International Journal of Pure and Applied Mathematics
Volume 77 No. 4 2012, 459-467
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

ON NORMALIZED SEMI PARALLEL T'-VECTOR FIELD IN FINSLER SPACE

Bankteshwar Tiwari
DST-CIMS, Faculty of Science
Banaras Hindu University
Varanasi, 221005, INDIA

Abstract: The semi-parallel vector field in Riemannian geometry has been introduced by Fulton [3], whereas in Finsler geometry by Singh and Prasad [9], for instance, concurrent vector fields and concircular vector fields are semi parallel. The purpose of the present paper is to introduce Normalized Semi Parallel T'-vector field in Finsler space and to study the properties of some special Finsler spaces with this vector field. For instance, there in no such vector field in non-Riemannian C-reducible Finsler space. The notations and terminologies are referred to the monograph [5].

AMS Subject Classification: 53B40, 53C60
Key Words: Finsler space, semi parallelism, normalized semi parallel vector field, T'-tensor, C-reducible Finsler space

1. Introduction

Let M^n be an $n(\geq 2)$ dimensional Finsler space endowed with a fundamental function $L = L(x, y)$, where $x = (x^i)$ is a point and $y = (y^i)$ is a supporting element of M^n. The metric tensor g_{ij}, angular metric tensor h_{ij} and (h)hv-torsion tensor C_{ijk} of M^n are respectively given by

Received: October 25, 2011

$$g_{ij}=\frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j},\ h_{ij}=L\frac{\partial^2 L}{\partial y^i \partial y^j}\ \textit{and}\ C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

So that in terms of normalized supporting element $l_i = \frac{g_{ij}y^j}{L}$, the angular metric tensor can be written as $h_{ij} = g_{ij} - l_i l_j$. The T- tensor T_{ijkh} of M^n is defined as [5]

$$T_{ijkh} = LC_{ijk}|_h + C_{ijk}l_h + C_{jkh}l_i + C_{khi}l_j + C_{hij}l_k, \tag{1.1}$$

where the symbol $|$ means the v-covariant derivative with respect to Cartan connection $C\Gamma$ of M^n. Transvection of (1.1) by the reciprocal metric tensor g^{kh} of g_{kh} gives

$$T_{ij} = LC_i|_j + C_i l_j + C_j l_i, \tag{1.2}$$

where $T_{ij}(= g^{kh}T_{ijkh})$ and $C_i(= g^{jk}C_{ijk})$ are called T'-tensor and the torsion vector of M^n respectively. If the T'-tensor T_{ij} of M^n vanishes, then M^n is called Finsler space with T'-condition. For instance, a C^v-reducible Finsler space [7] satisfies T-condition as well as T'-condition. If the T'-tensor T_{il} of a Finsler space M^n is written $T_{ij} = \alpha h_{ij}$, then M^n is called a Finsler space with semi-T'-condition [1] and if the T'-tensor T_{ij} of a Finsler space M^n is written as $T_{ij} = \alpha' h_{ij} - (\frac{\beta}{C^2})C_i C_j$, then M^n is called a Finsler space with quasi-T'-condition [2]. For instance, a C-reducible Finsler space satisfies semi-T'-condition and a semi-C-reducible Finsler space with constant coefficient satisfies quasi-T'-condition. The semi parallelism of vector fields in Finsler spaces has been introduced by Singh and Prasad [9] as follows.

Definition. A normalised vector field X_i in a Finsler space M^n is said to be semi parallel if:

(a) X_i is function of coordinate only,

(b) $C^i_{jk}X_i = 0$, and

(c) $X_{i|j} = \rho(g_{ij} - X_i X_j)$, where the symbol $_|$ means the h-covariant derivative with respect to Cartan connection $C\Gamma$ of M^n.

Further Izumi [10] introduced the h-vector field v_i which is v-covariant constant with respect to $C\Gamma$ and satisfying $LC^i_{jk}X_i = \sigma h_{jk}$. Pandey and Diwedi [8] studied normalized semi parallel Ch-vector field (X_i) satisfying the condition (a) $X_i|_j = 0$, (b) $LC^i_{jk}X_i = \alpha h_{jk} + \beta L^2 C_j C_k$ and (c) $X_{i|j} = \rho(g_{ij} - X_i X_j)$.

