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LINEAR INTERPOLATION IN MINKOWSKI SPACE

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Abstract: Linear interpolation have been done on sphere Euclidean using quaternions. In this paper,we have done the linear interpolation on Lorentzian sphere Minkowski space using split quaternions. This interpolation curve is called spherical linear interpolation in Minkowski space (slerp). Slerp forms a great arc on the split quaternion unit Lorentzian sphere. That also yields the shortest possible interpolation path between the two split quaternion on the unit Lorentzian sphere.

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1. Introduction

First proposed by shoemake[7], as an alternative to using matrix representations, Hamilton's quaternions have found favour within the computer animation community as a mean of representing orientations in $3D$.

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Typically, in animating characters, key orientations are set and intermediate orientations need to be computed to create an animation connecting the poses set by orientations.

Shoemake[7] suggested spherical linear interpolation (slerp) as a means for determining the intermediate orientations between two given ones.In computer graphis, slerp is shorth and for spherical linear interpolation, in the context of quaternion interpolation for the purpose of animating $3D$ rotation. It refers to constant speed motion along a unit radius great circle arc, given the ends and an interpolation parameter between 0 and 1. It is called spherical linear because the two quaternion rotations are interpolated uniformly along a geodesic in the surface of the 3-sphere[7].

In this paper, first we investigate interpolation on sphere Euclidean using quaternions and then we propose these interpolation on Lorentzian spheres.So far, interpolations is done on sphere Euclidean. Our aim is do interpolations on Lorentzian spheres using metric Lorentz and split quaternion.

2. Quaternion and Split Quaternion

Definition 1. The algebra H of quaternion is define as the 4-dimensional vector space over R having a basis$\{1, i, j, k\}$ with the following properties[3]

$$\begin{aligned} i^2 &= j^2 = k^2 = -1, \\ ij &= -ji = k, \quad ki = -ik = j, \quad jk = -kj = i. \end{aligned} \tag{1}$$

Form (1) it is clear that H is not commutative and 1 is the identity element of H. It also H is an associative algebra.

$$q = a_0 \cdot 1 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k \in H \ \ (a_0, a_1, a_2, a_3 \in R)$$

we define the conjugate $\overline{q}$ of q as $\overline{q} = a_0 \cdot 1 - a_1 \cdot i - a_2 \cdot j - a_3 \cdot k$. For evry $q = a_0 \cdot 1 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k \in H$ we have

$$q \cdot \overline{q} = \left(a_0^2 + a_1^2 + a_2^2 + a_3^2\right)$$

we define the norm N_q of the quaternion q to be the nonnegative real number $a_0^2 + a_1^2 + a_2^2 + a_3^2$. $N_q = 0$ if and only if $q = 0$.If $q = a_0 \cdot 1 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k$ and $p = b_0 + b_1 \cdot i + b_2 \cdot j + b_3 \cdot k$ be two quaternion and let $r = q * p$, then r is given by

$$r = S_q S_p - g\left(V_q, V_p\right) + S_q V_p + S_p V_q + V_q \wedge V_p,$$

where

$$\begin{aligned}
S_q &= a_{0,}\ S_p = b_0,\ g\left(V_q, V_p\right) = a_1 b_1 + a_2 b_2 + a_3 b_3, \\
V_q &= a_1 \cdot i + a_2 \cdot j + a_3 \cdot k,\ V_p = b_1 \cdot i + b_2 \cdot j + b_3 \cdot k \\
V_q \wedge V_p &= \left(a_2 b_3 - a_3 b_2\right) i + \left(a_3 b_1 - a_1 b_3\right) j + \left(a_1 b_2 - a_2 b_1\right) k.
\end{aligned}$$

The algebra H_1 of quaternion is define as unit quaternion, q, has a norm equal to one, that is

$$\mid q \mid = \mid q^{-1} \mid = 1 \qquad \text{and} \qquad N^2(q) = q^{-1} q = 1$$

Definition 2. The algebra H' of split quaternion is defined as the 4-dimensional vector space over R having a basis $\{1, i, j, k\}$ with the following properties[4, 5]

$$\begin{aligned}
i^2 &= -1\ , j^2 = k^2 = 1 \\
ij &= -ji = k, kj = -jk = i, ki = -ik = j.
\end{aligned} \tag{2}$$

form (2) it is clear that H' is not commutative and 1 is the identity element of H'. It also $H'$is an associative algebra.For

$$q = a_0 \cdot 1 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k \in H' \ \left(a_0, a_1, a_2, a_3 \in R\right)$$

we define the conjugate $\overline{q}$ of q as $\overline{q} = a_0 \cdot 1 - a_1 \cdot i - a_2 \cdot j - a_3 \cdot k \in H'$.For evry $q = a_0 \cdot 1 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k \in H'$ we have

