ON \((\varepsilon)\)-PARA SASAKIAN 3-MANIFOLDS

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Abstract: In this paper we study the 3-dimensional \((\varepsilon)\)-para Sasakian manifolds. We obtain a necessary and sufficient condition for an \((\varepsilon)\)-para Sasakian 3-manifold to be an indefinite space form. We show that a Ricci-semi-symmetric \((\varepsilon)\)-para Sasakian 3-manifold is an indefinite space form. We investigate the necessary and sufficient condition for an \((\varepsilon)\)-para Sasakian 3-manifold to be locally \(\varphi\)-symmetric. It is proved that in an \((\varepsilon)\)-para Sasakian 3-manifold with \(\eta\)-parallel Ricci tensor the scalar curvature is constant. It is also shown that every \((\varepsilon)\)-para Sasakian 3-manifolds is pseudosymmetric in the sense of R. Deszcz.

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1. Introduction

In 1976, Sato [25] introduced a structure \((\varphi, \xi, \eta)\) satisfying \(\varphi^2 = I - \eta \otimes \xi\) and \(\eta(\xi) = 1\) on a differentiable manifold, which is now well known as an almost paracontact structure. The structure is an analogue of the almost contact structure [24, 5] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, T. Takahashi [27] introduced almost contact manifolds equipped with associated pseudo-Riemannian metrics. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as \((\varepsilon)\)-almost contact metric manifolds and \((\varepsilon)\)-Sasakian manifolds respectively [2, 14, 15]. Also, in 1989, K. Matsumoto [18] replaced the structure vector field \(\xi\) by \(-\xi\) in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold. In a Lorentzian almost paracontact manifold given by Matsumoto, the semi-Riemannian metric has only index 1 and the structure vector field \(\xi\) is always timelike. These circumstances motivated the authors in [32] to associate a semi-Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, and they called this indefinite almost paracontact metric structure an \((\varepsilon)\)-almost paracontact structure, where the structure vector field \(\xi\) is spacelike or timelike according as \(\varepsilon = 1\) or \(\varepsilon = -1\).

In [32] the authors studied \((\varepsilon)\)-almost paracontact manifolds, and in particular, \((\varepsilon)\)-para Sasakian manifolds. They gave basic definitions, some examples of \((\varepsilon)\)-almost paracontact manifolds and introduced the notion of an \((\varepsilon)\)-para Sasakian structure. The basic properties, some typical identities for curvature tensor and Ricci tensor of the \((\varepsilon)\)-para Sasakian manifolds were also studied in [32]. The authors in [32] proved that if a semi-Riemannian manifold is one of flat, proper recurrent or proper Ricci-recurrent, then it can not admit an \((\varepsilon)\)-para Sasakian structure. Also they showed that, for an \((\varepsilon)\)-para Sasakian manifold, the conditions of being symmetric, semi-symmetric or of constant sectional curvature are all identical.

In this paper we study 3-dimensional \((\varepsilon)\)-para Sasakian manifolds. The paper organized as follows. Section 2 is devoted to the some basic definitions and curvature properties of \((\varepsilon)\)-para Sasakian manifolds. In Section 2, we also prove that an \((\varepsilon)\)-para Sasakian manifold is an indefinite space form if and only if
the scalar curvature $r$ of the manifold is equal to $-6\varepsilon$. In Section 3, we show that a Ricci-semi-symmetric $(\varepsilon)$-para Sasakian 3-manifold is an indefinite space form. In Section 4, a necessary and sufficient condition for an $(\varepsilon)$-para Sasakian 3-manifold to be locally $\phi$-symmetric is obtained. Section 5 contains some results on $(\varepsilon)$-para Sasakian 3-manifolds with $\eta$-parallel Ricci tensor. In last Section 6, it is shown that every $(\varepsilon)$-para Sasakian 3-manifolds is pseudosymmetric in the sense of R. Deszcz.