The purpose of the present paper is to introduce normalized semi parallel T'-vector field and to study the properties of some special Finsler spaces admitting this field.

Definition 1.1. A normalized vector field X_i in a Finsler space is said to be semi parallel T'-vector field if:

(a) $X_i|_j = 0$,

(b) $LC^i_{jk}X_i = T_{jk}$, and

(c) $X_{i|j} = \rho(x)(g_{ij} - X_iX_j)$.

Throughout the paper the vector field X_i is assumed to be positively homogeneous of degree zero in y^i.

Remark. A normalized semi parallel T'-vector field X_i, in a Finsler space with T or T'-condition, is a normalized semi parallel vector field. A normalized semi parallel T'-vector field X_i, in a Finsler space with semi-T'-condition and quasi-T'-condition, is normalized semi parallel h-vector field and Ch-vector field respectively.

Proposition 1.1. *There is no normalized semi parallel T'-vector field parallel or perpendicular to line element l^i.*

Proof. If possible let $X^i = \lambda(x,y)y^i$, where λ is homogenous of degee (-1) with respect to y. Differentiating v-covariantly with respect to y^j and using condition (a) of definition(1.1), we have $y^i\frac{\partial\lambda}{\partial y^j} + \lambda\delta^i_j = 0$. Summing with respect to i and j using homogenity of λ, we have $(n-1)\lambda = 0$, i.e., $\lambda = 0$, which is a contradiction, because $X^i \neq 0$.

Further if we assume $X_iy^i = 0$, then differentiating v-covariantly with respect to y^j and using condition (a) of definition(1.1), we have $X_i = 0$, which is again a contradiction.

Proposition 1.2. *The scalar ρ in definition(1.1) is function of position only.*

Proof. If possible let $\rho = \rho(x,y)$. Consider the second Ricci identity [5], for normalized semi parallel T'-vector field X_i

$$X_{i|j}|_k - X_i|_{k|j} = -X_hP^h_{ijk} - X_{i|h}C^h_{jk} + X_i|_hP^h_{jk} \tag{1.3}$$

Using (a) and (c) of definition(1.1) we have

$$\frac{\partial\rho}{\partial y^k}(g_{ij} - X_iX_j) = -X_hP^h_{ijk} - \rho(C_{ijk} - X_iL^{-1}T_{jk}) \tag{1.4}$$

Contracting by y^j and using $P^h_{ijk}y^j = 0$, $T_{jk}y^j = 0$ and $C_{ijk}y^j = 0$, we have $\frac{\partial\rho}{\partial y^k}(y_i - X_0X_i) = 0$. Since $(y_i - X_0X_i) = 0$, contradicts the proposition(1.1), we have $\frac{\partial\rho}{\partial y^k} = 0$, that is, ρ is function of position only.

Theorem 1.1. *If X_i be a normalized semi parallel T'-vector field in a Finsler space F^n then:*

(a) $X_h S^h_{ijk} = 0$,

(b) $X_h P^h_{ijk} = -\rho(C_{ijk} - X_i L^{-1} T_{jk})$, *and*

(c) $X_h R^h_{ijk} = -g_{ij}(\rho_k + \rho^2 X_k) + g_{ik}(\rho_j + \rho^2 X_j) + X_i(\rho_k X_j - \rho_j X_k)$, *where* ρ_k *stands for* $\rho_{|k}$.

Proof. Consider the third Ricci identity [5]

$$X_i|_j|_k - X_i|_k|_j = -X_h S^h_{ijk}$$

Using condition(a) of definition(1.1), we have $X_h S^h_{ijk} = 0$. Using the fact ρ is function of position only in equation (1.4), we have $X_h P^h_{ijk} = -\rho(C_{ijk} - X_i L^{-1} T_{jk})$. Finally consider the Rici first identity

$$X_{i|j|k} - X_{i|k|j} = -X_h R^h_{ijk} - X_i|_h R^h_{jk}.$$

Using conditions (a)and (c) of definition(1.1), we have

$$X_h R^h_{ijk} = -g_{ij}(\rho_k + \rho^2 X_k) + g_{ik}(\rho_j + \rho^2 X_j) + X_i(\rho_k X_j - \rho_j X_k).$$

