International Journal of Pure and Applied Mathematics
Volume 77 No. 4 2012, 579-593
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

GENERALIZED DERIVATIONS OF BIMODULES

Hiroaki Komatsu
Faculty of Computer Science and System Engineering
Okayama Prefectural University
Soja, 719-1197, JAPAN

Abstract: We define generalized derivations of bimodules over noncommutative rings, which extend the notion of derivations, and construct a module which generates all generalized derivations. Using generalized derivations, we characterize separable ring extensions. We also generalize Posner's theorem on the composition of two derivations of a prime ring to a theorem on generalized derivations.

Dedicated to Professor Masami Ito
on his 70th birthday.

AMS Subject Classification: 16W25
Key Words: differential operator, separable extension, Posner's theorem

1. Introduction

Differential operators on a commutative algebra were introduced by R.G. Heyneman and M.E. Sweedler [5], which extend the notion of derivations. M.E. Sweedler [13] introduced right differential operators on a noncommutative algebra, which extend the notion of right derivations. In [7] the author introduced right differential operators on a noncommutative ring extension.

Received: March 8, 2012

In this paper, we introduce the notion of generalized derivations from a bimodule to a bimodule over a ring which extend the notion of derivations from a ring to a bimodule.

Definition 1. Let R and S be rings and let M and N be R-S-bimodules. An additive mapping $f : M \to N$ is called a *generalized derivation* from M to N if it satisfies

$$f(rms) = f(rm)s + rf(ms) - rf(m)s \quad (r \in R,\ m \in M,\ s \in S).$$

We denote by $\mathcal{D}_{R,S}(M,N)$ the set of all generalized derivations from M to N. For an R-bimodule M, we put

$$\mathrm{GDer}(R,M) = \mathcal{D}_{R,R}(R,M).$$

Unlike [5], [7], [13], we do not consider high order types.

In the first half of this paper, we study $\mathrm{GDer}(R,M)$ for a ring R with or without an identity element. All derivations from R to an R-bimodule M belong to $\mathrm{GDer}(R,M)$. We show that elements in $\mathrm{GDer}(R,M)$ are closely related to generalized derivations of several types defined in [2], [9], [11] (Theorem 6).

We also generalize Posner's theorem [12, Theorem 1] on the composition of two derivations of a prime ring, which was already extended to generalized derivations of M. Brešar [2] by B. Hvala [6] and T.K. Lee and W.K. Shiue [8]. Our result is Theorem 8.

In the second half of this paper, We define generalized derivations on ring extensions. When a ring homomorphism from a ring K with an identity element 1 to a ring R with an identity element 1 which sends 1 to 1 is given, we say that R/K is a *unitary ring extension*. For R-S-bimodules M and N over rings R and S, we denote by $\mathrm{Hom}_{R\text{-}S}(M,N)$ the set of all R-S-homomorphisms from M to N.

Definition 2. Let $\mathcal{A} = (R/K,\ S/L)$ be a pair of unitary ring extensions. For unitary R-S-bimodules M and N, we put

$$\mathcal{D}_{\mathcal{A}}(M,N) = \mathcal{D}_{R,S}(M,N) \cap \mathrm{Hom}_{K\text{-}L}(M,N).$$

Also, for an R-bimodule M, we put

$$\mathrm{GDer}_K(R,M) = \mathcal{D}_{R,R}(R,M) \cap \mathrm{Hom}_{K\text{-}K}(R,M).$$

For a given pair $\mathcal{A} = (R/K,\ S/L)$ of unitary ring extensions, we construct the module $\mathcal{J}_{\mathcal{A}}$ which generates all elements in $\mathcal{D}_{\mathcal{A}}(M,N)$ for all unitary R-S-bimodules M and N (Theorem 12). Using $\mathcal{J}_{\mathcal{A}}$, we study the condition that

$$\mathcal{D}_{\mathcal{A}}(M,N) = \mathrm{Hom}_{R\text{-}L}(M,N) + \mathrm{Hom}_{K\text{-}S}(M,N)$$

for all unitary R-S-bimodules M and N. This condition is related to the separability. If R/K and S/L are separable extensions, then the above equalities hold for all unitary R-S-bimodules (Theorem 16). In case of $R/K = S/L$, the converse is also true (Theorem 17).

2. Notations and Elementary Results

Throughout this paper, $\mathbb{Z}$ represents the ring of integers. For left (resp. right) R-modules M and N over a ring R, we denote by $\mathrm{Hom}_R(M,N)$ the set of all R-homomorphisms from M to N.

Definition 3. Let R and S be rings. A module X is called an (R,S)-*quadrimodule* if X is a left R, right R, left S, and right S-module and these four actions commute each other. When R and S have identity elements, an (R,S)-quadrimodule X is said to be *unitary* if X is unitary as R-module and as S-module. An additive mapping from an (R,S)-quadrimodule X to an (R,S)-quadrimodule Y is called an (R,S)-*homomorphism* if it is a left R, right R, left S, and right S-homomorphism. The set of all (R,S)-homomorphisms from X to Y is denoted by $\mathrm{Hom}_{(R,S)}(X,Y)$. A *submodule* and a *factormodule* of an (R,S)-quadrimodule are defined by usual manner.