$$q \cdot \overline{q} = \left(a_0^2 + a_1^2 - a_2^2 - a_3^2\right)$$

we define the norm N_q and the inverse q^{-1} of the quaternion respectively the real number $N_q = a_0^2 + a_1^2 - a_2^2 - a_3^2$ and $q^{-1} = \frac{\overline{q}}{N_q}, N_q \neq 0$.If $q = a_0 \cdot 1 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k$ and $p = b_0 + b_1 \cdot i + b_2 \cdot j + b_3 \cdot k$ be two split quaternion and let $r = q * p$, then r is given by

$$r = S_q S_p + g\left(V_q, V_p\right) + S_q V_p + S_p V_q + V_q \wedge V_p,$$

where

$$\begin{aligned}
S_q &= a_{0,}\ S_p = b_0,\ g\left(V_q, V_p\right) = -a_1 b_1 + a_2 b_2 + a_3 b_3, \\
V_q &= a_1 \cdot i + a_2 \cdot j + a_3 \cdot k,\ V_p = b_1 \cdot i + b_2 \cdot j + b_3 \cdot k \\
V_q \wedge V_p &= \left(a_3 b_2 - a_2 b_3\right) i + \left(a_3 b_1 - a_1 b_3\right) j + \left(a_1 b_2 - a_2 b_1\right) k.
\end{aligned}$$

The algebra H_1' of split quaternion is called as unit split quaternion.

3. Some Properties of Minkowski 3-Space

In this section, we give some useful definition and propositions about Minkowski space.[5]

Definition 3. The Lorentz-Minkowski space is the metric space $E_1^3 = \left(R^3, \langle , \rangle\right)$, where the metric $\langle , \rangle$ is given

$$\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3 \qquad u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$$

The metric $\langle , \rangle$ is called the Lorentzian metric.

A vector $v \in E_1^3$ is called
1.spacelike if $\langle v, v\rangle > 0$ or $v = 0$
2.timelike if $\langle v, v\rangle < 0$
3.lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$

We point out that the null vector $v = 0$ is considered of spacelike type although it satisfies $\langle v, v\rangle = 0$. The light-cone of E_1^3 is defined as the set of all lightlike vectors of E_1^3, that is,

$$C = \left\{(x, y, z) \in E_1^3; x^2 + y^2 - z^2 = 0\right\} - \{(0, 0, 0)\}.$$

see figure (3.1) The set of timelike vectors will be denote by τ and it is the following set:

$$\tau = \left\{(x, y, z) \in E_1^3; x^2 + y^2 - z^2 < 0\right\}.$$

Proposition 4. *Two timelike vectors u and v lie in the same timelike cone if and only if $\langle u, v\rangle < 0$*

Theorem 5. *Let u and v two timelike vectors.*

$$|\langle u, v\rangle| \geq \sqrt{-\langle u, u\rangle}\sqrt{-\langle v, v\rangle}$$

and the equality holds if and if u and v are two proportional vectors. In the case that both vectors lie in the same timelike cone, there exists a unique number $\varphi \geq 0$ such that

$$\langle u, v\rangle = -\mid u \mid\mid v \mid \cosh\varphi$$

This number φ is called the hyperbolic angle between u and v.

Corollary 6. *If u,v are two timelike vectors that lie in the timelike cone, then*

$$\mid u + v \mid \geq \mid u \mid + \mid v \mid$$

and the equality holds if and if u and v are proportional.