2. Two and Three Dimensional Finsler Spaces with Normalized Semi Parallel T'-Vector Field

In this section we shall consider two and three dimensional Finsler spaces admits normalized semi parallel T'-vector field. Let F^2 be a two dimensional Finsler space with Berwald frame (l^i, m^i), where l^i is the normalised supporting element: $l^i = \frac{y^i}{L}$, m^i is the normalised torsion vector: $m^i = \frac{C^i}{C}$, see [5], so that the (h)hv torsion tensor of M^2 is written as

$$LC_{ijk} = I m_i m_j m_k$$

where I is called the main scalar of M^2. The T-tensor and T'-tensor of M^2 can be written as [5] $T_{ijkh} = I_{;2} m_i m_j m_k m_h$ and $T_{ij} = I_{;2} m_i m_j = \alpha h_{ij}$, where $I_{;2} = I|_j m^j$. Any vector X_i of F^2 can be written as $X_i = X_1 l_i + X_2 m_i$, where $X_1 = X_i l^i$ and $X_2 = X_i m^i$. Let X_i be normalized semi parallel T'-vector field. Substituting the values of LC_{ijk} and T_{ij} in condition (b) of definition (1.1), we have

$$X_2 = (\log I)_{;2} \tag{2.1}$$

where $(\log I)_{;2} = (\log I)|_t m^t$. Consider the h-torsion tensor of F^2 (see [5])

$$R_{ijk} = R_{hijk}y^h = LRm_i(l_j m_k - l_k m_j). \tag{2.2}$$

Contracting this equation by X^i, we have

$$X^i R_{ijk} = R_{hijk}y^h X^i = RX_2(y_j m_k - y_k m_j). \tag{2.3}$$

Further contracting equation(c) of theorem(1.1) by y^i and using proper dummy suffixes, we have

$$R_{hijk}y^h X^i = y_k(\rho_j + \rho^2 X_j) - y_j(\rho_k + \rho^2 X_k) + X_0(\rho_k X_j - \rho_j X_k) \tag{2.4}$$

Equating equations (2.3) and (2.5) and contracting by $l^k m^j$, we have

$$\rho_2 X_1^2 - (\log I)_{;2}[(R + \rho^2) + \rho_1 X_1] + \rho_2 = 0. \tag{2.5}$$

Theorem 2.1. *If X_i be a normalized semi parallel T'-vector field in F^2 then $X_i = X_1 l_i + (\log I)_{;2} m_i$, where X_1 is given by equation (2.5).*

Now consider a three dimensional Finsler space M^3 with Moor frame (l^i, m^i, n^i), where l^i is the normalised supporting element: $l^i = \frac{y^i}{L}$, m^i is the normalised torsion vector: $m^i = \frac{C^i}{C}$ and n^i constructed by $g_{ij}l^i n^j = 0 = g_{ij}m^i n^j$ and $g_{ij}n^i n^j = 1$, so that the (h)hv torsion tensor of M^3 is written as

$$LC_{ijk} = Hm_i m_j m_k - J\pi_{(ijk)}(m_i m_j n_k) + I\pi_{(ijk)}(m_i n_j n_k) + J(n_i n_j n_k) \tag{2.6}$$

where the functions H, I and J are main scalars of M^3 satisfying $LC = H + I$ and the notation $\pi_{(ijk)}$ indicates cyclic permutation of indices i,j,k and summation, see [5]. Contraction of (2.6) by g^{jk} gives

$$LC_i = (H + I)m_i = LCm_i. \tag{2.7}$$

Differentiation of (2.7) v-covariantly with respect to y^j gives

$$T_{ij} = LC_i|_j + C_i l_j + C_j l_i = (LC)_{;2}m_i m_j + Cv_3 n_i n_j + Cv_2(n_i m_j + n_j m_i), \tag{2.8}$$

where v_i is v-connection vector of M^3 given by $v_i = v_1 l_i + v_2 m_i + v_3 n_i$ satisfying $v_1 = 0$ and $v_2 = C^{-1}(LC)_{;3}$ (see [5]).

