International Journal of Pure and Applied Mathematics
Volume 77 No. 4 2012, 501-515
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

A GENERALIZATION OF Y. NOZAKI AND M. RIESZ'S ULTRAHYPERBOLIC KERNEL

Manuel A. Aguirre

Núcleo Consolidado Matemática Pura y Aplicada
Facultad de Ciencias Exactas
Universidad Nacional del Centro
Tandil, Provincia de Buenos Aires, ARGENTINA

Abstract: In this article, we introduce the distributional family $B_{\alpha+mn-n}(V)$ defined by (3). $B_{\alpha mn-n}(V)$ is a generalization of distributional family $R_\alpha(u)$ defined by (6), due to Y. Nozaki and this is called in [2] the Marcel Riesz Ultrahyperbolic Kernel.

We obtain the formula $L_a^k\{B_{\alpha+mn-n}(V)\} = C_{\alpha,m,n}B_{\alpha+mn-n-2km}(V)$ and evaluate $B_{\alpha+mn-n}(V)$ at $\alpha = -2m(\frac{n}{2}+k-\frac{n}{2m})$ for the cases: $\mu = 2ms+m$, $s = 1,2,...$, $v = 2mt$, $t = 0,1,2,...$; $\mu = 2ms$, $s = 0,1,2,...$, $v = 2mt+m$, $t = 1,2,...$, $\mu = 2ms$, $s = 0,1,2,...$, $v = 2mt$, $t = 0,1,2,..$ and $\mu = 2ms+m$ and $v = 2mt+m$, $s,t = 0,1,2,...$, where L_a is the operator defined by (41) and the constant $C_{\alpha,m,n}$ is defined by (26).

All the results are generalizations of Marcel Riesz ultrahyperbolic kernel and appear in [2] and [3].

1. Introduction

Let $x = (x_1,...,x_n)$ be a point of R^n, we shall write

$$V = V(x_1,...,x_n) = \sum_{j=1}^{n} a_j x_j^2 = a_1x_1^2+...+a_\mu x_\mu^2+a_{\mu+1}x_{\mu+1}^2+...+a_{\mu+v}x_{\mu+v}^2 \quad (1)$$

where a_j are real numbers $\mu+v=n$ dimension of the space.

Received: December 27, 2011

We designate the domain:

$$T_+ = \{x \in R^n : x_1 > 0, x_2 > 0, ..., x_\mu,\ V > 0\} \tag{2}$$

and $\overline{T_+}$ we designate it's closure.

We shall consider the following functions of the family $B_\alpha(x)$ defined by

$$B_{\alpha+mn-n}(V) = \begin{cases} \frac{V_+^{\frac{\alpha-n}{2m}}}{K_{m,n}(\alpha)} \text{ if } x \in T_+ \\ 0 \text{ if } x \notin T_+ \end{cases} \tag{3}$$

where α is a complex number, $m = 1, 2, ..$ and $K_{m,n(\alpha)}$ is defined by

$$K_{m,n(\alpha)} = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{\alpha-n}{2m} + 1)\Gamma(\frac{1}{2} - \frac{\alpha}{2m})\Gamma(\frac{\alpha}{m})}{\Gamma(\frac{\alpha-\mu}{2m} + 1)\Gamma(\frac{\mu-\alpha}{2m})} \tag{4}$$

and

$$< V_+^{\frac{\alpha-n}{2m}}, \varphi >= \int_{V>0} V^{\frac{\alpha-n}{2m}} \varphi(x) dx. \tag{5}$$

In(5), φ is a testing function in D(space of infinitely differentiable) with compact support.

$B_{\alpha+mn-n}(V)$, which is an ordinary function if $\text{Re}(\frac{\alpha}{2m}) \geq \frac{n}{2m}$ is a distribution of α. We shall call $B_{\alpha+mn-n}(V)$ the generalization of Nozaki Y. ultrahyperbolic Kernel.