Definition 4. Let X be an (R,S)-quadrimodule over rings R and S. For $x \in X$, $r \in R$, and $s \in S$, we use the notation

$$\{x,r\} = xr - rx \quad \text{and} \quad \langle x,s\rangle = xs - sx,$$

which play an important role in this paper.

It is well-known that

$$\{x,rr'\} = \{x,r\}r' + r\{x,r'\} \quad \text{and} \quad \langle x,ss'\rangle = \langle x,s\rangle s' + s\langle x,s'\rangle$$

for all $x \in X$, r, $r' \in R$, s, $s' \in S$. For $X' \subseteq X$ and $R' \subseteq R$, we define $\{X',R'\}$ to be the additive subgroup of X generated by the set $\big\{\{x,r\} \mid x \in X',\ r \in R'\big\}$. Similarly, we define $\langle X',S'\rangle$ for $S' \subseteq S$.

Let M and N be R-S-bimodules. As is well-known, $\mathrm{Hom}_{\mathbb{Z}}(M,N)$ is an (R,S)-quadrimodule. Actually, for $f \in \mathrm{Hom}_{\mathbb{Z}}(M,N)$, $r \in R$, $s \in S$, and $m \in M$, we have

$$(rf)(m) = rf(m),\ (fr)(m) = f(rm)$$
$$(sf)(m) = f(ms),\ (fs)(m) = f(m)s,$$

and therefore we can define $\{f,r\}$ and $\langle f,s\rangle$. Obviously, $\{f,R\}=0$ means that f is an R-homomorphism, and $\langle f,S\rangle=0$ means that f is an S-homomorphism. For another R-S-bimodule P and for $f\in\mathrm{Hom}_{\mathbb{Z}}(M,N)$, $g\in\mathrm{Hom}_{\mathbb{Z}}(N,P)$, $r\in R$, $s\in S$, we can see that

$$\{gf,\,r\}=\{g,r\}f+g\{f,r\}\quad\text{and}\quad\langle gf,\,s\rangle=\langle g,s\rangle f+g\langle f,s\rangle. \tag{1}$$

By definition, we have

$$\begin{aligned}\langle\{f,r\},s\rangle(m)&=\{\langle f,s\rangle,r\}(m)\\&=f(rm)s+rf(ms)-rf(m)s-f(rms)\end{aligned}$$

for all $m\in M$. Hence we get the next

Lemma 5. *Let R and S be rings and let M and N be R-S-bimodules. Then, for $f\in\mathrm{Hom}_{\mathbb{Z}}(M,N)$, the following conditions are equivalent:*

1. $f\in\mathcal{D}_{R,S}(M,N)$
2. $\langle\{f,\,R\},\,S\rangle=0.$
3. $\{\langle f,\,S\rangle,\,R\}=0.$

Elements in $\mathrm{GDer}(R,M)$ are closely related to generalized derivations of several types.

Theorem 6. *Let R be a ring, M an R-bimodule, and $f\in\mathrm{Hom}_{\mathbb{Z}}(R,M)$. We consider the following conditions:*

C_1 $f(xyz)=f(xy)z+xf(yz)-xf(y)z\quad(x,\ y,\ z\in R)$,
i.e. $f\in\mathrm{GDer}(R,M)$.

C_2 *There exist* $f',\,f''\in\mathrm{Hom}_{\mathbb{Z}}(R,M)$ *such that*

$$f''(xy)=f(x)y+xf'(y)\quad(x,\ y\in R),$$

i.e. f is a generalized derivation of G.F. Leger and E.M. Luks [9].

C_3 *There exists a derivation $d:R\to M$ such that*

$$f(xy)=f(x)y+xd(y)\quad(x,\ y\in R),$$

i.e. f is a generalized derivation of M. Brešar [2].

C_4 *There exists $m\in M$ such that*

$$f(xy)=f(x)y+xf(y)+xmy\quad(x,\ y\in R),$$

i.e. a pair (f,m) is a generalized derivation of A. Nakajima [11].

Then the following hold:

1. *The implications* $C_4 \Rightarrow C_3 \Rightarrow C_2$ *and the implication* $C_3 \Rightarrow C_1$ *are true.*
2. *If* $\{m \in M \mid mR = 0\} = 0$*, then the implication* $C_2 \Rightarrow C_1$ *is true.*
3. *If* R *has an identity element, then the implication* $C_1 \Rightarrow C_4$ *is true.*

Proof. (1) $C_3 \Rightarrow C_2$ is trivial.

$C_4 \Rightarrow C_3$. The mapping $d : R \to M$ defined by $d(x) = f(x) + mx$ is a derivation and satisfies C_3. This fact was proved in [11].