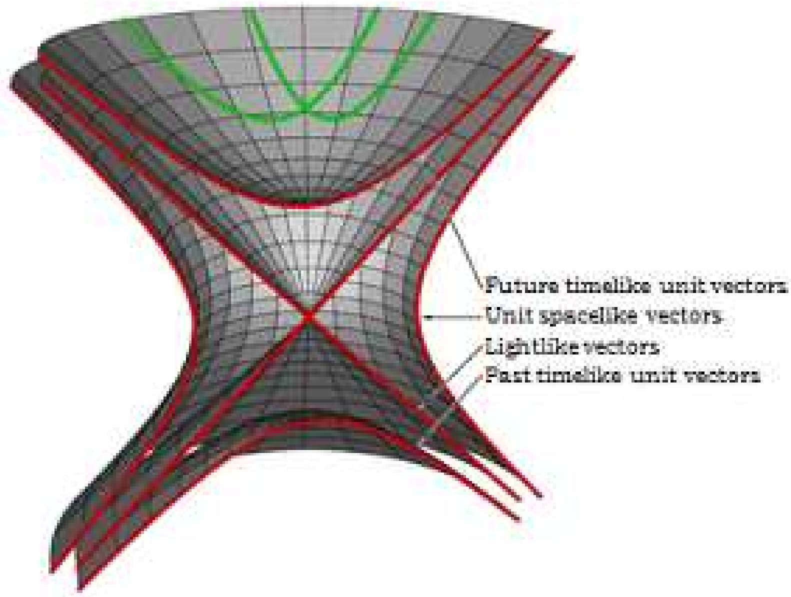

Figure 1: The causal character in Minkowski space

4. Visualizing an Approximation of Angular Velocity

We want to visualize the angular velocity of the interpolation curve. For example it will be interesting to see if some of the interpolation curves have constant angular velocity.In the following, q_i denotes the i'th frame, i.e. the i'th quaternion in a discrete quaternion interpolation curve. To produce a graph of the angular velocity, we must define a function that gives an approximation of the angular velocity. We will base our definition in mathematics, and use that we have defined a norm on quaternions. We can define the distance between two quaternions q_1, q_2 to be $d(q_1, q_2) = \| q_1 - q_2 \|$. Then the angular velocity V in the i'th quaternion q_i can be approximated by the centered average:[1]

$$V(q_i) = \frac{d(q_i, q_{i-1}) + d(q_i, q_{i+1})}{2} = \frac{\| q_i - q_{i-1} \| + \| q_i - q_{i+1} \|}{2}$$

Plotting V as a function of the interpolation parameter yields a graph of an approximation to the angular velocity.

5. Spherical Linear Quaternion Interpolation: Slerp

The interpolation curve for Slerp forms a great arc on the quaternion unit sphere. In differential geometry terms, the great arc is a geodesic corresponding to a straight line. Not only does Slerp follow a great arc it follows the shortest great arc. Thus Slerp yields the shortest possible interpolation path between the two quaternions on the unit sphere. Furthermore Slerp has constant angular velocity. All in all Slerp is the optimal interpolation curve between two rotations.[1]

Proposition 7. *Let $q, q' \in H$. Define q, q' as the corresponding four-dimensional vectors and let α be the angle between them. Then $q \cdot q' = \|q\|\|q'\| \cos\alpha$, see [1].*

Proposition 8. *Let $q = [s, v] \in H_1$.Then there exists $v' \in R^3$ and $\theta \in (-\pi, \pi]$ such that $q = [\cos\theta, v' \sin\theta]$, see [1].*

Proposition 9. *Let $q \in H_1$, $q = [\cos\theta, \sin\theta n]$. Let $r = (x, y, z) \in R^3$ and $p = [0, r] \in H$. Then $p' = qpq^{-1}$ is p rotated 2θ about the axis n.[1]*

Definition 10. The quaternion used for a starting rotation given by p and ending with rotation q, for $p, q \in H_1$, $q = p\left(p^{-1}q\right)^h$. This can be written[1, 7]

$$slerp\,(p, q, h) = p\left(p^{-1}q\right)^h, h \in [0, 1]$$

Remark 11. The quaternion product $p^{-1}q$ can be greatly simplified by use of the fact that, for a unit quaternion $u = [\cos\theta, w\sin\theta]$ and $u^t = [\cos t\theta, \sin t\theta]$. From the definition you can see that $t = 0$ gives rotation p, $t = 1$ the rotation q, and $t \epsilon\, (0, 1)$ gives all intermediate rotations.[1, 6, 7]

Proposition 12. *The curve $slerp\,(p, q, h) : H_1 \times H_1 \times [0, 1] \to H_1$ is a great arc on the unit quaternion sphere between p and q. The position vector function of Slerp has constant angular velocity.[1]*

Proof. To show proposition we must prove that the following four conditions are met :

$$\begin{aligned} slerp\,(p, q, 0) &= p && (3)\\ slerp\,(p, q, 1) &= q && (4)\\ \|slerp\,(p, q, h)\| &= 1, (h \in [0, 1]) && (5)\\ \frac{d^2}{dh^2} slerp\,(p, q, h) &= c\ slerp\,(p, q, h)\,,\ c \leq 0 \in R && (6) \end{aligned}$$

Conditions (3) and (4) are shown directly using the definitions for exp and log Condition (5) is met since exp maps into H_1 and since the norm of a product is the product of the norms

$$\| \, slerp\left(p,q,h\right) \|=\| \, p \, \| \| \left(p^{-1}q\right)^h \|= 1 \, \| \exp\left(h \log\left(p^{-1}q\right)\right) \|= 1$$

To show condition (6), we need the second derivative of Slerp.

$$\begin{aligned} \frac{d}{dt} slerp\left(p,q,h\right) &= \frac{d}{dt} p\left(p^{-1}q\right)^h \\ &= slerp\left(p,q,h\right)\log\left(p^{-1}q\right) \end{aligned} \tag{7}$$

$$\frac{d^2}{dh^2} slerp\left(p,q,h\right) = slerp\left(p,q,h\right)\log\left(p^{-1}q\right)^2$$