By making $m = 1$, $a_1 = a_2... = a_\mu = 1$ and $a_{\mu+1} = a_{\mu+2} = ... = a_{\mu+v} = -1$ in (3) and (4) the formula (3) is reduce to

$$B_\alpha(V) = R_\alpha(u) = \begin{cases} \frac{u_+^{\frac{\alpha-n}{2}}}{K_{1,n}(\alpha)} \text{ if } x \in T_+ \\ 0 \text{ if } x \notin T_+ \end{cases} \tag{6}$$

where

$$u = u(x_1, ..., x_n) = x_1^2 + ... + x_\mu^2 - x_{\mu+1}^2 - ... - x_{\mu+v}^2 \tag{7}$$

and

$$K_{1,n}(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{\alpha-n}{2} + 1)\Gamma(\frac{1}{2} - \frac{\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{\alpha-\mu}{2} + 1)\Gamma(\frac{\mu-\alpha}{2})}. \tag{8}$$

$R_\alpha(u)$ is introduced by Nozaki Y. in [1], p. 72 and $R_\alpha(u)$ is called in [2] the Marcel Riesz ultrahyperbolic Kernel.

By putting $\mu = 1$ in (6), (7) and (8), we obtain precisely the Marcel Riesz hyperbolic kernel (see [1], p. 72). We observe that by putting $\mu = 1$ in (6), (7) and (8) and remembering the Legendre's duplication formula of $\Gamma(z)$

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2}) \text{ (see [4], p. 5, formula (15))} \tag{9}$$

the formula (6)is reduced to

$$N_\alpha(u) = \begin{cases} \frac{\sigma_+^{\frac{\alpha-n}{2}}}{H_{n(\alpha)}} \text{ if } x \in T_+ \\ 0 \text{ if } x \notin T_+ \end{cases} \tag{10}$$

where

$$\sigma = x_1^2 - x_2^2 - ... - x_n^2 \tag{11}$$

and

$$H_n(\alpha) = 2^{\alpha-1}\pi^{\frac{n-2}{2}}\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha-n}{2}+1) \tag{12}$$

$N_\alpha(u)$ is the hyperbolic Kernel of Marcel Riesz (see [3], p. 31).

We observe that considering the formula (9) the formula (4)is reduced to

$$K_{m,n}(\alpha) = H_{m,n}(\alpha)X_m(\mu,\alpha) \tag{13}$$

where

$$H_{m,n}(\alpha) = \pi^{\frac{n-2}{2}}2^{\frac{\alpha}{m}-1}\Gamma(\frac{\alpha}{2m})\Gamma(\frac{\alpha-n}{2m}+1) \tag{14}$$

and

$$X_m(\mu,\alpha) = \frac{\Gamma(\frac{\alpha}{2m}+\frac{1}{2})\Gamma(\frac{1}{2}-\frac{\alpha}{2m})}{\Gamma(\frac{\alpha-\mu}{2m}+1)\Gamma(\frac{\mu-\alpha}{2m})}. \tag{15}$$

Now using the formulae

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)} \tag{16}$$

and

$$\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \frac{\pi}{\cos(z\pi)} \tag{17}$$

$X_m(\mu,\alpha)$ can be rewritten in the following form:

$$X_m(\mu,\alpha) = (-1)^{\frac{\mu-m}{2m}} \text{ if } \mu = 2ms+m, s = 1,2,.. \tag{18}$$

and

$$X_m(\mu,\alpha) = -(-1)^{\frac{\mu}{2m}}\frac{\sin(\frac{\alpha\pi}{2m})}{\cos(\frac{\alpha\pi}{2m})} \text{ if } \mu = 2ms, s = 1,2,.. \tag{19}$$

Therefore,from(15)and considering(18)and(19) we have,

$$K_{m,n}(\alpha) = (-1)^{\frac{\mu-n}{2m}} H_{m,n}(\alpha) \text{ if } \mu = 2ms + m \tag{20}$$

and

$$K_{n,n}(\alpha) = [(-1)(-1)^{\frac{\mu}{2m}} \frac{\sin(\frac{\alpha\pi}{2m})}{\cos(\frac{\alpha\pi}{2m})}] H_{m,n}(\alpha) \text{ if } \mu = 2ms \tag{21}$$