Now any vector X_i of F^3 can be written as $X_i = X_1 l_i + X_2 m_i + X_3 n_i$, where $X_1 = X_i l^i$, $X_2 = X_i m^i$and $X_3 = X_i n^i$. Let X_i be normalized semi parallel T'-vector field. Substituting the values of LC_{ijk} and T_{ij} in condition (b) of definition (1.1), we have

$$HX_2 - JX_3 - (LC)_{;2} = 0, \quad -J + IX_3 - Cv_2 = 0 \ \textit{and}\ IX_2 + JX_3 - Cv_3 = 0. \tag{2.9}$$

Eliminating X_2 and X_3 from three relations of equations(2.9), we have

$$(LC)_{;2}(J^2+I^2)+Cv_2(LCJ)+Cv_3(J^2-HI)=0. \tag{2.10}$$

In particular, if $v_3 = C^{-1}(LC)_{;2}$ then $Sv_3+LCJv_2 = 0$, where $S = 2J^2+I^2-HI$ is the v-scalar curvature of F^3.

Theorem 2.2. *If a three dimensional Finsler space F^3 admitting a normalized semi parallel T'-vector field X_i, then $(LC)_{;2}(J^2+I^2)+Cv_2(LCJ)+Cv_3(J^2-HI)=0$. In particular if $v_3 = C^{-1}(LC)_{;2}$ then $Sv_3+LCJv_2=0$, where $S=2J^2+I^2-HI$ is the v-scalar curvature of F^3.*

3. Special Finsler Spaces with Normalized Semi Parallel T'-Vector Field

In this section we consider the behaviour of some special Finsler spaces, for instance, Landsberg space, Finsler space with scalar curvature, C-reducible space, S-4 like Finsler space admitting a normalized semi parallel T'-vector field.

Definition. (see [5]) An n dimensional Finsler space F^n is Landsberg space iff its hv-curvature P_{hijk} vanishes.

Theorem 3.1. *If a Landsberg space F^n admits a normalized semi parallel T'-vector field X_i then the scalar ρ vanishes.*

Proof. Since F^n is Landsberg $P_{hijk}=0$. From condition (b) theorem(1.1), we have

$$\rho(C_{ijk}-X_iT_{jk})=0. \tag{3.1}$$

Contracting above equation by y^i, we have $\rho X_0T_{jk}=0$ but $X_0 \neq 0$ due to proposition (1.1). If possible let $\rho \neq 0$, then $T_{jk}=0$. From equation (3.1) we have $\rho C_{ijk}=0$, which gives $C_{ijk}=0$ i.e., the space is Riemannian, which is a contradiction. Hence $\rho=0$.

Now consider a Finsler space with scalar curvature which is characterized by [5]

$$R_{hjk} = -y^iR_{hijk} = \frac{1}{3}L^2(K|_jh_{hk}-K|_kh_{hj})+K(y_jh_{hk}-y_kh_{hj}). \tag{3.2}$$

where K is positively homogeneous of degree zero in y^i and called scalar curvature of F^n. Contracting above equatioin by X^h and using condition(c) of therem(1.1), we have

$$\begin{aligned} & y_k(\rho_j + \rho^2 X_j) - y_j(\rho_k + \rho^2 X_k) + X_0(\rho_k X_j - \rho_j X_k) \\ = & \frac{1}{3} L^2(K|_j h_{hk} X^h - K|_k h_{hj} X^h) + K(y_j h_{hk} X^h - y_k h_{hj} X^h) \end{aligned} \tag{3.3}$$

Again contraction by y^k gives

$$(L^2(\rho^2 + K) + X_0\rho_0)X_j - (\rho_0 + X_0(K + \rho^2))y_j + (L^2 - X_0^2)\rho_j = 0 \tag{3.4}$$

i.e., $X_i = \lambda y_i + \mu \rho_i$ provided $L^2(\rho^2 + K) + X_0\rho_0 \neq 0$, where $\lambda = \frac{(\rho_0 + X_0(K+\rho^2))}{(L^2(\rho^2+K)+X_0\rho_0)}$ and $\mu = \frac{(X_0^2 - L^2)}{(L^2(\rho^2+K)+X_0\rho_0)}$.