$C_3 \Rightarrow C_1$. For any x, y, $z \in R$, we see that

$$\begin{aligned} &f(xy)z + xf(yz) - xf(y)z \\ &\quad = (f(x)y + xd(y))z + x(f(y)z + yd(z)) - xf(y)z \\ &\quad = f(x)yz + xd(yz) \\ &\quad = f(xyz). \end{aligned}$$

(2) For any x, y, z, $r \in R$, we see that

$$\begin{aligned} &(f(xyz) - f(xy)z - xf(yz) + xf(y)z)r \\ &\quad = f(xyz)r - f(xy)zr - xf(yz)r + xf(y)zr \\ &\quad = (f''(xyzr) - xyzf'(r)) - (f''(xyzr) - xyf'(zr)) \\ &\qquad - x(f''(yzr) - yzf'(r)) + x(f''(yzr) - yf'(zr)) \\ &\quad = 0. \end{aligned}$$

Hence, f satisfies C_1.

(3) Putting $y = 1$ in C_1, we obtain that $f(xz) = f(x)z + xf(z) - xf(1)z$. □

Example 7. We give an example of $f \in \mathrm{GDer}(R, R)$ which does not satisfy the condition C_2 in Theorem 6. Let R be a vector space with basis $\{e_{ij} \mid i,\ j \in \mathbb{Z}\}$ over a field K. We define the multiplication on R by

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \quad (i,\ j,\ k,\ l \in \mathbb{Z}),$$

where δ_{jk} is the Kronecker delta. Then R becomes a K-algebra. We define $f \in \mathrm{Hom}_K(R, R)$ by

$$f(e_{ij}) = e_{i0} \quad (i,\ j \in \mathbb{Z}).$$

It is easy to see that $f(xy) = xf(y)$ for all x, $y \in R$, and hence $f \in \mathrm{GDer}(R, R)$. Now suppose that f satisfies C_2. Then there exist f', $f'' \in \mathrm{Hom}_{\mathbb{Z}}(R, R)$ such that $f''(xy) = f(x)y + xf'(y)$ for all x, $y \in R$. We can write

$$f'(e_{00}) = \sum_{i,j} \alpha_{ij}e_{ij} \quad (\alpha_{ij} \in K).$$

For any $0 \neq k \in \mathbb{Z}$, we see that

$$0 = f''(e_{0k}e_{00}) = f(e_{0k})e_{00} + e_{0k}f'(e_{00}) = e_{00} + \sum_j \alpha_{kj}e_{0j}.$$

Hence we have $\alpha_{k0} = -1$ for all $0 \neq k \in \mathbb{Z}$, a contradiction.

3. Posner's Theorem on Generalized Derivations

For an R-bimodule M, we denote by $\mathrm{End}({}_RM_R)$ the set of all R-R-endomorphisms of M, by $\mathrm{End}({}_RM)$ the set of all left R-endomorphisms of M, and by $\mathrm{End}(M_R)$ the set of all right R-endomorphisms of M. For a mapping $f : X \to Y$ and $A \subseteq X$, $f|_A$ denotes the restriction mapping of f to A.

Theorem 8. *Let R be a prime ring with extended centroid C and f, $g \in \mathrm{GDer}(R, R)$. Then the following conditions are equivalent:*

1. $gf|_{R^2} \in \mathrm{GDer}(R^2, R)$.
2. $gf|_I \in \mathrm{GDer}(I, R)$ *for some nonzero ideal I of R.*
3. *One of the following conditions holds.*
   - (a) $f \in \mathrm{End}({}_RR_R)$.
   - (b) $g \in \mathrm{End}({}_RR_R)$.
   - (c) $f, g \in \mathrm{End}({}_RR)$.
   - (d) $f, g \in \mathrm{End}(R_R)$.
   - (e) *There exist $\varphi \in \mathrm{End}({}_{RC}RC)$, $\psi \in \mathrm{End}(RC_{RC})$, and $\lambda \in C$ such that $f(x) = (\varphi + \psi)(x)$ and $g(x) = \lambda(\varphi - \psi)(x)$ for all $x \in R$.*
   - (f) *R is of characteristic 2 and there exist λ, $\mu \in C$ such that $g(x) = \lambda f(x) + \mu x$ for all $x \in R$.*

To prove this theorem we need the following two well-known results. For the sake of completeness, we give their proofs.

Lemma 9. *Let R be a prime ring with extended centroid C, I a nonzero ideal of R, and f an additive mapping from R to R or RC. If f is a left (resp. right) I-homomorphism, then f is a left (resp. right) R-homomorphism.*

Proof. We consider the (R, R)-quadrimodules $\mathrm{Hom}_{\mathbb{Z}}(R, R)$ and $\mathrm{Hom}_{\mathbb{Z}}(R, RC)$. Suppose that f is a left I-homomorphism. Then, for any $a \in I$ and $x \in R$, we see that

$$a\{f, x\} = \{f, ax\} - \{f, a\}x = 0,$$

and so $I\{f, R\}(R) = 0$. Hence we have $\{f, R\}(R) = 0$, which means that f is a left R-homomorphism. Similarly, if f is a right I-homomorphism, then f is a right R-homomorphism. □

Lemma 10. *Let R be a ring, $\varphi \in \mathrm{End}({}_RR)$, and $\psi \in \mathrm{End}(R_R)$. Then $\varphi\psi(a) = \psi\varphi(a)$ hold for all $a \in R^2$.*

Proof. We see that

$$\varphi(\psi(xy)) = \varphi(\psi(x)y) = \psi(x)\varphi(y) = \psi(x\varphi(y)) = \psi(\varphi(xy))$$

for all x, $y \in R$. □

Proof of Theorem 8. (1) ⇒ (2) is trivial.