From(3)and using(20)and(21), $B_{\alpha+mn-n}(V)$ can be rewritten in the following form

$$B_{\alpha+mn-n}(V) = \frac{V_+^{\frac{\alpha-n}{2m}}}{(-1)^{\frac{\mu-m}{2m}} H_{m,n}(\alpha)} \text{ if } \mu = 2ms + m \tag{22}$$

and

$$B_{\alpha+mn-n}(V) = \frac{V_+^{\frac{\alpha-n}{2m}} \cos(\frac{\alpha\pi}{2m})}{[(-1)(-1)^{\frac{\mu}{2m}} \sin(\frac{\alpha\pi}{2m})] H_{m,n}(\alpha)} \text{ if } \mu = 2ms \tag{23}$$

Lemma 1. *Let $B_{\alpha+mn-n}(V)$ be the distributional family functional defined by (3) and L_a the operator defined by*

$$L_a = \frac{1}{a_1}\frac{\partial^2}{\partial x_1^2} + ... + \frac{1}{a_\mu}\frac{\partial^2}{\partial x_\mu^2} + \frac{1}{a_{\mu+1}}\frac{\partial^2}{\partial x_{\mu+1}^2} + ... + \frac{1}{a_{\mu+v}}\frac{\partial^2}{\partial x_{\mu+v}^2} \tag{24}$$

if $a_1, ..., a_\mu > 0$ and $a_{\mu+1}, ..., a_{\mu+v} < 0$, then the following formula is valid

$$L_a^k\{B_{\alpha+mn-n}(V)\} = C_{\alpha,m,k} B_{\alpha+mn-n-2km}(V) \tag{25}$$

where

$$C_{\alpha,m,k} = \frac{\Gamma(\frac{\alpha-n}{2m} + \frac{n}{2})\Gamma(1 - \frac{\alpha}{2m})(-1)^k}{\Gamma(\frac{\alpha-n}{2m} + \frac{n}{2} - k)\Gamma(-\frac{\alpha}{2m} + k + 1)}. \tag{26}$$

Proof. From (1) a simple calculation show that

$$L_a\{V_+^\lambda\} = 2^2\lambda(\lambda - 1 + \frac{n}{2})V_+^{\lambda-1} \tag{27}$$

where L_a is the operator defined by (24).By k-fold iterator of (27) we have,

$$L_a^k\{V_+^\lambda\} = 2^{2k}\lambda(\lambda-1)...(\lambda-(k-1))(\lambda+\frac{n}{2}-1)(\lambda+\frac{n}{2}-2)...(\lambda+\frac{n}{2}-k)V_+^{\lambda-k}.$$

Now using the formulae

$$(-1)^l z(z-1)...(z-l+1) = \frac{(-1)^l\Gamma(z+1)}{\Gamma(z-l+1)}, l = 1, 2, ...\text{(see [4])} \tag{28}$$

and

$$(z-1)...(z-l) = \frac{\Gamma(z)}{\Gamma(z-l)} = \frac{(-1)^l \Gamma(-z+l+1)}{\Gamma(1-z)}.l = 1,2,...(\text{see [4]}) \tag{29}$$

we have,

$$L_a^k\{V_+^\lambda\} = 2^{2k}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-k)}.\frac{\Gamma(\lambda+\frac{n}{2})}{\Gamma(\lambda+\frac{n}{2}-k)}V_+^{\lambda-k} \tag{30}$$

By putting $\lambda = \frac{\alpha-n}{2m}$ in (30) and using (22) and (23) we have,

$$\begin{aligned} L_a^k\{B_{\alpha+mn-n}(V_+)\} &= 2^{2k}\frac{\Gamma(\frac{\alpha-n}{2m}+1)}{\Gamma(\frac{\alpha-n}{2m}+1-k)}.\frac{\Gamma(\frac{\alpha-n}{2m}+\frac{n}{2})}{\Gamma(\frac{\alpha-n}{2m}+\frac{n}{2}-k)}. \\ &.\frac{V_+^{\frac{\alpha-n}{2m}-k}}{(-1)^{\frac{\mu-m}{2m}}H_{m,n}(\alpha)} \text{ if } \mu = 2ms+m \end{aligned} \tag{31}$$

and

$$\begin{aligned} L^k\{B_{\alpha+mn-n}(V)\} &= \frac{\cos(\frac{\alpha\pi}{2m})}{[(-1)(-1)^{\frac{\mu}{2m}}]\sin(\frac{\alpha\pi}{2m})}.\frac{2^{2k}\Gamma(\frac{\alpha-n}{2m}+1)}{\Gamma(\frac{\alpha-n}{2m}+1-k)}. \\ &.\frac{\Gamma(\frac{\alpha-n}{2m}+\frac{n}{2})}{\Gamma(\frac{\alpha-n}{2m}+\frac{n}{2}-k)}V_+^{\frac{\alpha-n}{2m}-k} \text{ if } \mu = 2ms. \end{aligned} \tag{32}$$

