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ON (ε)-PARA SASAKIAN 3-MANIFOLDS

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Abstract: In this paper we study the 3-dimensional (ε)-para Sasakian manifolds. We obtain a necessary and sufficient condition for an (ε)-para Sasakian 3-manifold to be an indefinite space form. We show that a Ricci-semi-symmetric (ε)-para Sasakian 3-manifold is an indefinite space form. We investigate the necessary and sufficient condition for an (ε)-para Sasakian 3-manifold to be locally φ-symmetric. It is proved that in an (ε)-para Sasakian 3-manifold with η-parallel Ricci tensor the scalar curvature is constant. It is also shown that every (ε)-para Sasakian 3-manifolds is pseudosymmetric in the sense of R. Deszcz.

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1. Introduction

In 1976, Sāto [25] introduced a structure (φ, ξ, η) satisfying $\varphi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold, which is now well known as an almost paracontact structure. The structure is an analogue of the almost contact structure [24, 5] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, T. Takahashi [27] introduced almost contact manifolds equipped with associated pseudo-Riemannian metrics. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as (ε)-almost contact metric manifolds and (ε)-Sasakian manifolds respectively [2, 14, 15]. Also, in 1989, K. Matsumoto [18] replaced the structure vector field ξ by $-\xi$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold. In a Lorentzian almost paracontact manifold given by Matsumoto, the semi-Riemannian metric has only index 1 and the structure vector field ξ is always timelike. These circumstances motivated the authors in [32] to associate a semi-Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, and they called this indefinite almost paracontact metric structure an (ε)-almost paracontact structure, where the structure vector field ξ is spacelike or timelike according as $\varepsilon = 1$ or $\varepsilon = -1$.

In [32] the authors studied (ε)-almost paracontact manifolds, and in particular, (ε)-para Sasakian manifolds. They gave basic definitions, some examples of (ε)-almost paracontact manifolds and introduced the notion of an (ε)-para Sasakian structure. The basic properties, some typical identities for curvature tensor and Ricci tensor of the (ε)-para Sasakian manifolds were also studied in [32]. The authors in [32] proved that if a semi-Riemannian manifold is one of flat, proper recurrent or proper Ricci-recurrent, then it can not admit an (ε)-para Sasakian structure. Also they showed that, for an (ε)-para Sasakian manifold, the conditions of being symmetric, semi-symmetric or of constant sectional curvature are all identical.

In this paper we study 3-dimensional (ε)-para Sasakian manifolds. The paper organized as follows. Section 2 is devoted to the some basic definitions and curvature properties of (ε)-para Sasakian manifolds. In Section 2, we also prove that an (ε)-para Sasakian manifold is an indefinite space form if and only if the scalar curvature r of the manifold is equal to -6ε. In Section 3, we show that a Ricci-semi-symmetric (ε)-para Sasakian 3-manifold is an indefinite space form. In Section 4, a necessary and sufficient condition for an (ε)-para Sasakian 3-manifold to be locally φ-symmetric is obtained. Section 5 contains some results on (ε)-para Sasakian 3-manifolds with η-parallel Ricci tensor. In last Section 6, it is shown that every (ε)-para Sasakian 3-manifolds is pseudosymmetric in the sense of R. Deszcz.

2. Preliminaries

Let M be an n-dimensional almost paracontact manifold [25] equipped with an almost paracontact structure (φ, ξ, η) consisting of a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = I - \eta \otimes \xi, \tag{1}$$

$$\eta(\xi) = 1, \tag{2}$$

$$\varphi\xi = 0, \tag{3}$$

$$\eta \circ \varphi = 0. \tag{4}$$

Throughout this paper we assume that $X, Y, Z, U, V, W \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields in M, unless specifically stated otherwise. By a semi-Riemannian metric [23] on a manifold M, we understand a non-degenerate symmetric tensor field g of type $(0,2)$. In particular, if its index is 1, it becomes a Lorentzian metric [1]. Let g be a semi-Riemannian metric with index$(g) = \nu$ in an n-dimensional almost paracontact manifold M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \tag{5}$$

where $\varepsilon = \pm 1$. Then M is called an (ε)-almost paracontact metric manifold equipped with an (ε)-almost paracontact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$ [32]. In particular, if index$(g) = 1$, then an (ε)-almost paracontact metric manifold will be called a Lorentzian almost paracontact manifold. In particular, if the metric g is positive definite, then an (ε)-almost paracontact metric manifold is the usual almost paracontact metric manifold [25].