2. Preliminaries

Let $M$ be an $n$-dimensional almost paracontact manifold [25] equipped with an almost paracontact structure $(\varphi, \xi, \eta)$ consisting of a tensor field $\varphi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying

\begin{align*}
\varphi^2 &= I - \eta \otimes \xi, \\
\eta(\xi) &= 1, \\
\varphi \xi &= 0, \\
\eta \circ \varphi &= 0.
\end{align*}

Throughout this paper we assume that $X, Y, Z, U, V, W \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields in $M$, unless specifically stated otherwise. By a semi-Riemannian metric [23] on a manifold $M$, we understand a non-degenerate symmetric tensor field $g$ of type $(0,2)$. In particular, if its index is 1, it becomes a Lorentzian metric [1]. Let $g$ be a semi-Riemannian metric with index($g$) = $\nu$ in an $n$-dimensional almost paracontact manifold $M$ such that

\[ g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X) \eta(Y), \]

where $\varepsilon = \pm 1$. Then $M$ is called an $(\varepsilon)$-almost paracontact metric manifold equipped with an $(\varepsilon)$-almost paracontact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$ [32]. In particular, if index($g$) = 1, then an $(\varepsilon)$-almost paracontact metric manifold will be called a Lorentzian almost paracontact manifold. In particular, if the metric $g$ is positive definite, then an $(\varepsilon)$-almost paracontact metric manifold is the usual almost paracontact metric manifold [25].

The equation (5) is equivalent to

\[ g(X, \varphi Y) = g(\varphi X, Y) \]
along with
\[ g(X, \xi) = \varepsilon \eta(X). \]  \hspace{1cm} (7)

From (7) it follows that
\[ g(\xi, \xi) = \varepsilon, \]  \hspace{1cm} (8)

that is, the structure vector field \( \xi \) is never lightlike. Defining
\[ \Phi(X, Y) \equiv g(X, \varphi Y), \]  \hspace{1cm} (9)

we note that
\[ \Phi(X, \xi) = 0. \]  \hspace{1cm} (10)

Let \((M, \varphi, \xi, \eta, g, \varepsilon)\) be an \((\varepsilon)\)-almost paracontact metric manifold (resp. a Lorentzian almost paracontact manifold). If \(\varepsilon = 1\), then \(M\) will be said to be a spacelike \((\varepsilon)\)-almost paracontact metric manifold (resp. a spacelike Lorentzian almost paracontact manifold). Similarly, if \(\varepsilon = -1\), then \(M\) will be said to be a timelike \((\varepsilon)\)-almost paracontact metric manifold (resp. a timelike Lorentzian almost paracontact manifold) [32]. Note that a timelike Lorentzian almost paracontact structure is a Lorentzian almost paracontact structure in the sense of Mihai and Rosca [20, 19], which differs in the sign of the structure vector field of the Lorentzian almost paracontact structure given by Matsumoto [18].

An \((\varepsilon)\)-almost paracontact metric structure is called an \((\varepsilon)\)-para Sasakian structure if
\[ (\nabla_X \varphi) Y = -g(\varphi X, \varphi Y)\xi - \varepsilon \eta(Y) \varphi^2 X, \]  \hspace{1cm} (11)

where \(\nabla\) is the Levi-Civita connection with respect to \(g\). A manifold endowed with an \((\varepsilon)\)-para Sasakian structure is called an \((\varepsilon)\)-para Sasakian manifold [32]. In an \((\varepsilon)\)-para Sasakian manifold we have [32]
\[ \nabla \xi = \varepsilon \varphi, \]  \hspace{1cm} (12)
\[ \Phi(X, Y) = g(\varphi X, Y) = \varepsilon g(\nabla_X \xi, Y) = (\nabla_X \eta) Y. \]  \hspace{1cm} (13)