Condition (6) holds if $\log\left(p^{-1}q\right)^2$is a non-positive real number. Since $p^{-1}, q \in H_1$, then $p^{-1}, q \in H_1$. there exists $\theta \in R$ and $v \in R^3$, $\mid v \mid= 1$ such that $p^{-1}q = [\cos\theta, \sin\theta v]$. Then:

$$\begin{aligned} \log\left(p^{-1}q\right)^2 &= [0,\theta v]^2 \\ &= \left[-\theta^2 v\cdot v, \theta^2 v\times v\right] \\ &= \left[-\theta^2, 0\right] \end{aligned}$$

Thus $\frac{d^2}{dh^2} slerp\left(p,q,h\right) = c \; slerp\left(p,q,h\right)$ where $c = -\theta^2 \le 0$.

Having shown that $slerp\left(p,q,h\right), h \in [0,1]$ spans a great arc between p and q, there are still two possible curves depending on which direction around the unit sphere Slerp takes. The following proposition states that Slerp behaves as desired. □

Proposition 13. *Let $p, q \in H_1$. Then $slerp\left(p,q,h\right), h \in [0,1]$, spans the shortest great arc between p and q on the unit quaternion sphere.[1]*

Proof. Let $q_{\frac{1}{2}} = slerp\left(p,q,\frac{1}{2}\right)$ and let α denote the angle between p and $q_{\frac{1}{2}}$.Slerp yields the shortest arc if and only if $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This is equivalent to $\cos\left(\alpha\right) \in [0,1]$. We therefore examine the sign of $\cos\left(\alpha\right)$.

$$\begin{aligned} \cos\left(\alpha\right) &= p\cdot q_{\frac{1}{2}} \\ &= p\cdot slerp\left(p,q,\frac{1}{2}\right) \\ &= p\cdot \left(p\left(p^{-1}q\right)^{\frac{1}{2}}\right) \end{aligned}$$

Since $p^{-1}, q \in H_1$ it follows that $p^{-1}q \in H_1$. there exists $w \in R^3$, $\mid w \mid= 1$ and $\psi \in (-\pi, \pi]$ such that $p^{-1}q = [\cos\psi, \sin\psi w]$.

$$\begin{aligned} \cos\alpha &= p \cdot \left(p \left[\cos(\psi), \sin(\psi)\, w \right]^{\frac{1}{2}} \right) \\ &= \cos\left(\frac{\psi}{2}\right) \end{aligned}$$

Now $\psi \in (-\pi, \pi]$ yields $\cos\left(\frac{\psi}{2}\right) \geq 0$ and therefore $\cos(\alpha) \geq 0$. Thus Slerp spans the shortest great arc between p and q. □

A formula for spherical linear interpolation form p to q with parameter h moving from 0 to 1, can be obtained two different ways. From the group structure we find

$$slerp\,(p, q, h) = p\left(p^{-1}q\right)^h \ , \ h \in [0, 1]$$

While from thr $4 - D$ geometry comes

$$slerp\,(q_0, q_1, h) = \frac{q_0 \sin((1-h)\,\theta) + q_1 \sin(h\theta)}{\sin\theta} \tag{8}$$

where $q_0 \cdot q_1 = \cos\theta$. Notice that this expression is not defined for $q_0 = \pm q_1$. The obvious patch is $slerp\,(q, q, h) \equiv q$.

The correctness of the expression above (8) can be shown in the plane. The interpolation between p_0 and p_1 can be written :

$$q(h) = \begin{pmatrix} \cos(v + ht) \\ \sin(v + ht) \end{pmatrix}$$

The expression from (8) can - through applying the addition formulas for sin and cos successively -be written as:

$$\begin{aligned} slerp\,(p_0, p_1, h) &= \frac{p_0 \sin((1-h)\,t) + p_1 \sin(ht)}{\sin(t)} \\ &= \begin{pmatrix} \frac{\cos(v)\sin((1-h)t) + \cos(v+t)\sin(ht)}{\sin(t)} \\ \frac{\sin(v)\sin((1-h)t) + \sin(v+t)\sin(ht)}{\sin(t)} \end{pmatrix} \\ &= \begin{pmatrix} \cos(v)\cos(ht) - \sin(v)\sin(ht) \\ \sin(v)\cos(ht) + \cos(v)\sin(ht) \end{pmatrix} \\ &= \begin{pmatrix} \cos(v + ht) \\ \sin(v + ht) \end{pmatrix} \\ &= q(h) \end{aligned}$$

Thus, the correctness of the expression has been proven in the plane. This result can be generalized directly to four dimensions thereby proving (8) [1, 6, 7]