The equation (5) is equivalent to

$$g(X, \varphi Y) = g(\varphi X, Y) \tag{6}$$

along with

$$g(X,\xi) = \varepsilon\eta(X). \tag{7}$$

From (7) it follows that

$$g(\xi,\xi) = \varepsilon, \tag{8}$$

that is, the structure vector field ξ is never lightlike. Defining

$$\Phi(X,Y) \equiv g(X,\varphi Y), \tag{9}$$

we note that

$$\Phi(X,\xi) = 0. \tag{10}$$

Let $(M,\varphi,\xi,\eta,g,\varepsilon)$ be an (ε)-almost paracontact metric manifold (resp. a Lorentzian almost paracontact manifold). If $\varepsilon = 1$, then M will be said to be a spacelike (ε)-almost paracontact metric manifold (resp. a spacelike Lorentzian almost paracontact manifold). Similarly, if $\varepsilon = -1$, then M will be said to be a timelike (ε)-almost paracontact metric manifold (resp. a timelike Lorentzian almost paracontact manifold) [32]. Note that a timelike Lorentzian almost paracontact structure is a Lorentzian almost paracontact structure in the sense of Mihai and Rosca [20, 19], which differs in the sign of the structure vector field of the Lorentzian almost paracontact structure given by Matsumoto [18].

An (ε)-almost paracontact metric structure is called an (ε)-para Sasakian structure if

$$(\nabla_X\varphi)Y = -g(\varphi X,\varphi Y)\xi - \varepsilon\eta(Y)\varphi^2 X, \tag{11}$$

where ∇ is the Levi-Civita connection with respect to g. A manifold endowed with an (ε)-para Sasakian structure is called an (ε)-para Sasakian manifold [32]. In an (ε)-para Sasakian manifold we have [32]

$$\nabla\xi = \varepsilon\varphi, \tag{12}$$

$$\Phi(X,Y) = g(\varphi X,Y) = \varepsilon g(\nabla_X\xi,Y) = (\nabla_X\eta)Y. \tag{13}$$

An (ε)-almost paracontact metric manifold is called η-Einstein if its Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{14}$$

The k-nullity distribution $N(k)$ of a semi-Riemannian manifold M is defined by

$$N(k) : p \quad\rightarrow\quad N_p(k) = \{Z \in T_pM :$$

$$R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}, \tag{15}$$

for all $X, Y \in \chi(M)$, where k is some smooth function (see [29]). If M is an η-Einstein (ε)-para Sasakian manifold and the structure vector field ξ belongs to the k-nullity distribution $N(k)$ for some smooth function k, then we say that M is an $N(k)$-η-Einstein (ε)-para Sasakian manifold (see [31]).

In an (ε)-para Sasakian manifold, the Riemann curvature tensor R and the Ricci tensor S satisfy the following equations [32]:

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{16}$$

$$R(X,Y,Z,\xi) = -\eta(X)g(Y,Z) + \eta(Y)g(X,Z), \tag{17}$$

$$\eta(R(X,Y)Z) = -\varepsilon\eta(X)g(Y,Z) + \varepsilon\eta(Y)g(X,Z), \tag{18}$$

$$R(\xi,X)Y = -\varepsilon g(X,Y)\xi + \eta(Y)X, \tag{19}$$

$$S(X,\xi) = -(n-1)\eta(X). \tag{20}$$

It is known that in a semi-Riemannian 3-manifold

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}\left(g(Y,Z)X - g(X,Z)Y\right), \end{aligned} \tag{21}$$

where Q is the Ricci operator and r is the scalar curvature of the manifold. If we substitute Z by ξ in (21) and use (16), we get

$$\varepsilon(\eta(Y)QX - \eta(X)QY) = \left(1 + \frac{\varepsilon r}{2}\right)(\eta(Y)X - \eta(X)Y). \tag{22}$$