(2) ⇒ (3). We consider the (R, R)-quadrimodule $\mathrm{Hom}_{\mathbb{Z}}(R, R)$. By Lemma 5 and the equations (1) in §2, for any a, $b \in I$, we see that

$$\begin{aligned} \langle\{gf, a\}, b\rangle &= \langle\{g, a\}f + g\{f, a\}, b\rangle \\ &= \langle\{g, a\}, b\rangle f + \{g, a\}\langle f, b\rangle + \langle g, b\rangle\{f, a\} + g\langle\{f, a\}, b\rangle \\ &= \{g, a\}\langle f, b\rangle + \langle g, b\rangle\{f, a\}. \end{aligned} \tag{2}$$

Since $\{f, a\}$ and $\{g, a\}$ belong to $\mathrm{End}(R_R)$ and $\langle f, b\rangle$ and $\langle g, b\rangle$ belong to $\mathrm{End}({}_RR)$, for any $c \in I$ and any x, $y \in R$, we see that

$$\begin{aligned} &\{g, a\}\langle f, b\rangle(xcy) = \{g, a\}(xc\langle f, b\rangle(y)) = \{g, a\}(x)c\langle f, b\rangle(y) \quad \text{and} \\ &\langle g, b\rangle\{f, a\}(xcy) = \langle g, b\rangle(\{f, a\}(x)cy) = \{f, a\}(x)c\langle g, b\rangle(y). \end{aligned}$$

Hence, we have

$$\{g, a\}(x)\, c\, \langle f, b\rangle(y) + \{f, a\}(x)\, c\, \langle g, b\rangle(y) = 0 \quad (a,\ b,\ c \in I,\ x,\ y \in R). \tag{3}$$

First assume that $\{g, I\} = 0$ or $\langle f, I\rangle = 0$. This means that $g \in \mathrm{End}({}_IR)$ or $f \in \mathrm{End}(R_I)$. Hence we have $g \in \mathrm{End}({}_RR)$ or $f \in \mathrm{End}(R_R)$ by Lemma 9. The equation (3) implies that

$$\{f, I\}(R)I\{g, I\}(R) = 0.$$

It follows that $\{f, I\} = 0$ or $\langle g, I\rangle = 0$, and therefore $f \in \mathrm{End}({}_RR)$ or $g \in \mathrm{End}(R_R)$ by Lemma 9. Hence one of conditions (a), (b), (c), and (d) holds.

Next assume that $\{g, I\} \neq 0$ and $\langle f, I\rangle \neq 0$. Then there exist a_0, $b_0 \in I$ and x_0, $y_0 \in R$ such that

$$\{g, a_0\}(x_0) \neq 0 \quad \text{and} \quad \langle f, b_0\rangle(y_0) \neq 0.$$

By the equation (3), we have

$$\{g, a_0\}(x_0)\, c\, \langle f, b_0\rangle(y_0) + \{f, a_0\}(x_0)\, c\, \langle g, b_0\rangle(y_0) = 0 \quad (c \in I). \tag{4}$$

Since the Martindale quotient ring of I coincides with the Martindale quotient ring of R, we can apply [3, Lemma 1] to the equation (4). Hence there exists $\lambda \in C$ such that

$$\{g,\ a_0\}(x_0) = \lambda\{f,\ a_0\}(x_0).$$

By the equation (3), we have

$$\{f,\ a_0\}(x_0)c(\lambda\langle f,\ b\rangle(y) + \langle g, b\rangle(y)) = 0.$$

Hence, we have

$$\langle g, b\rangle(y) = -\lambda\langle f, b\rangle(y) \quad (b \in I,\ y \in R). \tag{5}$$

Again by the equation (3), we have

$$(\{g, a\}(x) - \lambda\{f, a\}(x))c\langle f,\ b_0\rangle(y_0) = 0.$$

Hence, we have

$$\{g, a\}(x) = \lambda\{f, a\}(x) \quad (a \in I,\ x \in R). \tag{6}$$

We define φ_0, $\psi_0 : R \to RC$ by $\varphi_0(x) = g(x) - \lambda f(x)$ and $\psi_0(x) = g(x) + \lambda f(x)$ $(x \in R)$. Then the equation (6) means that $\{\varphi_0,\ I\} = 0$ and the equation (5) means that $\langle \psi_0,\ I\rangle = 0$ in $\mathrm{Hom}_{\mathbb{Z}}(R,\ RC)$. Hence, by Lemma 9, φ_0 is a left R-homomorphism and ψ_0 is a right R-homomorphism. As was shown in the proof of [3, Lemma 5A], φ_0 and ψ_0 can be extended to $\varphi \in \mathrm{End}({}_{RC}RC)$ and $\psi \in \mathrm{End}(RC_{RC})$ respectively. If R is not of characteristic 2, then we have $f = (2\lambda)^{-1}(\psi - \varphi)$ and $g = 2^{-1}(\varphi + \psi)$ on R. If R is of characteristic 2, then we have $\varphi = \psi \in \mathrm{End}({}_{RC}RC_{RC})$. Therefore, by [1, Theorem 2.3.2], there exists $\mu \in C$ such that $\varphi(x) = \mu x$ for all $x \in R$.