An \((\varepsilon)\)-almost paracontact metric manifold is called \(\eta\)-Einstein if its Ricci tensor \(S\) satisfies the condition
\[ S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y). \]  \hspace{1cm} (14)

The \(k\)-nullity distribution \(N(k)\) of a semi-Riemannian manifold \(M\) is defined by
\[ N(k) : p \rightarrow N_p(k) = \{Z \in T_pM : \]
\[ R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y), \]  
\[ (15) \]

for all \( X, Y \in \chi(\cal M) \), where \( k \) is some smooth function (see [29]). If \( M \) is an \( \eta \)-Einstein \((\varepsilon)\)-para Sasakian manifold and the structure vector field \( \xi \) belongs to the \( k \)-nullity distribution \( N(k) \) for some smooth function \( k \), then we say that \( M \) is an \( N(k) \)-\( \eta \)-Einstein \((\varepsilon)\)-para Sasakian manifold (see [31]).

In an \((\varepsilon)\)-para Sasakian manifold, the Riemann curvature tensor \( R \) and the Ricci tensor \( S \) satisfy the following equations [32]:

\[ R(X, Y) \xi = \eta(X)Y - \eta(Y)X, \]
\[ (16) \]
\[ R(X, Y, Z, \xi) = -\eta(X)g(Y, Z) + \eta(Y)g(X, Z), \]
\[ (17) \]
\[ \eta(R(X, Y) Z) = -\varepsilon\eta(X)g(Y, Z) + \varepsilon\eta(Y)g(X, Z), \]
\[ (18) \]
\[ R(\xi, X)Y = -\varepsilon g(X, Y)\xi + \eta(Y)X, \]
\[ (19) \]
\[ S(X, \xi) = -(n - 1)\eta(X). \]
\[ (20) \]

It is known that in a semi-Riemannian 3-manifold

\[ R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \]
\[ -\frac{r}{2} (g(Y, Z)X - g(X, Z)Y), \]
\[ (21) \]

where \( Q \) is the Ricci operator and \( r \) is the scalar curvature of the manifold. If we substitute \( Z \) by \( \xi \) in (21) and use (16), we get

\[ \varepsilon(\eta(Y)QX - \eta(X)QY) = \left(1 + \frac{\varepsilon r}{2}\right)(\eta(Y)X - \eta(X)Y). \]
\[ (22) \]

By putting \( Y = \xi \) in (22) and using (2) and (20) for \( n = 3 \), we obtain

\[ QX = \frac{1}{2}\{(r + 2\varepsilon)X - (r + 6\varepsilon)\eta(X)\xi\}, \]

that is,

\[ S(X, Y) = \frac{1}{2}\{(r + 2\varepsilon)g(X, Y) - \varepsilon(r + 6\varepsilon)\eta(X)\eta(Y)\}. \]
\[ (23) \]

By using (23) in (21), we obtain

\[ R(X, Y)Z = \left(\frac{r}{2} + 2\varepsilon\right)\{g(Y, Z)X - g(X, Z)Y\} \]
\[ -\left(\frac{r}{2} + 3\varepsilon\right)\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \]
\[ +\varepsilon\eta(Y)\eta(Z)X - \varepsilon\eta(X)\eta(Z)Y\}. \]
\[ (24) \]

If an \((\varepsilon)\)-para Sasakian manifold is a space of constant curvature then it is an indefinite space form.
Lemma 1. An $(\varepsilon)$-para Sasakian 3-manifold is an indefinite space form if and only if the scalar curvature $r = -6\varepsilon$.

Proof. Let a 3-dimensional $(\varepsilon)$-para Sasakian manifold be an indefinite space form. Then
\[ R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}, \quad X, Y, Z \in \chi(M), \quad (25) \]
where $c$ is the constant curvature of the manifold. By using the definition of Ricci curvature and (25) we have
\[ S(X, Y) = 2c g(X, Y). \quad (26) \]
If we use (26) in the definition of the scalar curvature we get
\[ r = 6c. \quad (27) \]
From (26) and (27) one can easily see that
\[ S(X, Y) = \frac{r}{3} g(X, Y). \quad (28) \]
By putting $X = Y = \xi$ in (23) and using (28) we obtain
\[ r = -6\varepsilon. \]

Conversely, if $r = -6\varepsilon$ then from the equation (24) we can easily see that the manifold is an indefinite space form. This completes the proof.

