k X_j - \rho_j X_k).
\]

**2. Two and Three Dimensional Finsler Spaces with Normalized Semi Parallel \( T' \)-Vector Field**

In this section we shall consider two and three dimensional Finsler spaces admits normalized semi parallel \( T' \)-vector field. Let \( F^2 \) be a two dimensional Finsler space with Berwald frame \((l^i, m^i)\), where \( l^i \) is the normalised supporting element: \( l^i = \frac{\nu^i}{L} \), \( m^i \) is the normalised torsion vector: \( m^i = \frac{C^i}{\alpha} \), see [5], so that the (h)hv torsion tensor of \( M^2 \) is written as

\[
LC_{ijk} = Im_im_jm_k
\]

where \( I \) is called the main scalar of \( M^2 \). The \( T \)-tensor and \( T' \)-tensor of \( M^2 \) can be written as [5] \( T_{ijkh} = I_2 m_im_jm_km_h \) and \( T_{ij} = I_2 m_im_j = \alpha h_{ij} \), where \( I_2 = I m^j \). Any vector \( X_i \) of \( F^2 \) can be written as \( X_i = X_1 l_i + X_2 m_i \), where \( X_1 = X_i l^i \) and \( X_2 = X_i m^i \). Let \( X_i \) be normalized semi parallel \( T' \)-vector field. Substituting the values of \( LC_{ijk} \) and \( T_{ij} \) in condition (b) of definition (1.1), we have

\[
X_2 = (\log I)_2
\]
where \((\log I)_2 = (\log I)|_m t\). Consider the h-torsion tensor of \(F^2\) (see [5])
\[ R_{ijk} = R_{hijkl}^t = LRm_i(l_jm_k - l_km_j). \] (2.2)

Contracting this equation by \(X^i\), we have
\[ X^iR_{ijk} = R_{hijkl}^t X^i = RX_2(y_jm_k - y_km_j). \] (2.3)

Further contracting equation (c) of theorem (1.1) by \(y^i\) and using proper dummy suffixes, we have
\[ R_{hijkl}^t X^i = y_k(\rho_j + \rho^2X_j) - y_j(\rho_k + \rho^2X_k) + X_0(\rho_kX_j - \rho_jX_k) \] (2.4)

Equating equations (2.3) and (2.5) and contracting by \(l^k m^j\), we have
\[ \rho_2X_2^2 - (\log I)_2[(R + \rho^2) + \rho_1X_1] + \rho_2 = 0. \] (2.5)

**Theorem 2.1.** If \(X_i\) be a normalized semi parallel \(T^i\)-vector field in \(F^2\) then \(X_i = X_1l_i + (\log I)_2m_i\), where \(X_1\) is given by equation (2.5).

Now consider a three dimensional Finsler space \(M^3\) with Moore frame \((l^i, m^i, n^i)\), where \(l^i\) is the normalised supporting element: \(l^i = \frac{y^i}{l}\), \(m^i\) is the normalised torsion vector: \(m^i = \frac{C^i}{I}\) and \(n^i\) constructed by \(g_{ij}l^in^j = 0 = g_{ij}m^in^j\) and \(g_{ij}n^in^j = 1\), so that the \((h)hv\) torsion tensor of \(M^3\) is written as
\[ LC_{ijk} = Hm_im_jm_k - J\pi_{(ijk)}(m_in_jn_k) + l\pi_{(ijk)}(m_in_jn_k) + J(n_in_jn_k) \] (2.6)

where the functions \(H, I\) and \(J\) are main scalars of \(M^3\) satisfying \(LC = H + I\) and the notation \(\pi_{(ijk)}\) indicates cyclic permutation of indices \(i, j, k\) and summation, see [5]. Contraction of (2.6) by \(g^{jk}\) gives
\[ LC_i = (H + I)m_i = LCm_i. \] (2.7)

Differentiation of (2.7) \(v\)-covariantly with respect to \(y^j\) gives
\[ T_{ij} = LC_i\big|j + C_il_j + C_jl_i = (LC)_2m_im_j + Cv_3n_in_j + Cv_2(n_im_j + n_jm_i), \] (2.8)

where \(v_i\) is \(v\)-connection vector of \(M^3\) given by \(v_i = v_1l_i + v_2m_i + v_3n_i\) satisfying \(v_1 = 0\) and \(v_2 = C^{-1}(LC)_3\) (see [5]).