By putting $Y = \xi$ in (22) and using (2) and (20) for $n = 3$, we obtain

$$QX = \frac{1}{2}\{(r+2\varepsilon)X - (r+6\varepsilon)\eta(X)\xi\},$$

that is,

$$S(X,Y) = \frac{1}{2}\{(r+2\varepsilon)g(X,Y) - \varepsilon(r+6\varepsilon)\eta(X)\eta(Y)\}. \tag{23}$$

By using (23) in (21), we obtain

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2} + 2\varepsilon\right)\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad -\left(\frac{r}{2} + 3\varepsilon\right)\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\varepsilon\eta(Y)\eta(Z)X - \varepsilon\eta(X)\eta(Z)Y\}. \end{aligned} \tag{24}$$

If an (ε)-para Sasakian manifold is a space of constant curvature then it is an indefinite space form.

Lemma 1. *An* (ε)*-para Sasakian* 3*-manifold is an indefinite space form if and only if the scalar curvature* $r = -6\varepsilon$*.*

Proof. Let a 3-dimensional (ε)-para Sasakian manifold be an indefinite space form. Then

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \qquad X,Y,Z \in \chi(M), \tag{25}$$

where c is the constant curvature of the manifold. By using the definition of Ricci curvature and (25) we have

$$S(X,Y) = 2c\, g(X,Y). \tag{26}$$

If we use (26) in the definition of the scalar curvature we get

$$r = 6\, c. \tag{27}$$

From (26) and (27) one can easily see that

$$S(X,Y) = \frac{r}{3}\, g(X,Y). \tag{28}$$

By putting $X = Y = \xi$ in (23) and using (28) we obtain

$$r = -\, 6\, \varepsilon.$$

Conversely, if $r = -6\varepsilon$ then from the equation (24) we can easily see that the manifold is an indefinite space form. This completes the proof.

Theorem 2. *Every* (ε)*-para Sasakian* 3*-manifold is an* $N(-\varepsilon)$*-*η*-Einstein manifold.*

Proof. The proof follows from (23) and (16).

3. Ricci-Semi-Symmetric (ε)-para Sasakian 3-Manifolds

A semi-Riemannian manifold M is said to be Ricci-semi-symmetric [21] if its Ricci tensor S satisfies the condition

$$R(X,Y) \cdot S = 0, \qquad X,Y \in \chi(M), \tag{29}$$

where $R(X,Y)$ acts as a derivation on S.

Let M be a Ricci-semi-symmetric (ε)-para Sasakian 3-manifold. From (29) we have

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \tag{30}$$

If we put $Y = \xi$ and use (19), then we get

$$\begin{aligned} 0 &= \varepsilon g(X,U)S(\xi,V) - \eta(U)S(X,V) \\ &\quad +\varepsilon g(X,V)S(U,\xi) - \eta(V)S(U,X). \end{aligned} \tag{31}$$

By using (20) in (31) we obtain

$$\begin{aligned} 0 &= 2\varepsilon g(X,U)\eta(V) + S(X,V)\eta(U) \\ &\quad +2\varepsilon g(X,V)\eta(U) + S(X,U)\eta(V). \end{aligned} \tag{32}$$

Consider that $\{e_1, e_2, e_3\}$ be an orthonormal basis of the T_pM, $p \in M$. Then by putting $X = U = e_i$ in (32) and taking the summation for $1 \leq i \leq 3$, we have

$$S(\xi,V) + 8\varepsilon\eta(V) + r\eta(V) = 0. \tag{33}$$

Again by using (20) in (33), we get

$$(r + 6\varepsilon)\eta(V) = 0,$$

which gives $r = -6\varepsilon$. This implies, in view of Lemma 1, that the manifold is an indefinite space form.

Therefore, we can state the following

Theorem 3. *A Ricci-semi-symmetric (ε)-para Sasakian 3-manifold is an indefinite space form.*

4. Locally φ-Symmetric (ε)-Para Sasakian 3-Manifolds

Analogous to the notion introduced by Takahashi [28] for Sasakian manifolds, we give the following definition.