From (14) and using (29) we have,

$$H_{m,n}(\alpha-2jm) = \frac{2^{-2j}2^{2k}\Gamma(\frac{n-\alpha}{2m})\Gamma(1-\frac{\alpha}{2m})}{\Gamma(\frac{n-\alpha}{2m}+j)\Gamma(\frac{-\alpha}{2m}+j+1)}.H_{m.n}(\alpha) \tag{33}$$

From (31) ,(32) and using the formulae (16) and (33) we obtain

$$\begin{aligned} L^k\{B_{\alpha+mn-n}(V)\} &= 2^{2k}\frac{\Gamma(\frac{\alpha-n}{2m}+1)}{\Gamma(\frac{\alpha-n}{2m}+1-k)}.\frac{\Gamma(\frac{\alpha-n}{2m}+\frac{n}{2})}{\Gamma(\frac{\alpha-n}{2m}+\frac{n}{2}-k)}\frac{2^{-2k}\Gamma(\frac{n-\alpha}{2m})\Gamma(1-\frac{\alpha}{2m})}{(-1)^{\frac{\mu-m}{2m}}\Gamma(\frac{n-\alpha}{2m}+k)\Gamma(\frac{-\alpha}{2m}+k+1)} \\ \frac{V_+^{\frac{\alpha-n}{2m}-k}}{H_{m,n}(\alpha-2km)} &= \frac{\Gamma(\frac{\alpha-n}{2m}+\frac{n}{2})}{\Gamma(\frac{\alpha-n}{2m}+\frac{n}{2}-k)}.\frac{\Gamma(1-\frac{\alpha}{2m})(-1)^k}{\Gamma(-\frac{\alpha}{2m}+k+1)}B_{\alpha+mn-n-2km}(V) \text{ if } \mu = 2ms+m \end{aligned} \tag{34}$$

and considering that

$$\cos(\frac{\alpha-2km}{2m})\pi = (-1)^k\cos(\frac{\alpha}{2m}) \tag{35}$$

and

$$\sin(\frac{\alpha-2km}{2m})\pi = (-1)^k\sin(\frac{\alpha}{2m})\pi \tag{36}$$

we have,

$$L^k\{B_{\alpha+mn-n}(V)\} = \frac{\Gamma(\frac{\alpha-n}{2m}+\frac{n}{2})}{\Gamma(\frac{\alpha-n}{2m}+\frac{n}{2}-k)}.\frac{\Gamma(1-\frac{\alpha}{2m})(-1)^k}{\Gamma(-\frac{\alpha}{2m}+k+1)}B_{\alpha-2km}(V) \text{ if } \mu = 2ms \tag{37}$$

From (34) and (37) we obtain the formulae (25) and (26). □

The formulae (25) and (26) is a generalization of the property of Marcel Riesz Kernel

$$L^k\{R_\alpha(u)\} = R_{\alpha-2k}(u) \tag{38}$$

which is due to S.E.Trione and apper in [2], p. 11, formula (V,2)) and [6], p. 123, formula (I,2,29).

In fact, by putting $m=1$, $a_1 = ... = a_\mu = 1$ and $a_{\mu+1} = ... = a_{\mu+v} = -1$ in(25)and(26) and considering the formula(16) and $C_{\alpha,1,k} = 1$,we have,

$$L^k\{R_\alpha(u)\} = C_{\alpha,1,k}B_{\alpha-2k}(u) = R_{\alpha-2k}(u) \tag{39}$$

where $u = u(x_1...x_n) = x_1^2 + ... + x_\mu^2 - x_{\mu+1}^2 - ... - x_{\mu+v}^2$, $\mu + v = n$ dimension of the space and L is the operator defined by(24).