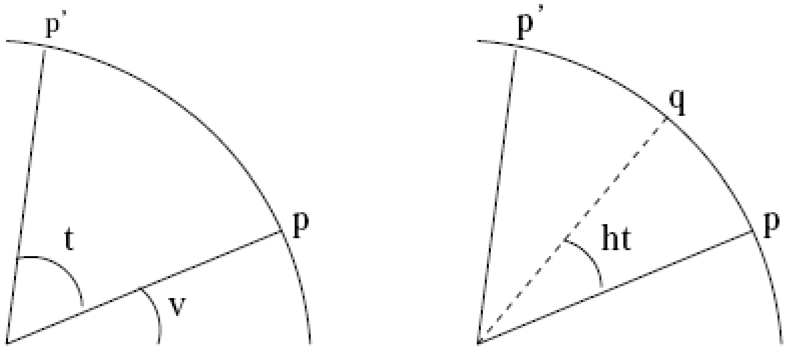

Figure 2: Slerp in the plane: a) The interpolation goes from p to p' across the angle t; b) A step in the interpolation, where ($h \in [0, 1]$), q moves from p to p'

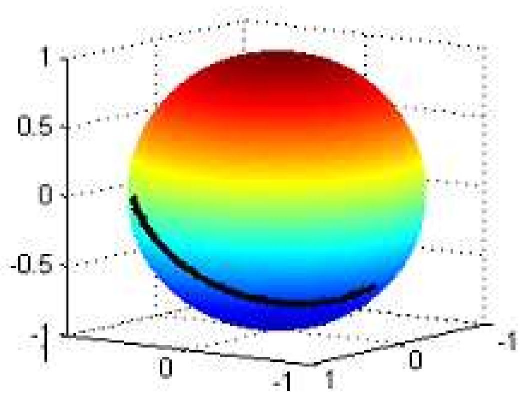

Figure 3: The shapes of interpolation are simulated with MATLAB R2010a: a) Quaternion interpolation between the two key frames in Euclidean space, there are 50 interpolated frames; b) Velocity graph quaternion interpolation.

6. Visualizing an Approximation of Angular Velocity in Minkowski Space

We want to visualize the angular velocity of the interpolation curve in Minkowski space. In the following, q_i denotes the i'th frame, i.e. the i'th split quaternion in a discrete split quaternion interpolation curve. We will base our definition in mathematics, and use that we have defined a norm on split quaternions. We can define the distance between two split quaternions q_1, q_2 to be $d(q_1, q_2) = \| q_1 - q_2 \|$. Then the angular velocity V in the i'th split quaternion q_i can be approximated by the centered average:

$$V(q_i) = \frac{d(q_i, q_{i-1}) + d(q_i, q_{i+1})}{2} = \frac{\| q_i - q_{i-1} \| + \| q_i - q_{i+1} \|}{2}$$

Plotting V as a function of the interpolation parameter yields a graph of an approximation to the angular velocity in Minkowski space.

7. Linear Interpolation in Minkowski Space

In this section, we comput the interpolation on Lorentzian sphere. We have done this interpolations using metric Lorentz and split quaternion.

Lemma 14. *let us express any split quaternion in polar form similar to quaternions and complex numbers.[4]*

(i) *Every spacelike quaternion can be written in the form*

$$q = N_q (\sinh\varphi + \overrightarrow{\varepsilon_0} \cosh\varphi)$$

where $\sinh\varphi = \frac{q_1}{N_q}$, $\cosh\varphi = \frac{\sqrt{-q_2^2+q_3^2+q_4^2}}{N_q}$ *and* $\overrightarrow{\varepsilon_0} = \frac{q_2 i + q_3 j + q_4 k}{\sqrt{-q_2^2+q_3^2+q_4^2}}$ *is a spacelike unit vector in* E_1^3.

(ii) *Every timelike quaternion with spacelike vector part can be written in the form*

$$q = N_q (\cosh\varphi + \overrightarrow{\varepsilon_0} \sinh\varphi)$$

where $\cosh\varphi = \frac{q_1}{N_q}$, $\sinh\varphi = \frac{\sqrt{-q_2^2+q_3^2+q_4^2}}{N_q}$ *and* $\overrightarrow{\varepsilon_0} = \frac{q_2 i + q_3 j + q_4 k}{\sqrt{-q_2^2+q_3^2+q_4^2}}$ *is a spacelike unit vector in* E_1^3 *and* $\overrightarrow{\varepsilon_0} \times \overrightarrow{\varepsilon_0} = 1$.

(iii) Every timelike quaternion with timelike vector part can be written in the form

$$q = N_q (\cos\varphi + \overrightarrow{\varepsilon_0} \sin\varphi)$$

where $\cos\varphi = \frac{q_1}{N_q}$, $\sin\varphi = \frac{\sqrt{q_2^2-q_3^2-q_4^2}}{N_q}$ *and* $\overrightarrow{\varepsilon_0} = \frac{q_2 i + q_3 j + q_4 k}{\sqrt{q_2^2-q_3^2-q_4^2}}$ *is a timelike unit vector in* E_1^3 *and* $\overrightarrow{\varepsilon_0} \times \overrightarrow{\varepsilon_0} = -1$.