Theorem 2. Every $(\varepsilon)$-para Sasakian 3-manifold is an $N(-\varepsilon)\eta$-Einstein manifold.

Proof. The proof follows from (23) and (16).

3. Ricci-Semi-Symmetric $(\varepsilon)$-para Sasakian 3-Manifolds

A semi-Riemannian manifold $M$ is said to be Ricci-semi-symmetric [21] if its Ricci tensor $S$ satisfies the condition
\[ R(X, Y) \cdot S = 0, \quad X, Y \in \chi(M), \quad (29) \]
where $R(X, Y)$ acts as a derivation on $S$. 
Let $M$ be a Ricci-semi-symmetric $(\varepsilon)$-para Sasakian 3-manifold. From (29) we have
\[ S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0. \] (30)
If we put $Y = \xi$ and use (19), then we get
\[
0 = \varepsilon g(X, U)S(\xi, V) - \eta(U)S(X, V) + \varepsilon g(X, V)S(U, \xi) - \eta(V)S(U, X). \] (31)
By using (20) in (31) we obtain
\[
0 = 2\varepsilon g(X, U)\eta(V) + S(X, V)\eta(U) + 2\varepsilon g(X, V)\eta(U) + S(U, V)\eta(U). \] (32)

Consider that $\{e_1, e_2, e_3\}$ be an orthonormal basis of the $T_pM$, $p \in M$. Then by putting $X = U = e_i$ in (32) and taking the summation for $1 \leq i \leq 3$, we have
\[ S(\xi, V) + 8\varepsilon\eta(V) + r\eta(V) = 0. \] (33)
Again by using (20) in (33), we get
\[ (r + 6\varepsilon)\eta(V) = 0, \]
which gives $r = -6\varepsilon$. This implies, in view of Lemma 1, that the manifold is an indefinite space form.

Therefore, we can state the following

**Theorem 3.** A Ricci-semi-symmetric $(\varepsilon)$-para Sasakian 3-manifold is an indefinite space form.

4. **Locally $\varphi$-Symmetric $(\varepsilon)$-Para Sasakian 3-Manifolds**

Analogous to the notion introduced by Takahashi [28] for Sasakian manifolds, we give the following definition.

**Definition 4.** An $(\varepsilon)$-para Sasakian manifold is said to be locally $\varphi$-symmetric if
\[ \varphi^2(\nabla_W R)(X, Y, Z) = 0, \]
for all vector fields $X, Y, Z$ orthogonal to $\xi$. 
Now by taking covariant derivative of (24) with respect to $W$ and using (9) and (10) we have

$$\left(\nabla_W R\right)(X, Y, Z) = \frac{1}{2}(\nabla_W r) \left\{ g(Y, Z)X - g(X, Z)Y \
- g(Y, Z)\eta(X)\xi + g(X, Z)\eta(Y)\xi \\
- \varepsilon\eta(Y)\eta(Z)X + \varepsilon\eta(X)\eta(Z)Y \right\}$$

$$+ \left(\frac{r}{2} + 3\varepsilon\right) \left\{ -g(Y, Z)\left(\Phi(X, W)\xi + \varepsilon\eta(X)\varphi W\right) \\
+ g(X, Z)\left(\Phi(Y, W)\xi + \varepsilon\eta(Y)\varphi W\right) \\
- \varepsilon\left(\Phi(Y, W)\eta(Z) + \Phi(Z, W)\eta(Y)\right) \right\} X \\
+ \varepsilon\left(\Phi(X, W)\eta(Z) + \Phi(Z, W)\eta(X)\right) Y \right\},$$