(3) $\Rightarrow$ (1). The cases (a), (b), (c), and (d) are trivial.

In case of (e), for any $a \in R^2$, we see that

$$gf(a) = \lambda(\varphi - \psi)(\varphi + \psi)(a) = \lambda(\varphi^2 - \psi^2)(a)$$

by Lemma 10. Since both $\lambda\varphi^2$ and $\lambda\psi^2$ belong to $\mathrm{GDer}(RC, RC)$, we conclude that $gf|_{R^2} \in \mathrm{GDer}(R^2, R)$.

In case of (f), we have $gf(x) = \lambda f^2(x) + \mu f(x)$ $(x \in R)$. Similar to the equation (2), we get

$$\langle\{f^2,\, a\},\, b\rangle = \{f, a\}\langle f, b\rangle + \langle f, b\rangle\{f, a\} \quad (a,\ b \in R)$$

Since R is of characteristic 2, Lemma 10 implies that

$$\langle\{f^2, a\}, b\rangle(R^2) = 0,$$

and so $f^2|_{R^2} \in \mathrm{GDer}(R^2, R)$. Hence the mapping $R^2 \ni a \mapsto \lambda f^2(a) \in RC$ belongs to $\mathrm{GDer}(R^2,\, RC)$. It is clear that the mapping $R^2 \ni a \mapsto \mu f(a) \in RC$ belongs to $\mathrm{GDer}(R^2,\, RC)$. Hence, we have $gf|_{R^2} \in \mathrm{GDer}(R^2,\, R)$. □

4. The module $\mathcal{J}_{\mathcal{A}}$

For a given pair $\mathcal{A} = (R/K,\, S/L)$ of unitary ring extensions, we construct the unitary (R, S)-quadrimodule $\mathcal{J}_{\mathcal{A}}$ which generates all elements in $\mathcal{D}_{\mathcal{A}}(M, N)$ for all unitary R-S-bimodules M and N. We put

$$T_{\mathcal{A}} = R \otimes_K R \otimes_{\mathbb{Z}} S \otimes_L S.$$

Then $T_{\mathcal{A}}$ is a unitary (R, S)-quadrimodule by the following manner:

$$\begin{gathered} r(r' \otimes r'' \otimes s' \otimes s'') = rr' \otimes r'' \otimes s' \otimes s'', \\ (r' \otimes r'' \otimes s' \otimes s'')r = r' \otimes r''r \otimes s' \otimes s'', \\ s(r' \otimes r'' \otimes s' \otimes s'') = r' \otimes r'' \otimes ss' \otimes s'', \\ (r' \otimes r'' \otimes s' \otimes s'')s = r' \otimes r'' \otimes s' \otimes s''s \\ (r,\ r',\ r'' \in R,\ s,\ s',\ s'' \in S). \end{gathered}$$

Let $U_{\mathcal{A}}$ denote the submodule of $T_{\mathcal{A}}$ generated by

$$\langle\{1 \otimes 1 \otimes 1 \otimes 1,\, R\},\, S\rangle$$

as an (R, S)-quadrimodule.

Definition 11. Under above notations, we put

$$\mathcal{J}_{\mathcal{A}} = T_{\mathcal{A}}/U_{\mathcal{A}} \quad \text{and}$$
$$\delta_{\mathcal{A}} = 1 \otimes 1 \otimes 1 \otimes 1 + U_{\mathcal{A}} \quad \text{in} \quad \mathcal{J}_{\mathcal{A}}.$$

Theorem 12. *Let $\mathcal{A} = (R/K,\, S/L)$ be a pair of unitary ring extensions. Then, for any unitary R-S-bimodules M and N, the additive mapping*

$$\Phi : \mathrm{Hom}_{(R,S)}(\mathcal{J}_{\mathcal{A}},\, \mathrm{Hom}_{\mathbb{Z}}(M,N)) \to \mathcal{D}_{\mathcal{A}}(M,N)$$

defined by

$$\Phi(\varphi) = \varphi(\delta_{\mathcal{A}}) \quad (\varphi \in \mathrm{Hom}_{(R,S)}(\mathcal{J}_{\mathcal{A}},\, \mathrm{Hom}_{\mathbb{Z}}(M,N)))$$

is a natural isomorphism.