Theorem 3.2. *If F^n be a Finsler space with scalar curvature admitting a normalized semi parallel T'-vector field X_i then $X_i = \lambda y_i + \mu \rho_i$ provided $L^2(\rho^2 + K) + X_0\rho_0 \neq 0$ where $\lambda = \frac{(\rho_0 + X_0(K+\rho^2))}{(L^2(\rho^2+K)+X_0\rho_0)}$ and $\mu = \frac{(X_0^2 - L^2)}{(L^2(\rho^2+K)+X_0\rho_0)}$.*

Now consider a C-reducible Finsler space, which is characterized as follows.

Definition. (see [5],[6]) An $n(n \geq 3)$ dimensional Finsler space is called C-reducible if the tensor C_{ijk} is written in the form

$$C_{ijk} = \frac{1}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j). \tag{3.5}$$

The T'-tensor T_{jk} of a C-reducible Finsler space can be written as [5]

$$T_{jk} = LC_j|_k + C_j l_k + C_j l_k = \alpha h_{jk} \tag{3.6}$$

where $\alpha = \frac{LC^i|_i}{n-1}$. Substituting the values of C_{ijk} and T_{jk} in equation (b) of definition (1.1), we have $\alpha h_{jk} = \frac{L}{n+1}(Y_j C_k + Y_k C_j + (C_i X^i)h_{jk})$, where $Y_k = h_{jk}X^j$ but from (3.6) and condition (b) of definition(1.1) we have $LX_k C^k = T_{jk}g^{jk} = \alpha(n-1)$ therefore $\alpha h_{jk} = \frac{L}{n+1}(Y_j C_k + Y_k C_j + \frac{\alpha(n-1)}{L} h_{jk})$ i.e., $2\alpha h_{jk} = L(Y_j C_k + Y_k C_j)$. Contrating by C^j we have

$$2\alpha C_k = L[(C_i X^i)C_k + C^2 Y_k] \tag{3.7}$$

Again contracting by C^k, we have

$$(\alpha - L(X_k C^k))C^2 = 0 \tag{3.8}$$

i.e., $\alpha(n-2)C^2 = 0$ but $n \geq 3$. If $\alpha = 0$ then $T_{jk} = 0$ and hence T-tensor of C-reducible Finsler space vanishes, which is a Riemannian space, see [7]. Further if $C^2 = 0$, from equation(3.7), $(n-3)\alpha C_k = 0$. But a C-reducible Finsler space with $C_k = 0$ or $\alpha = 0$ both reduces to a Riemannian space, therefore we are rest only with $n = 3$. For $n = 3$ from equation (2.10) using $C^2 = 0$, we have $T_{ij} = 0$ and hence T-tensor vanishes and therefore the space is again Riemannian. Summarising all, we have

Theorem 3.3. *There exist no normalized semi parallel T'-vector field in a non Riemannian C-reducible Finsler space.*

Finally we consider an S-4 like Finsler spaces which is characterized by

Definition. (see [5]) An $n(n \geq 5)$ dimensional Finsler space is said to be S-4 like if v-curvature tensor of $C\Gamma$ can be written in the form

$$S_{hijk} = h_{hj}M_{ik} + h_{ik}M_{hj} - h_{hk}M_{ij} - h_{ij}M_{hk}), \tag{3.9}$$

where M_{ij} is symmetric and indicatric tensor.

Contracting above equation by X^k and using theorem(1.1)(a), we have

$$M_i h_{hj} + M_{hj}Y_i - M_{ij}Y_h - M_h h_{ij} = 0 \tag{3.10}$$

where $Y_i = h_{ik}X^k$ and $M_i = M_{ij}X^j$. Contracting equation(3.10) by g^{ij}, we have $M_h = -\frac{M}{n-3}Y_h$ and by X^h, we have

$$M_{ij} = \lambda_1 h_{ij} + \lambda_2 Y_i Y_j. \tag{3.11}$$

where $\lambda_1 = -\frac{M_h X^h}{X^h Y_h}$ and $\lambda_2 = -\frac{2M}{(n-3)X^h Y_h}$. Thus we have

Theorem 3.4. *If an S-4 like Finsler space F^n admits a normalized semi parallel T'-vector field X^i then the indicatric tensor M_{ij} of F^n is given by equation (3.11).*

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