Now any vector \(X_i\) of \(F^3\) can be written as \(X_i = X_1l_i + X_2m_i + X_3n_i\), where \(X_1 = X_1l^i\), \(X_2 = X_im^i\) and \(X_3 = X_in^i\). Let \(X_i\) be normalized semi parallel \(T^i\)-vector field. Substituting the values of \(LC_{ijk}\) and \(T_{ij}\) in condition (b) of definition (1.1), we have
\[ HX_2 - JX_3 - (LC)_2 = 0, \quad -J + IX_3 - Cv_2 = 0 \quad \text{and} \quad IX_2 + JX_3 - Cv_3 = 0. \] (2.9)
Eliminating $X_2$ and $X_3$ from three relations of equations (2.9), we have

$$(LC)_2(J^2 + I^2) + Cv_2(LCJ) + Cv_3(J^2 - HI) = 0. \quad (2.10)$$

In particular, if $v_3 = C^{-1}(LC)_2$ then $Sv_3 + LCJv_2 = 0$, where $S = 2J^2 + I^2 - HI$ is the v-scalar curvature of $F^3$.

**Theorem 2.2.** If a three dimensional Finsler space $F^3$ admitting a normalized semi parallel $T'$-vector field $X_i$, then $(LC)_2(J^2 + I^2) + Cv_2(LCJ) + Cv_3(J^2 - HI) = 0$. In particular if $v_3 = C^{-1}(LC)_2$ then $Sv_3 + LCJv_2 = 0$, where $S = 2J^2 + I^2 - HI$ is the v-scalar curvature of $F^3$.

### 3. Special Finsler Spaces with Normalized Semi Parallel $T'$-Vector Field

In this section we consider the behaviour of some special Finsler spaces, for instance, Landsberg space, Finsler space with scalar curvature, C-reducible space, S-4 like Finsler space admitting a normalized semi parallel $T'$-vector field.

**Definition.** (see [5]) An n dimensional Finsler space $F^n$ is Landsberg space iff its hv-curvature $P_{hijk}$ vanishes.

**Theorem 3.1.** If a Landsberg space $F^n$ admits a normalized semi parallel $T'$-vector field $X_i$ then the scalar $\rho$ vanishes.

**Proof.** Since $F^n$ is Landsberg $P_{hijk} = 0$. From condition (b) theorem(1.1), we have

$$\rho(C_{ijk} - X_iT_{jk}) = 0. \quad (3.1)$$

Contracting above equation by $y^i$, we have $\rho X_0 T_{jk} = 0$ but $X_0 \neq 0$ due to proposition (1.1). If possible let $\rho \neq 0$, then $T_{jk} = 0$. From equation (3.1) we have $\rho C_{ijk} = 0$, which gives $C_{ijk} = 0$ i.e., the space is Riemannian, which is a contradiction. Hence $\rho = 0$.

Now consider a Finsler space with scalar curvature which is characterized by [5]

$$R_{hijk} = -y^iR_{hijk} = \frac{1}{3}L^2(K)_j h_{hk} - K_{kj} h_{hj} + K(y_j h_{hk} - y_k h_{hj}). \quad (3.2)$$

where K is positively homogeneous of degree zero in $y^i$ and called scalar curvature of $F^n$. Contracting above equation by $X^h$ and using condition(c) of
If $F^n$ be a Finsler space with scalar curvature admitting a normalized semi parallel $T'$-vector field $X_i$ then $X_i = \lambda y_i + \mu \rho_i$ provided $L^2(\rho^2 + K) + X_0 \rho_0 \neq 0$ where $\lambda = \frac{(\rho_0 + X_0(K + \rho^2))}{(L^2(\rho^2 + K) + X_0 \rho_0)}$ and $\mu = \frac{(X_0^2 - L^2)}{(L^2(\rho^2 + K) + X_0 \rho_0)}$.