Proof. Since $\langle\{\delta_{\mathcal{A}}, R\}, S\rangle = 0$, $\varphi \in \mathrm{Hom}_{(R,S)}(\mathcal{J}_{\mathcal{A}},\, \mathrm{Hom}_{\mathbb{Z}}(M,N))$ implies that $\varphi(\delta_{\mathcal{A}}) \in \mathcal{D}_{\mathcal{A}}(M,N)$. Hence Φ is well-defined. Since $\mathcal{J}_{\mathcal{A}}$ is generated by $\delta_{\mathcal{A}}$ as an (R,S)-quadrimodule, Φ is a monomorphism. Let f be an arbitrary element in $\mathcal{D}_{\mathcal{A}}(M,N)$. We define $\tilde{f} \in \mathrm{Hom}_{(R,S)}(T_{\mathcal{A}},\, \mathrm{Hom}_{\mathbb{Z}}(M,N))$ by putting $\tilde{f}(r \otimes r' \otimes s \otimes s') = s(rfr')s'$. Then, for any $r \in R$ and $s \in S$, we see that

$$\begin{aligned}
&\tilde{f}(\langle\{1 \otimes 1 \otimes 1 \otimes 1,\, r\},\, s\rangle) \\
&= \tilde{f}(1 \otimes r \otimes 1 \otimes s - r \otimes 1 \otimes 1 \otimes s - 1 \otimes r \otimes s \otimes 1 + r \otimes 1 \otimes s \otimes 1) \\
&= (fr)s - rfs - sfr + s(rf) \\
&= \langle\{f, r\}, s\rangle \\
&= 0.
\end{aligned}$$

It follows that $\tilde{f}(U_{\mathcal{A}}) = 0$. Hence there exists

$$\varphi \in \mathrm{Hom}_{(R,S)}(\mathcal{J}_{\mathcal{A}},\, \mathrm{Hom}_{\mathbb{Z}}(M,N))$$

such that $\Phi(\varphi) = f$. Thus Φ is an isomorphism. The naturality is easy. □

The S-R-bimodule structure of $\mathcal{J}_{\mathcal{A}}$ induces the R-S-bimodule structure of $\mathrm{Hom}_{R\text{-}S}(\mathcal{J}_{\mathcal{A}},\, N)$ for any R-S-bimodule N.

Corollary 13. *Let $\mathcal{A} = (R/K,\, S/L)$ be a pair of unitary ring extensions. Then, for any unitary R-S-bimodules M and N, the additive mapping*

$$\Psi : \mathrm{Hom}_{R\text{-}S}(M,\, \mathrm{Hom}_{R\text{-}S}(\mathcal{J}_{\mathcal{A}},\, N)) \to \mathcal{D}_{\mathcal{A}}(M,N)$$

defined by

$$\Psi(\psi)(m) = \psi(m)(\delta_{\mathcal{A}}) \quad (\psi \in \mathrm{Hom}_{R\text{-}S}(M, \mathrm{Hom}_{R\text{-}S}(\mathcal{J}_{\mathcal{A}}, N)), \ m \in M)$$

is a natural isomorphism.

Proof. Ψ is the composition of Φ in Theorem 12 and the natural isomorphism

$$\nu : \mathrm{Hom}_{R\text{-}S}(M, \mathrm{Hom}_{R\text{-}S}(\mathcal{J}_{\mathcal{A}}, N)) \to \mathrm{Hom}_{(R,S)}(\mathcal{J}_{\mathcal{A}}, \mathrm{Hom}_{\mathbb{Z}}(M, N))$$

defined by $\nu(\psi)(x)(m) = \psi(m)(x)$ $(x \in \mathcal{J}_{\mathcal{A}}, m \in M)$. □

Let M be a unitary R-S-bimodule. Using the S-R-bimodule structure of $\mathcal{J}_{\mathcal{A}}$, we get $\mathcal{J}_{\mathcal{A}} \otimes_{R\otimes_{\mathbb{Z}} S^\circ} M$ where S° is the opposite ring of S. Furthermore, the R-S-module structure of $\mathcal{J}_{\mathcal{A}}$ induces the R-S-bimodule structure of $\mathcal{J}_{\mathcal{A}} \otimes_{R\otimes_{\mathbb{Z}} S^\circ} M$.

Corollary 14. *Let $\mathcal{A} = (R/K, S/L)$ be a pair of unitary ring extensions. Then, for any unitary R-S-bimodules M and N, the additive mapping*

$$\Theta : \mathrm{Hom}_{R\text{-}S}(\mathcal{J}_{\mathcal{A}} \otimes_{R\otimes_{\mathbb{Z}} S^\circ} M, N) \to \mathcal{D}_{\mathcal{A}}(M, N)$$

defined by

$$\Theta(\theta)(m) = \theta(\delta_{\mathcal{A}} \otimes m) \quad (\theta \in \mathrm{Hom}_{R\text{-}S}(\mathcal{J}_{\mathcal{A}} \otimes_{R\otimes_{\mathbb{Z}} S^\circ} M, N), \ m \in M)$$

is a natural isomorphism.