Definition 4. An (ε)-para Sasakian manifold is said to be locally φ-symmetric if

$$\varphi^2(\nabla_W R)(X,Y,Z) = 0,$$

for all vector fields X, Y, Z orthogonal to ξ.

Now by taking covariant derivative of (24) with respect to W and using (9) and (10) we have

$$\begin{aligned}(\nabla_W R)(X,Y,Z) \;=\; & \frac{1}{2}(\nabla_W r)\,\{g(Y,Z)X - g(X,Z)Y \\ & -g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi \\ & -\varepsilon\eta(Y)\eta(Z)X + \varepsilon\eta(X)\eta(Z)Y\} \\ & +\left(\frac{r}{2}+3\varepsilon\right)\{-g(Y,Z)\left(\Phi(X,W)\xi + \varepsilon\eta(X)\varphi W\right) \\ & +g(X,Z)\left(\Phi(Y,W)\xi + \varepsilon\eta(Y)\varphi W\right) \\ & -\varepsilon\left(\Phi(Y,W)\eta(Z) + \Phi(Z,W)\eta(Y)\right)X \\ & +\varepsilon\left(\Phi(X,W)\eta(Z) + \Phi(Z,W)\eta(X)\right)Y\},\end{aligned} \tag{34}$$

where $X,Y \in \chi(M)$. Then by taking X, Y, Z orthogonal to ξ and using (1), (3), (4) and (7), from (34) we obtain

$$\varphi^2(\nabla_W R)(X,Y,Z) = \frac{1}{2}(\nabla_W r)\left(g(Y,Z)X - g(X,Z)Y\right). \tag{35}$$

Hence from (35) we can state the following theorem:

Theorem 5. *A 3-dimensional* (ε)*-para Sasakian manifold is locally* φ*-symmetric if and only if the scalar curvature* r *is constant.*

If a 3-dimensional (ε)-para Sasakian manifold is Ricci-semi-symmetric then we have showed that $r = -6\varepsilon$ that is r is constant. Therefore from (35), we have

Theorem 6. *A 3-dimensional Ricci-semi-symmetric* (ε)*-para Sasakian manifold is locally* φ*-symmetric.*

In particular, by taking $Z = \xi$ in (34) we have

$$\begin{aligned}(\nabla_W R)(X,Y,\xi) \;=\; & \left(\frac{\varepsilon r}{2}+3\right)\{-\eta(Y)\Phi(X,W)\xi + \eta(X)\Phi(Y,W)\xi \\ & - \Phi(Y,W)X + \Phi(X,W)Y\}.\end{aligned} \tag{36}$$

Applying φ^2 to the both sides of (36) we get

$$\begin{aligned}\varphi^2(\nabla_W R)(X,Y,\xi) \;=\; & \left(\frac{\varepsilon r}{2}+3\right)\{-\Phi(Y,W)\varphi^2 X \\ & +\Phi(X,W)\varphi^2 Y\}.\end{aligned} \tag{37}$$

If we take X, Y, W orthogonal to ξ in (36) and (37) we have

$$\varphi^2(\nabla_W R)(X,Y,\xi) = (\nabla_W R)(X,Y,\xi).$$

Now we can state the following:

Theorem 7. *Let M be an (ε)-para Sasakian 3-manifold such that*

$$\varphi^2(\nabla_W R)(X,Y,\xi) = 0$$

for all $X, Y, W \in \chi(M)$, orthogonal to ξ. Then M is an indefinite space form.

5. η-Parallel (ε)-Para Sasakian 3-Manifolds

Motivated by the definitions of Ricci η-parallelity for Sasakian manifolds and LP-Sasakian manifolds were given by Kon [16] and Shaikh and De [26], respectively, we give the following

Definition 8. Let M be an (ε)-para Sasakian manifold. If the Ricci tensor S satisfies

$$(\nabla_X S)(\varphi Y, \varphi Z) = 0, \qquad X, Y, W \in \chi(M)\,, \tag{38}$$

then the manifold M is said to be η-parallel.