**Theorem 3.2.** If $F^n$ be a Finsler space with scalar curvature admitting a normalized semi parallel $T'$-vector field $X_i$ then $X_i = \lambda y_i + \mu \rho_i$ provided $L^2(\rho^2 + K) + X_0 \rho_0 \neq 0$ where $\lambda = \frac{(\rho_0 + X_0(K + \rho^2))}{(L^2(\rho^2 + K) + X_0 \rho_0)}$ and $\mu = \frac{(X_0^2 - L^2)}{(L^2(\rho^2 + K) + X_0 \rho_0)}$.

Now consider a C-reducible Finsler space, which is characterized as follows.

**Definition.** (see [5],[6]) An $n(n \geq 3)$ dimensional Finsler space is called C-reducible if the tensor $C_{ijk}$ is written in the form

$$C_{ijk} = \frac{1}{n+1} (h_{ij}C_k + h_{jk}C_i + h_{ki}C_j).$$

(3.5)

The $T'$-tensor $T_{jk}$ of a C-reducible Finsler space can be written as [5]

$$T_{jk} = LC_j|k + C_jl_k + C_jl_k = \alpha h_{jk}$$

(3.6)

where $\alpha = \frac{LC_{i|j}}{n-1}$. Substituting the values of $C_{ijk}$ and $T_{jk}$ in equation (b) of definition (1.1), we have $\alpha h_{jk} = \frac{L}{n+1}(Y_j C_k + Y_k C_j + (C_i X^i)h_{jk})$, where $Y_k = h_{jk}X^j$ but from (3.6) and condition (b) of definition(1.1) we have $LX_k C^k = T_{jk} g^{jk} = \alpha(n-1)$ therefore $\alpha h_{jk} = \frac{L}{n+1}(Y_j C_k + Y_k C_j + \frac{\alpha(n-1)}{L} h_{jk})$ i.e., $2\alpha h_{jk} = L(Y_j C_k + Y_k C_j)$. Contrating by $C^j$ we have

$$2\alpha C_k = L[(C_i X_i)C_k + C^2 Y_k]$$

(3.7)

Again contracting by $C^k$, we have

$$(\alpha - L(X_k C^k))C^2 = 0$$

(3.8)

i.e., $\alpha(n-2)C^2 = 0$ but $n \geq 3$. If $\alpha = 0$ then $T_{jk} = 0$ and hence $T$-tensor of C-reducible Finsler space vanishes, which is a Riemannian space, see [7]. Further
if $C^2 = 0$, from equation (3.7), $(n - 3)\alpha C_k = 0$. But a C-reducible Finsler space with $C_k = 0$ or $\alpha = 0$ both reduces to a Riemannian space, therefore we are rest only with $n = 3$. For $n = 3$ from equation (2.10) using $C^2 = 0$, we have $T_{ij} = 0$ and hence $T$-tensor vanishes and therefore the space is again Riemannian. Summarising all, we have

**Theorem 3.3.** There exist no normalized semi parallel $T'$-vector field in a non Riemannian C-reducible Finsler space.

Finally we consider an S-4 like Finsler spaces which is characterized by

**Definition.** (see [5]) An $n(n \geq 5)$ dimensional Finsler space is said to be S-4 like if $\nu$-curvature tensor of $C\Gamma$ can be written in the form

$$S_{hijk} = h_{hj}M_{ik} + h_{ik}M_{hj} - h_{hk}M_{ij} - h_{ij}M_{hk}),$$

where $M_{ij}$ is symmetric and indicatric tensor.

Contracting above equation by $X^k$ and using theorem (1.1)(a), we have

$$M_i h_{hj} + M_{hj} Y_i - M_{ij} Y_h - M_h h_{ij} = 0 \quad (3.10)$$

where $Y_i = h_{ik}X^k$ and $M_i = M_{ij}X^j$. Contracting equation (3.10) by $g^{ij}$, we have $M_h = -\frac{M}{n-3}Y_h$ and by $X^h$, we have

$$M_{ij} = \lambda_1 h_{ij} + \lambda_2 Y_i Y_j. \quad (3.11)$$

where $\lambda_1 = -\frac{M_h X^h}{X^h Y_h}$ and $\lambda_2 = -\frac{2M}{(n-3)X^h Y_h}$. Thus we have

**Theorem 3.4.** If an S-4 like Finsler space $F^n$ admits a normalized semi parallel $T'$-vector field $X^i$ then the indicatric tensor $M_{ij}$ of $F^n$ is given by equation (3.11).

**References**