(34)

where $X, Y \in \chi(M)$. Then by taking $X, Y, Z$ orthogonal to $\xi$ and using (1), (3), (4) and (7), from (34) we obtain

$$\varphi^2(\nabla_W R)(X, Y, Z) = \frac{1}{2}(\nabla_W r) \left( g(Y, Z)X - g(X, Z)Y \right).$$

(35)

Hence from (35) we can state the following theorem:

**Theorem 5.** A 3-dimensional $(\varepsilon)$-para Sasakian manifold is locally $\varphi$-symmetric if and only if the scalar curvature $r$ is constant.

If a 3-dimensional $(\varepsilon)$-para Sasakian manifold is Ricci-semi-symmetric then we have showed that $r = -6\varepsilon$ that is $r$ is constant. Therefore from (35), we have

**Theorem 6.** A 3-dimensional Ricci-semi-symmetric $(\varepsilon)$-para Sasakian manifold is locally $\varphi$-symmetric.

In particular, by taking $Z = \xi$ in (34) we have

$$\left(\nabla_W R\right)(X, Y, \xi) = \left(\frac{\varepsilon r}{2} + 3\right) \left\{ -\eta(Y)\Phi(X, W)\xi + \eta(X)\Phi(Y, W)\xi \\
- \Phi(Y, W)X + \Phi(X, W)Y \right\}.$$

(36)

Applying $\varphi^2$ to the both sides of (36) we get

$$\varphi^2(\nabla_W R)(X, Y, \xi) = \left(\frac{\varepsilon r}{2} + 3\right) \left\{ -\Phi(Y, W)\varphi^2 X \\
+ \Phi(X, W)\varphi^2 Y \right\}.$$

(37)

If we take $X, Y, W$ orthogonal to $\xi$ in (36) and (37) we have

$$\varphi^2(\nabla_W R)(X, Y, \xi) = (\nabla_W R)(X, Y, \xi).$$

Now we can state the following:
Theorem 7. Let $M$ be an $(\varepsilon)$-para Sasakian 3-manifold such that
\[ \varphi^2(\nabla_WR)(X, Y, \xi) = 0 \]
for all $X, Y, W \in \chi(M)$, orthogonal to $\xi$. Then $M$ is an indefinite space form.

5. $\eta$-Parallel $(\varepsilon)$-Para Sasakian 3-Manifolds

Motivated by the definitions of Ricci $\eta$-parallelity for Sasakian manifolds and $LP$-Sasakian manifolds were given by Kon [16] and Shaikh and De [26], respectively, we give the following

Definition 8. Let $M$ be an $(\varepsilon)$-para Sasakian manifold. If the Ricci tensor $S$ satisfies
\[ (\nabla_X S)(\varphi Y, \varphi Z) = 0, \quad X, Y, W \in \chi(M), \]
then the manifold $M$ is said to be $\eta$-parallel.

Proposition 9. Let $M$ be an $(\varepsilon)$-para Sasakian 3-manifold with $\eta$-parallel Ricci tensor. Then the scalar curvature $r$ is constant.

Proof. From (23) by using (5) and (4)
\[ S(\varphi X, \varphi Y) = \left( \frac{r}{2} + \varepsilon \right) (g(X, Y) - \varepsilon \eta(X)\eta(Y)). \]
If we take the covariant derivative of (39) with respect to $Z$ and (13), we get
\[ (\nabla_Z S)(\varphi X, \varphi Y) = \frac{1}{2} \{(\nabla_Z r) (g(X, Y) - \varepsilon \eta(X)\eta(Y)) \]
\[ -\varepsilon(r + 2\varepsilon)(\Phi(X, Z)\eta(Y) + \Phi(Y, Z)\eta(X)) \}. \]
Since $M$ is an $(\varepsilon)$-para Sasakian 3-manifold with $\eta$-parallel Ricci tensor, then from (38) we have
\[ 0 = (\nabla_Z r)(g(X, Y) - \varepsilon \eta(X)\eta(Y)) \]
\[ -\varepsilon(r + 2\varepsilon)(\Phi(X, Z)\eta(Y) + \Phi(Y, Z)\eta(X)). \]

This completes the proof.