Proposition 9. *Let M be an (ε)-para Sasakian 3-manifold with η-parallel Ricci tensor. Then the scalar curvature r is constant.*

Proof. From (23) by using (5) and (4)

$$S(\varphi X, \varphi Y) = \left(\frac{r}{2} + \varepsilon\right)(g(X,Y) - \varepsilon\eta(X)\eta(Y))\,. \tag{39}$$

If we take the covariant derivative of (39) with respect to Z and (13), we get

$$\begin{aligned}(\nabla_Z S)(\varphi X, \varphi Y) &= \frac{1}{2}\{(\nabla_Z r)\,(g(X,Y) - \varepsilon\eta(X)\eta(Y)) \\ &\quad -\varepsilon(r + 2\varepsilon)\,(\Phi(X,Z)\eta(Y) + \Phi(Y,Z)\eta(X))\}\,.\end{aligned}$$

Since M is an (ε)-para Sasakian 3-manifold with η-parallel Ricci tensor, then from (38) we have

$$\begin{aligned}0 &= (\nabla_Z r)\{g(X,Y) - \varepsilon\eta(X)\eta(Y)\} \\ &\quad -\varepsilon(r + 2\varepsilon)\{\Phi(X,Z)\eta(Y) + \Phi(Y,Z)\eta(X)\}.\end{aligned} \tag{40}$$

Consider that $\{e_1, e_2, e_3\}$ be an orthonormal basis of the T_pM, $p \in M$. Then by substituting both X and Y by e_i, $1 \le i \le 3$, in (40) and then taking summation over i and using (10) we obtain

$$\nabla_Z r = 0, \qquad Z \in \chi(M)\,.$$

This completes the proof.

In view of Theorem 5 and Proposition 9 we have the following:

Theorem 10. *An (ε)-para Sasakian 3-manifold with η-parallel Ricci tensor is locally φ-symmetric.*

Remark 11. An (ε)-para Sasakian manifold is called Lorentzian para Sasakian manifold if $\varepsilon = -1$ and $\mathrm{index}(g) = 1$. Therefore, some results we obtained in the previous three sections can be considered as a generalization of the some results obtained by the authors in [26].

6. Pseudosymmetric (ε)-Para Sasakian 3-Manifolds

Now, we consider a well known generalization of the concept of an η-Einstein almost paracontact metric manifold in the following

Definition 12. [8] A non-flat n-dimensional Riemannian manifold (M, g) is said to be a *quasi Einstein manifold* if its Ricci tensor S satisfies

$$S = ag + b\eta \otimes \eta \tag{41}$$

or equivalently, its Ricci operator Q satisfies

$$Q = aI + b\eta \otimes \xi \tag{42}$$

for some smooth functions a and b, where η is a nonzero 1-form such that

$$g(X, \xi) = \eta(X), \qquad g(\xi, \xi) = \eta(\xi) = 1 \tag{43}$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the quasi Einstein manifold.

B. Y. Chen and K. Yano [9] defined a Riemannian manifold (M, g) to be of *quasi-constant curvature* if it is conformally flat manifold and its Riemann-Christoffel curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned} R(X,Y,Z,W) &= a\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &\quad + b\{g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &\quad + g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)\}, \end{aligned} \tag{44}$$

for all $X, Y, Z, W \in \chi(M)$, where a, b are some smooth functions and T is a non-zero 1-form defined by

$$g(X, \rho) = T(X), \qquad X \in \chi(M)$$

for a unit vector field ρ. On the other hand, Gh. Vrănceanu [33] defined a Riemannian manifold (M,g) to be of *almost constant curvature* if M satisfies (44). Later on, it was pointed out by A. L. Mocanu [22] that the manifold introduced by Chen and Yano and the manifold introduced by Vrănceanu were identical, as it can be verified that if the curvature tensor R is of the form (44), then the manifold is conformally flat. Thus, a Riemannian manifold is said to be of *quasi-constant curvature* if the curvature tensor R satisfies (44). If $b=0$, then the manifold reduces to a manifold of constant curvature.

Example 13. A manifold of quasi-constant curvature is a quasi Einstein manifold [11, Example 1]. Conversely, a conformally flat quasi Einstein manifold of dimension n $(n>3)$ is a manifold of quasi-constant curvature [12, Theorem 4].