We know (see [8]) that the distributional family $(V \pm i0)^{\frac{\alpha-n}{2m}}$ has simple poles at $\alpha = n - mn - 2mk$ and we have, the following formula

$$\underset{\alpha=n-mn-2mk}{\mathrm{Res}}(V \pm i0)^{\frac{\alpha-n}{2m}} = \frac{\pi^{\frac{n}{2}} L_a^k\{\delta(x)\}}{2^{2k}\Gamma(\frac{n}{2}+k)k!e^{\pm\frac{v\pi i}{2}}\sqrt{|\Delta|}}$$

(see [8], formulae (27) and (29)) (40)

where $h = 1, 2, ...$, Δ is the determinant of the coeffcients of $V(x_1, ..., x_n)$, L_a is the operator defined by

$$L_a = \frac{1}{a_1}\frac{\partial^2}{\partial x_1^2} + ... + \frac{1}{a_\mu}\frac{\partial^2}{\partial x_\mu^2} + \frac{1}{a_{\mu+1}}\frac{\partial^2}{\partial x_{\mu+1}^2} + ... + \frac{1}{a_{\mu+v}}\frac{\partial^2}{\partial x_{\mu+v}^2} \text{ (see [8], formula (30))} \tag{41}$$

if $a_1, ..., a_\mu > 0$ and $a_{\mu+1}, ..., a_{\mu+v} < 0$ and $(V \pm i0)^{\frac{\alpha-n}{2m}}$ is defined in analogues form of $(P \pm i0)^\lambda$(see [9], p. 275). Now using the property

$$(V \pm i0)^{\frac{\alpha-n}{2h}} = V_+^{\frac{\alpha-n}{2h}} e^{\pm\frac{v\pi i}{2h}} V_-^{\frac{\alpha-n}{2h}} \text{ (see [2], formulae (44) and (45))} \tag{42}$$

where

$$V_+^{\frac{\alpha-n}{2h}} = \begin{cases} V^{\frac{\alpha-n}{2h}} \text{if } V \geq 0 \\ 0 \text{ if } V < 0, \end{cases} \tag{43}$$

$$V_-^{\frac{\alpha-n}{2h}} = \begin{cases} (-V)^{\frac{\alpha-n}{2h}} \text{if } V \leq 0 \\ 0 \text{ if } V > 0 \end{cases} \tag{44}$$

we have,

$$V_+^{\frac{\alpha-n}{2m}} = \frac{\Gamma(1+\frac{\alpha-n}{2m})\Gamma(\frac{n-\alpha}{2m})}{2\pi i}[e^{-\pi i(\frac{\alpha-n}{2m})}(V+i0)^{\frac{\alpha-n}{2m}} - e^{\pi i(\frac{\alpha-n}{2m})}(V-i0)^{\frac{\alpha-n}{2m}}]. \tag{45}$$

From (45) and using (3) and (4) we have,

$$B_{\alpha+mn-n}(V)=\frac{V_{+}^{\frac{\alpha-n}{2m}}}{K_{m,n}(\alpha)}=\frac{\Gamma(\frac{n-\alpha}{2m})\Gamma(\frac{\alpha-\mu}{2m}+1)\Gamma(\frac{\mu-\alpha}{2m})}{2\pi i\pi^{\frac{n-2}{2}}\Gamma(\frac{1}{2}-\frac{\alpha}{2m})2^{\frac{\alpha}{m}-1}\pi^{-\frac{1}{2}}\Gamma(\frac{\alpha}{2m})\Gamma(\frac{\alpha}{2m}+\frac{1}{2})}$$
$$\left[e^{-\pi i(\frac{\alpha-n}{2m})}(V+i0)^{\frac{\alpha-n}{2m}}-e^{\pi i(\frac{\alpha-n}{2m})}(V-i0)^{\frac{\alpha-n}{2m}}\right]$$
$$=\frac{\Gamma(\frac{n-\alpha}{2m})}{2\pi i\pi^{\frac{n-2}{2}}2^{\frac{\alpha}{m}-1}}\frac{\Gamma(\frac{\alpha-\mu}{2m}+1)\Gamma(\frac{\mu-\alpha}{2m})}{\Gamma(\frac{1}{2}-\frac{\alpha}{2m})\Gamma(\frac{1}{2}+\frac{\alpha}{2m})}\left[e^{-\pi i(\frac{\alpha-n}{2m})}\frac{(V+i0)^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha}{2m})}-e^{\pi i(\frac{\alpha-n}{2m})}\frac{(V-i0)^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha}{2m})}\right]. \tag{46}$$