Proof. Θ is the composition of Φ in Theorem 12 and the natural isomorphism

$$\eta : \mathrm{Hom}_{R\text{-}S}(\mathcal{J}_{\mathcal{A}} \otimes_{R\otimes_{\mathbb{Z}} S^\circ} M, N) \to \mathrm{Hom}_{(R,S)}(\mathcal{J}_{\mathcal{A}}, \mathrm{Hom}_{\mathbb{Z}}(M, N))$$

defined by $\eta(\theta)(x)(m) = \theta(x \otimes m)$ $(x \in \mathcal{J}_{\mathcal{A}}, m \in M)$. □

5. Separability

For any pair $\mathcal{A} = (R/K,\ S/L)$ of unitary ring extensions, it is obvious by definition that

$$\mathcal{D}_{\mathcal{A}}(M,N) \supseteq \mathrm{Hom}_{R\text{-}L}(M,N) + \mathrm{Hom}_{K\text{-}S}(M,N)$$

for all unitary R-S-bimodules M and N. In this section, we study a pair $\mathcal{A}$ which satisfies the following condition $C_{\mathcal{A}}$.

$C_{\mathcal{A}}$ $\mathcal{D}_{\mathcal{A}}(M,N) = \mathrm{Hom}_{R\text{-}L}(M,N) + \mathrm{Hom}_{K\text{-}S}(M,N)$ for all unitary R-S-bimodules M and N.

Lemma 15. *Let $\mathcal{A} = (R/K,\ S/L)$ be a pair of unitary ring extensions. If there exist $\rho,\ \sigma \in \mathcal{J}_{\mathcal{A}}$ such that $\delta_{\mathcal{A}} = \rho + \sigma$ and $\{\rho, R\} = \langle \rho, L\rangle = \{\sigma, K\} = \langle \sigma, S\rangle = 0$, then $\mathcal{A}$ satisfies the condition $C_{\mathcal{A}}$.*

Proof. For any $\varphi \in \mathrm{Hom}_{(R,S)}(\mathcal{J}_{\mathcal{A}},\ \mathrm{Hom}_{\mathbb{Z}}(M,N))$, we can see that $\varphi(\rho) \in \mathrm{Hom}_{R\text{-}L}(M,N)$ and $\varphi(\sigma) \in \mathrm{Hom}_{K\text{-}S}(M,N)$. Therefore, the assertion is immediate from Theorem 12. □

According to Y. Miyashita [10], a unitary ring extension R/K is said to be *separable* if the R-R-homomorphism $R \otimes_K R \ni r \otimes r' \mapsto rr' \in R$ splits. As was mentioned in [10], R/K is separable if and only if there exists $\sum a_i \otimes b_i \in R \otimes_K R$ such that $\sum a_i b_i = 1$ and $\sum r a_i \otimes b_i = \sum a_i \otimes b_i r$ for all $r \in R$.

Theorem 16. *If $\mathcal{A} = (R/K,\ S/L)$ is a pair of separable ring extensions, then $\mathcal{A}$ satisfies the condition $C_{\mathcal{A}}$.*

Proof. There exists $\sum a_i \otimes b_i \in R \otimes_K R$ such that $\sum a_i b_i = 1$ and $\sum r a_i \otimes b_i = \sum a_i \otimes b_i r$ for all $r \in R$, and there exists $\sum c_j \otimes d_j \in S \otimes_L S$ such that $\sum c_j d_j = 1$ and $\sum s c_j \otimes d_j = \sum c_j \otimes d_j s$ for all $s \in S$. We put

$$\rho = \Big(\sum_i a_i \otimes b_i\Big) \otimes 1 \otimes 1 + U_{\mathcal{A}} \quad \text{and}$$
$$\sigma = \Big(1 \otimes 1 - \sum_i a_i \otimes b_i\Big) \otimes \Big(\sum_j c_j \otimes d_j\Big) + U_{\mathcal{A}}$$

in $\mathcal{J}_{\mathcal{A}}$. Then we can see that $\{\rho, R\} = \langle \rho, L\rangle = \{\sigma, K\} = \langle \sigma, S\rangle = 0$. Furthermore, we see that

$$\Big(1 \otimes 1 - \sum_i a_i \otimes b_i\Big) \otimes \Big(1 \otimes 1 - \sum_j c_j \otimes d_j\Big)$$

$$
\begin{aligned}
&= \Big(\sum_i a_i b_i \otimes 1 - \sum_i a_i \otimes b_i\Big) \otimes \Big(1 \otimes \sum_j c_j d_j - \sum_j c_j \otimes d_j\Big) \\
&= \sum_{i,j} a_i (b_i \otimes 1 - 1 \otimes b_i) \otimes (1 \otimes c_j - c_j \otimes 1) d_j \\
&= -\sum_{i,j} a_i \langle \{1 \otimes 1 \otimes 1 \otimes 1,\, b_i\},\, c_j \rangle d_j \\
&\in U_{\mathcal{A}}.
\end{aligned}
$$

Hence we have $\delta_{\mathcal{A}} = \rho + \sigma$. Thus the assertion is immediate from Lemma 15. □

In case of $R/K = S/L$, we obtain the next theorem which generalize [4, Satz 4.2].