Let (M,g) be a semi-Riemannian manifold with its Levi-Civita connection ∇. A tensor field F of type $(1,3)$ is known to be *curvature-like* provided that F satisfies the symmetric properties of the curvature tensor R. For example, the tensor R_g given by

$$R_g\,(X,Y)\,Z \equiv (X\wedge_g Y)\,Z = g(Y,Z)X - g(X,Z)Y, \tag{45}$$

where $X,Y\in\chi\,(M)$, is a trivial example of a curvature like tensor. Sometimes, the symbol R_g seems to be much more convenient than the symbol $(X\wedge_g Y)\,Z$. For example, a semi-Riemannian manifold (M,g) is of constant curvature c if and only if $R=cR_g$.

It is well known that every curvature-like tensor field F acts on the algebra $\mathcal{T}_s^1\,(M)$ of all tensor fields on M of type $(1,s)$ as a derivation [23, p. 44]:

$$\begin{aligned}(F\cdot P)\,(X_1,...,X_s;Y,X) &= F(X,Y)\,\{P\,(X_1,...,X_s)\}\\ &\quad -\sum_{j=1}^{s} P(X_1,...,F\,(X,Y)\,X_j,...,X_s)\end{aligned}$$

for all $X_1,\ldots,X_s\in\chi\,(M)$, $P\in\mathcal{T}_s^1\,(M)$. The derivative $F\cdot P$ of P by F is a tensor field of type $(1,s+2)$. A semi-Riemannian manifold (M,g) is said to be *semi-symmetric* if $R\cdot R=0$. Obviously, locally symmetric spaces $(\nabla R=0)$ are semi-symmetric. More generally, a semi-Riemannian manifold (M,g) is said to be *pseudo-symmetric* (in the sense of R. Deszcz) [13] if $R\cdot R$ and $R_g\cdot R$ in M are linearly dependent, that is, if there exists a real valued smooth function L on M such that

$$R\cdot R = L\,R_g\cdot R$$

is true on the set

$$U = \left\{ x \in M : R \neq \frac{r}{n(n-1)} R_g \text{ at } x \right\}.$$

A pseudo-symmetric space is said to be *proper* if it is not semi-symmetric. For details we refer to [6, 3].

In the literature, there is also another notion of pseudo-symmetry. A semi-Riemannian manifold (M, g) is said to be *pseudo-symmetric* in the sense of Chaki [7] if

$$\begin{aligned}(\nabla R)(X_1, X_2, X_3, X_4; X) &= 2\omega(X) R(X_1, X_2, X_3, X_4) \\ &\quad + \omega(X_1) R(X, X_2, X_3, X_4) \\ &\quad + \omega(X_2) R((X_1, X, X_3, X_4) \\ &\quad + \omega(X_3) R((X_1, X_2, X, X_4) \\ &\quad + \omega(X_4) R((X_1, X_2, X_3, X),\end{aligned}$$

for all $X_1, X_2, X_3, X_4; X \in \chi(M)$, where ω is a 1-form on (M, g). Of course, both the definitions of pseudo-symmetry for a semi-Riemannian manifold are not equivalent. For example, in contact geometry, every Sasakian space form is pseudo-symmetric in the sense of Deszcz [4, Theorem 2.3], but a Sasakian manifold cannot be pseudo-symmetric in the sense of Chaki [30, Theorem 1]. We assume the pseudo-symmetry always in the sense of Deszcz, unless specifically stated otherwise.

For Riemannian 3-manifolds, the following characterization of pseudosymmetry is known (cf. [17, 10]).

Proposition 14. *A 3-dimensional Riemannian manifold (M, g) is pseudo-symmetric if and only if it is quasi-Einstein, that is, if and only if there exists a 1-form η such that the Ricci tensor field S satisfies $S = ag + b\eta \otimes \eta$ for some smooth functions a and b.*

In view of the above Proposition, we can state the following:

Theorem 15. *Every 3-dimensional η-Einstein (ε)-almost paracontact metric manifold is always pseudo-symmetric. In particular, each 3-dimensional (ε)-para Sasakian manifold is pseudo-symmetric.*

Problem 16. It would be interesting to know whether an (ε)-almost para Sasakian manifold is pseudo-symmetric in the sense of Chaki or not.

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