Theorem 15. *Let* $q = \cosh\varphi + \overrightarrow{\varepsilon_0} \sinh\varphi$ *be a timelike quaternion with spacelike vector part and* $\overrightarrow{\varepsilon}$ *be a Lorentzian vector. Then the transformation* $R_q = q \times \overrightarrow{\varepsilon} \times q^{-1}$ *is a rotation through hyperbolic angle* 2φ *about the spacelike axis* $\overrightarrow{\varepsilon_0}$*.[4]*

Theorem 16. *Let* $q = \cos\varphi + \overrightarrow{\varepsilon_0}\sin\varphi$ *be a timelike quaternion with timelike vector part and* $\overrightarrow{\varepsilon}$ *be a Lorentzian vector. Then the transformation* $R_q(\varepsilon) = q\times \overrightarrow{\varepsilon} \times q^{-1}$ *is a rotation through* 2φ *about the timelike axis* $\overrightarrow{\varepsilon_0}$*.[4]*

Definition 17. The split quaternion used for a starting rotation given by p and ending with rotation q, for $p, q \in H_1^{\square}$, $q = p\left(p^{-1}q\right)^n$. This can be written

$$slerp\,(p,q,n) = p\left(p^{-1}q\right)^n,\ n \in [0,1]$$

slerp is spherical linear interpolation in Minkowski space.

Remark 18. The split quaternion, $p, q \in H_1^{\square}$, product $p^{-1}q$ can be greatly simplified by use of the fact that, for a unit quaternion $u = [\cosh\varphi, w\sinh\varphi]$ and $u^t = [\cosh(t\varphi), \sinh(t\varphi)]$. From the definition you can see that $t = 0$ gives rotation p, $t = 1$ the rotation q, and $t\epsilon\,(0,1)$ gives all intermediate rotations.

Proposition 19. *The curve* $slerp\,(p,q,n) : H_1^{\square} \times H_1^{\square} \times [0,1] \to H_1^{\square}$ *is a great arc on the unit split quaternion Lorentzian sphere between* p *and* q*.*

Proof. To show proposition we must prove that the following four conditions are met :

$$\begin{aligned}
slerp\,(p,q,0) &= p && (9)\\
slerp\,(p,q,1) &= q && (10)\\
\|slerp\,(p,q,n)\| &= 1, (n \in [0,1]) && (11)\\
\frac{d^2}{dh^2}slerp\,(p,q,n) &= cslerp\,(p,q,), c > 0 \in R\overrightarrow{w} * \overrightarrow{w} = 1 && (12)\\
\frac{d^2}{dh^2}slerp\,(p,q,n) &= cslerp\,(p,q,), c \leq 0 \in R && (13)\\
\overrightarrow{w} * \overrightarrow{w} &= -1
\end{aligned}$$

Conditions(9) and(10) are shown directly using the definitions for exp and log. Condition (11) is met since exp maps into $H_1^{\square}$ and since the norm of a product is the product of the norms

$$\|\ slerp\,(p,q,n)\ \| = \|\ p\ \|\|\ \left(p^{-1}q\right)^n\ \| = 1\ \|\ \exp\left(n\log\left(p^{-1}q\right)\right)\ \| = 1$$

To show condition (12), we need the second derivative of Slerp.

$$\begin{aligned}
\frac{d}{dt}slerp\,(p,q,n) &= \frac{d}{dt}p\left(p^{-1}q\right)^n && (14)\\
&= slerp\,(p,q,n)\log\left(p^{-1}q\right)
\end{aligned}$$

$$\frac{d^2}{dh^2} slerp\,(p,q,n) = slerp\,(p,q,n)\log\left(p^{-1}q\right)^2$$

Condition (12) holds if $\left(p^{-1}q\right)^2$is a positive real number. Since $p^{-1}, q \in H_1^{\square}$, then $p^{-1}, q \in H_1^{\square}$. there exists $\varphi \in R$ and $v \in E_1^3$, $\overrightarrow{w} * \overrightarrow{w} = 1$ $\overrightarrow{w}$ is spacelike unit vector such that $p^{-1}q = [\cosh\varphi, \sinh\varphi w]$. Then:

$$\begin{aligned} \log\left(p^{-1}q\right)^2 &= [0, h\varphi w]^2 \\ &= \left[h^2\varphi^2 v\cdot v, h^2\varphi^2 v\times v\right] \\ &= \left[h^2\varphi^2, 0\right] \end{aligned}$$