In view of Theorem 5 and Proposition 9 we have the following:
Theorem 10. An \((\varepsilon)\)-para Sasakian 3-manifold with \(\eta\)-parallel Ricci tensor is locally \(\varphi\)-symmetric.

Remark 11. An \((\varepsilon)\)-para Sasakian manifold is called Lorentzian para Sasakian manifold if \(\varepsilon = -1\) and \(\text{index}(g) = 1\). Therefore, some results we obtained in the previous three sections can be considered as a generalization of the some results obtained by the authors in [26].

6. Pseudosymmetric \((\varepsilon)\)-Para Sasakian 3-Manifolds

Now, we consider a well known generalization of the concept of an \(\eta\)-Einstein almost paracontact metric manifold in the following

Definition 12. [8] A non-flat \(n\)-dimensional Riemannian manifold \((M, g)\) is said to be a \textit{quasi Einstein manifold} if its Ricci tensor \(S\) satisfies

\[
S = a g + b \eta \otimes \eta
\]

or equivalently, its Ricci operator \(Q\) satisfies

\[
Q = a I + b \eta \otimes \xi
\]

for some smooth functions \(a\) and \(b\), where \(\eta\) is a nonzero 1-form such that

\[
g(X, \xi) = \eta(X), \quad g(\xi, \xi) = \eta(\xi) = 1
\]

for the associated vector field \(\xi\). The 1-form \(\eta\) is called the associated 1-form and the unit vector field \(\xi\) is called the generator of the quasi Einstein manifold.

B. Y. Chen and K. Yano [9] defined a Riemannian manifold \((M, g)\) to be of \textit{quasi-constant curvature} if it is conformally flat manifold and its Riemann-Christoffel curvature tensor \(R\) of type \((0, 4)\) satisfies the condition

\[
R(X, Y, Z, W) = a \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
+ b \{g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W) \\
+ g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)\},
\]

for all \(X, Y, Z, W \in \chi(M)\), where \(a, b\) are some smooth functions and \(T\) is a non-zero 1-form defined by

\[
g(X, \rho) = T(X), \quad X \in \chi(M)
\]
for a unit vector field $\rho$. On the other hand, Gh. Vrânceanu [33] defined a Riemannian manifold $(M, g)$ to be of *almost constant curvature* if $M$ satisfies (44). Later on, it was pointed out by A. L. Mocanu [22] that the manifold introduced by Chen and Yano and the manifold introduced by Vrânceanu were identical, as it can be verified that if the curvature tensor $R$ is of the form (44), then the manifold is conformally flat. Thus, a Riemannian manifold is said to be of *quasi-constant curvature* if the curvature tensor $R$ satisfies (44). If $b = 0$, then the manifold reduces to a manifold of constant curvature.

**Example 13.** A manifold of quasi-constant curvature is a quasi Einstein manifold [11, Example 1]. Conversely, a conformally flat quasi Einstein manifold of dimension $n$ ($n > 3$) is a manifold of quasi-constant curvature [12, Theorem 4].

Let $(M, g)$ be a semi-Riemannian manifold with its Levi-Civita connection $\nabla$. A tensor field $F$ of type $(1,3)$ is known to be *curvature-like* provided that $F$ satisfies the symmetric properties of the curvature tensor $R$. For example, the tensor $R_\rho$ given by

$$R_\rho(X,Y)Z \equiv (X \wedge g Y)Z = g(Y,Z)X - g(X,Z)Y,$$

where $X,Y \in \chi(M)$, is a trivial example of a curvature like tensor. Sometimes, the symbol $R_\rho$ seems to be much more convenient than the symbol $(X \wedge g Y)Z$.