Theorem 17. *Let R/K be a unitary ring extension. Then the following conditions are equivalent:*

1. *R/K is separable.*
2. *The pair $\mathcal{A} = (R/K,\ R/K)$ satisfies the condition $C_{\mathcal{A}}$.*
3. *For any unitary R-bimodule M and any $f \in \mathrm{GDer}_K(R, M)$, there exist $u, v \in M^K$ such that $f(r) = ru + vr$ for all $r \in R$, where M^K is the set of all $m \in M$ such that $\alpha m = m\alpha$ for all $\alpha \in K$.*

Proof. (1) ⇒ (2) is a special case of Theorem 16.

(2) ⇒ (3). For a unitary R-bimodule M and $f \in \mathrm{GDer}_K(R, M)$, there exist $g \in \mathrm{Hom}_{R\text{-}K}(R, M)$ and $h \in \mathrm{Hom}_{K\text{-}R}(R, M)$ such that $f = g + h$. We have $f(r) = rg(1) + h(1)r$ for all $r \in R$. It is easy to see that $g(1), h(1) \in M^K$.

(3) ⇒ (1). Let M be a unitary R-bimodule and $d : R \to M$ a K-derivation. Since $d \in \mathrm{GDer}_K(R, M)$, there exist $u, v \in M^K$ such that $d(r) = ru + vr$ for all $r \in R$. Since $d(1) = 0$, we have $v = -u$, and therefore d is an inner K-derivation. Hence, by [4, Satz 4.2], R/K is separable. □

In case of algebras, we obtain the next theorem stronger than Theorem 16.

Theorem 18. *Let R and S be algebras with identity elements over a commutative ring K with an identity element. Put $\mathcal{A} = (R/K,\ S/K)$. If either R or S is a separable K-algebra, then the equalities*

$$\mathcal{D}_{\mathcal{A}}(M, N) = \mathrm{Hom}_R(M, N) + \mathrm{Hom}_S(M, N)$$

hold for all unitary algebra R-S-bimodules M and N.

Proof. By symmetrical viewing, we may suppose that R is a separable K-algebra. Then there exists $\sum a_i \otimes b_i \in R \otimes_K R$ such that $\sum a_i b_i = 1$ and $\sum r a_i \otimes b_i = \sum a_i \otimes b_i r$ for all $r \in R$. In the (R, S)-quadrimodule $\mathrm{Hom}_{\mathbb{Z}}(M, N)$, we see that $f = \sum a_i b_i f = \sum a_i f b_i - \sum a_i \{f, b_i\}$. Since $\{f, b_i\} \in \mathrm{Hom}_S(M, N)$, we have $a_i\{f, b_i\} \in \mathrm{Hom}_S(M, N)$. As an image of R-bimodule homomorphism $R \otimes_K R \ni x \otimes y \mapsto x f y \in \mathrm{Hom}_{\mathbb{Z}}(M, N)$, we get $\sum r a_i f b_i = \sum a_i f b_i r$ for all $r \in R$. Hence $\sum a_i f b_i \in \mathrm{Hom}_R(M, N)$. □

6. Acknowledgment

The author would like to thank the referee for his valuable suggestions.

References

[1] K.I. Beidar, W.S. Martindale III, A.V. Mikhalev, *Rings with Generalized Identities*, Marcel Dekker Inc., New York (1996).

[2] M. Brešar, On the distance of the composition of two derivations to the generalized derivations, *Glasgow Math. J.*, **33** (1991), 89-93.

[3] M. Brešar, Functional identities of degree two, *J. Algebra*, **172** (1995), 690-720.

[4] S. Elliger Über Automorphismen und Derivationen von Ringen, *J. Reine Angew. Math.*, **277** (1975), 155-177.

[5] R.G. Heyneman, M.E. Sweedler, Affine Hopf algebras, I, *J. Algebra*, **13** (1969), 192-241.

[6] B. Hvala, Generalized derivations in rings, *Comm. Algebra*, **26** (1998), 1147-1166.

[7] H. Komatsu, High order Kähler modules of noncommutative ring extensions, *Comm. Algebra*, **29** (2001), 5499-5524.

[8] T.K. Lee, W.K. Shiue, Identities with generalized derivations, *Comm. Algebra*, **29** (2001), 4437-4450.

[9] G.F. Leger, E.M. Luks, Generalized derivations of Lie algebras, *J. Algebra*, **228** (2000), 165-203.

[10] Y. Miyashita, Finite outer Galois theory of non-commutative rings, *J. Fac. Sci. Hokkaido Univ. Ser. I*, **19** (1966), 114-134.

[11] A. Nakajima, On categorical properties of generalized derivations, *Scientiae Math.*, **2** (1999), 345-352.

[12] E. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8** (1957), 1093-1100.

[13] M.E. Sweedler, Right derivations and right differential operators, *Pacific J. Math.*, **86** (1980), 327-360.