Thus $\frac{d^2}{dh^2} slerp\,(p,q,n) = c\ slerp\,(p,q,n)$ where $c = h^2\varphi^2 > 0$

To show condition (13), we need the second derivative of Slerp.

$$\begin{aligned} \frac{d}{dt} slerp\,(p,q,n) &= \frac{d}{dt} p\left(p^{-1}q\right)^n \\ &= slerp\,(p,q,n)\log\left(p^{-1}q\right) \end{aligned} \tag{15}$$

$$\frac{d^2}{dh^2} slerp\,(p,q,n) = slerp\,(p,q,n)\log\left(p^{-1}q\right)^2$$

Condition (13) holds if $\left(p^{-1}q\right)^2$is a non-positive real number. Since $p^{-1}, q \in H_1^{\square}$, then $p^{-1}, q \in H_1^{\square}$. there exists $\varphi \in R$ and $v \in E_1^3$, $\overrightarrow{w} * \overrightarrow{w} = -1$ $\overrightarrow{w}$ is timelike unit vector such that $p^{-1}q = [\cos\varphi, \sin\varphi w]$. Then:

$$\begin{aligned} \log\left(p^{-1}q\right)^2 &= [0, \varphi w]^2 \\ &= \left[\varphi^2 v\cdot v, \varphi^2 v\times v\right] \\ &= \left[-\varphi^2, 0\right] \end{aligned}$$

Thus $\frac{d^2}{dh^2} slerp\,(p,q,n) = c\ slerp\,(p,q,n)$ where $c = \varphi^2 \le 0$. Having shown that $slerp\,(p,q,n)\,, n \in [0,1]$ spans a great arc between p and q. □

Proposition 20. *Let $p, q \in H_1^{\square}$. Then $slerp\,(p,q,n)\,, n \in [0,1]\,,$ spans the shortest great arc between p and q on the unit split quaternion Lorentzian sphere.*

Proof. Let $q_{\frac{1}{2}} = slerp\left(p, q, \frac{1}{2}\right)$ and let α denote the hiperbolic angle between p and $q_{\frac{1}{2}}$.Slerp yields the shortest arc if $\alpha \in R^+$, $\cosh\alpha \ge 0, \cosh\alpha \in [0,\infty)$.

$$\cosh(\alpha) = p\cdot q_{\frac{1}{2}}$$

$$
\begin{aligned}
&= p \cdot slerp\left(p, q, \frac{1}{2}\right) \\
&= p \cdot (p\left(p^{-1}q\right)^{\frac{1}{2}})
\end{aligned}
$$

Since $p^{-1}, q \in H_1^{\square}$ it follows that $p^{-1}q \in H_1^{\square}$. there exists $w \in E_1^3$, $\mid w \mid = 1$ such that $p^{-1}q = [\cosh\varphi, \sinh\varphi w]$. Then

$$
\begin{aligned}
\cosh\alpha &= p \cdot \left(p\left[\cosh\left(\varphi\right), \sinh\left(\varphi\right)w\right]^{\frac{1}{2}}\right) \\
&= p \cdot \left(p \exp\left(\left(\frac{1}{2}\right)\log\left[\cosh\varphi, \sinh\varphi w\right]\right)\right) \\
&= p \cdot \left(p \exp\left(\left[0, h\left(\frac{\varphi}{2}\right)\right]w\right)\right) \\
&= p \cdot \left(p\left[\cosh\left(\frac{\varphi}{2}\right), \sinh\left(\frac{\varphi}{2}\right)w\right]\right) \\
&= (p \cdot [1, 0]) \cdot \left(p\left[\cosh\left(\frac{\varphi}{2}\right), \sinh\left(\frac{\varphi}{2}\right)w\right]\right) \\
&= \parallel p \parallel^2 \left([1, 0] \cdot \left[\cosh\left(\frac{\varphi}{2}\right), \sinh\left(\frac{\varphi}{2}\right)w\right]\right) \\
&= \parallel p \parallel^2 \cosh\left(\frac{\varphi}{2}\right) \\
&= \cosh\left(\frac{\varphi}{2}\right)
\end{aligned}
$$

Now $\cosh\left(\frac{\varphi}{2}\right) \geq 0$ and therefore $\cosh(\alpha) \geq 0$. Thus Slerp spans the shortest great arc between p and q on the unit split quaternion Lorentzian sphere $\square$

A formula for interpolation in Minkowski space form p to q with parameter n moving from 0 to 1, can be obtained two different ways. From the group structure we find

$$
slerp\,(p, q, n) = p\left(p^{-1}q\right)^n, \; n \in [0, 1] \;\;, \; p, q \in H_1^{\square}
$$

While from thr $4 - D$ geometry comes

$$
\begin{aligned}
slerp\,(q_0, q_1, n) &= \frac{q_0 \sinh\left(\left(1 - n\right)\varphi\right) + q_1 \sinh\left(n\varphi\right)}{\sinh(\varphi)}, \qquad (16) \\
p, q &\in H_1^{\square}, \; n \in [0, 1]
\end{aligned}
$$

where $-\langle q_0, q_1\rangle = \cosh\varphi$. Slerp is spherical linear interpolation in Minkowski space.