For example, a semi-Riemannian manifold $(M, g)$ is of constant curvature $c$ if and only if $R = cR_\rho$.

It is well known that every curvature-like tensor field $F$ acts on the algebra $T^1_s(M)$ of all tensor fields on $M$ of type $(1,s)$ as a derivation [23, p. 44]:

$$(F \cdot P)(X_1, \ldots, X_s; Y, X) = F(X,Y)\{P(X_1, \ldots, X_s)\}$$

$$- \sum_{j=1}^s P(X_1, \ldots, F(X,Y)X_j, \ldots, X_s)$$

for all $X_1, \ldots, X_s \in \chi(M), P \in T^1_s(M)$. The derivative $F \cdot P$ of $P$ by $F$ is a tensor field of type $(1, s + 2)$. A semi-Riemannian manifold $(M, g)$ is said to be *semi-symmetric* if $R \cdot R = 0$. Obviously, locally symmetric spaces ($\nabla R = 0$) are semi-symmetric. More generally, a semi-Riemannian manifold $(M, g)$ is said to be *pseudo-symmetric* (in the sense of R. Deszcz) [13] if $R \cdot R$ and $R_\rho \cdot R$ in $M$ are linearly dependent, that is, if there exists a real valued smooth function $L$ on $M$ such that

$$R \cdot R = L R_\rho \cdot R$$
is true on the set

\[ U = \left\{ x \in M : R \neq \frac{r}{n(n-1)} R_g \text{ at } x \right\}. \]

A pseudo-symmetric space is said to be proper if it is not semi-symmetric. For details we refer to [6, 3].

In the literature, there is also another notion of pseudo-symmetry. A semi-Riemannian manifold \((M, g)\) is said to be pseudo-symmetric in the sense of Chaki [7] if

\[
(\nabla R)(X_1, X_2, X_3, X_4; X) = 2\omega(X)R(X_1, X_2, X_3, X_4) \\
+ \omega(X_1)R(X, X_2, X_3, X_4) \\
+ \omega(X_2)R((X_1, X, X_3, X_4) \\
+ \omega(X_3)R((X_1, X_2, X, X_4) \\
+ \omega(X_4)R((X_1, X_2, X_3, X),
\]

for all \(X_1, X_2, X_3, X_4; X \in \chi(M)\), where \(\omega\) is a 1-form on \((M, g)\). Of course, both the definitions of pseudo-symmetry for a semi-Riemannian manifold are not equivalent. For example, in contact geometry, every Sasakian space form is pseudo-symmetric in the sense of Deszcz [4, Theorem 2.3], but a Sasakian manifold cannot be pseudo-symmetric in the sense of Chaki [30, Theorem 1]. We assume the pseudo-symmetry always in the sense of Deszcz, unless specifically stated otherwise.

For Riemannian 3-manifolds, the following characterization of pseudosymmetry is known (cf. [17, 10]).

**Proposition 14.** A 3-dimensional Riemannian manifold \((M, g)\) is pseudo-symmetric if and only if it is quasi-Einstein, that is, if and only if there exists a 1-form \(\eta\) such that the Ricci tensor field \(S\) satisfies \(S = ag + b\eta \otimes \eta\) for some smooth functions \(a\) and \(b\).

In view of the above Proposition, we can state the following:

**Theorem 15.** Every 3-dimensional \(\eta\)-Einstein \((\varepsilon)\)-almost paracontact metric manifold is always pseudo-symmetric. In particular, each 3-dimensional \((\varepsilon)\)-para Sasakian manifold is pseudo-symmetric.

**Problem 16.** It would be interesting to know whether an \((\varepsilon)\)-almost para Sasakian manifold is pseudo-symmetric in the sense of Chaki or not.
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