The correctness of the expression above(16) can be shown in the plane. The interpolation between p_0 and p_1 can be written :

$$q(n) = \begin{pmatrix} \cosh(v + nt) \\ \sinh(v + nt) \end{pmatrix}$$

The expression from (16) can - through applying the addition formulas for sinh and cosh successively -be written as:

$$\begin{aligned}
(p_0, p_1, n) &= \frac{p_0 \sinh((1-n)t) + p_1 \sinh(nt)}{\sinh(t)} \\
&= \begin{pmatrix} \frac{\cosh(v)\sinh((1-n)t)+\cosh(v+t)\sinh(nt)}{\sinh(t)} \\ \frac{\sinh(v)\sinh((1-n)t)+\sinh(v+t)\sinh(nt)}{\sinh(t)} \end{pmatrix} \\
&= \begin{pmatrix} \cosh(v)\cosh(nt) + \sinh(v)\sinh(nt) \\ \sinh(v)\cosh(nt) + \cosh(v)\sinh(nt) \end{pmatrix} \\
&= \begin{pmatrix} \cosh(v + nt) \\ \sinh(v + nt) \end{pmatrix} \\
&= q(n)
\end{aligned}$$

Thus, the correctness of the expression has been proven in the plane. This result can be generalized directly to four dimensions thereby proving (16).

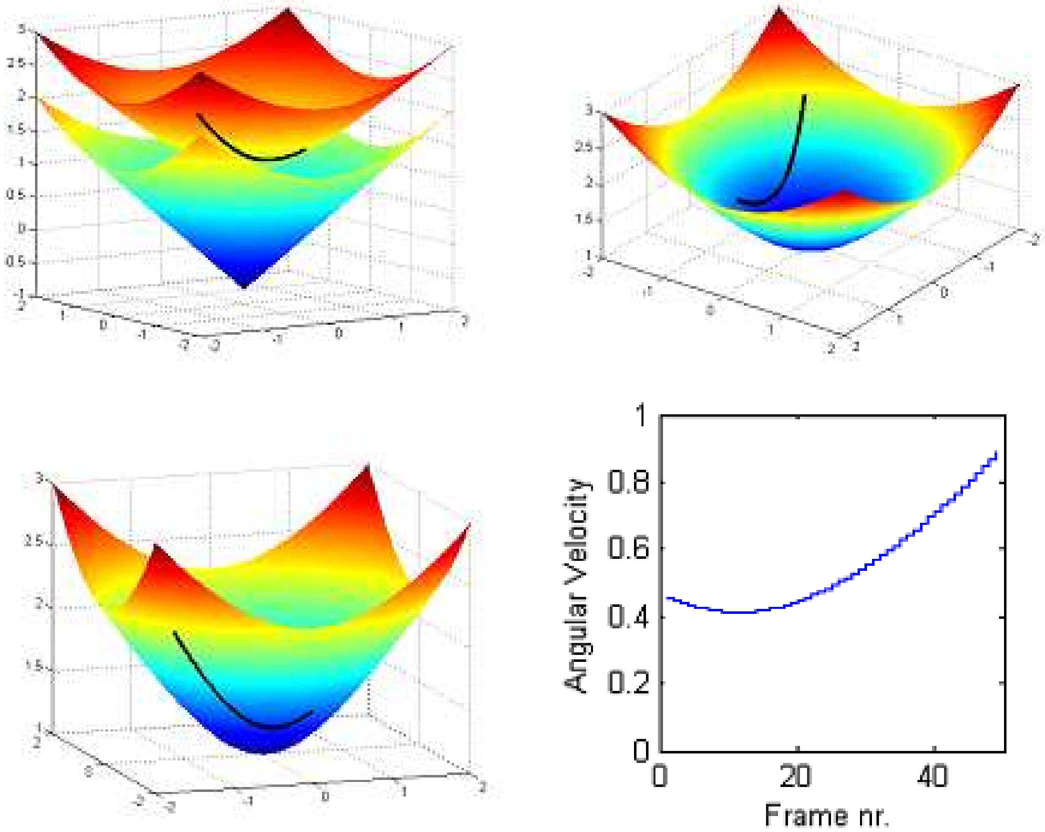

Figure 4: The shapes of interpolation are simulated with MATLAB R2010a: a) Quaternion interpolation between the two key frames in Minkowski space, there are 50 interpolated frames; b) Inside scope; c) Outside scope, d) Velocity graph split quaternion interpolation.

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