From (46) we have,

$$B_{n-mn-2km}(V)=\lim_{\alpha\to n-mn-2km}B_{\alpha+mn-n}(V)=\frac{\Gamma(\frac{n}{2}+k)}{2\pi i2^{\frac{n}{m}-n-2k-1}\pi^{\frac{n-2}{2}}}\cdot$$
$$\cdot\lim_{\alpha\to n-mn-2km}\frac{\Gamma(\frac{\alpha-\mu}{2m}+1)\Gamma(\frac{\mu-\alpha}{2m})}{\Gamma(\frac{1}{2}-\frac{\alpha}{2m})\Gamma(\frac{1}{2}+\frac{\alpha}{2m})}\cdot$$
$$\left[e^{\pi i(\frac{n}{2}+k)}\lim_{\alpha\to n-mn-2km}\frac{(V+i0)^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha}{2m})}-\right.$$
$$\left.-e^{-(\frac{n}{2}+k)\pi i}\lim_{\alpha\to n-mn-2km}\frac{(V-i0)^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha}{2m})}\right]. \tag{47}$$

We observe that the expression

$$(\frac{n}{2}+k-\frac{n}{2m}) \tag{48}$$

is non-negative integer for all $m=1,2,3,...$ in the following cases:$\mu=2ms+m$, $v=2mt+m$ and $\mu=2ms$ and $v=2mt$.

For the cases: $\mu=2ms+m$, and $v=2mt$, and $\mu=2ms$ and $v=2mt+m$ the expression

$$(\frac{n}{2}+k-\frac{n}{2m}) \tag{49}$$

is non-negative integer if m is odd. Therefore,by using the formula (29) and taking into account (48) and (49) we have,

$$\operatorname*{Res}_{\alpha=-mn-2km+n}\Gamma(\frac{\alpha}{2m})=\lim_{\alpha\to-(mn+2km-n)}(\alpha-n+mn+2km)\Gamma(\frac{\alpha}{2m})$$
$$=\lim_{\beta\to0}\beta\Gamma(\frac{\beta}{2m}-(\frac{n}{2}+k-\frac{n}{2m}))=$$
$$=2m\lim_{\beta\to0}\frac{\beta}{2m}\frac{\Gamma(\frac{\beta}{2m})\Gamma(1-\frac{\beta}{2m})}{(-1)^{\frac{n}{2}+k-\frac{n}{2m}}\Gamma(-\frac{\beta}{2m}+k-\frac{n}{2m}+1+\frac{n}{2})}$$
$$=2m\lim_{\beta\to0}\frac{\Gamma(1+\frac{\beta}{2m})\Gamma(1-\frac{\beta}{2m})}{(-1)^{\frac{n}{2}+k-\frac{n}{2m}}\Gamma(-\frac{\beta}{2m}+k-\frac{n}{2m}+1+\frac{n}{2})}$$
$$=\frac{2m}{(-1)^{\frac{n}{2}+k-\frac{n}{2m}}\Gamma(k-\frac{n}{2m}+1+\frac{n}{2})}. \tag{50}$$

Now taking into account the formulae (16) and (17) we have,

$$\lim_{\alpha\to n-mn-2km} \frac{\Gamma(\frac{\alpha-\mu}{2m}+1)\Gamma(\frac{\mu-\alpha}{2m})}{\Gamma(\frac{1}{2}-\frac{\alpha}{2m})\Gamma(\frac{1}{2}+\frac{\alpha}{2m})} = \lim_{\alpha\to n-mn-2km} \frac{\pi}{\sin(\frac{\mu-\alpha}{2m})\pi}.\frac{\cos(\frac{\alpha}{2m})\pi}{\pi} = \frac{1}{(-1)^{\frac{\mu-m}{2m}}} \tag{51}$$

if $\mu = 2ms + m, s = 0, 1, 2, ..$ and $m = 0, 1, 2, ..$

Now by considering that the gamma function $\Gamma(z)$ has simple poles at $z = 0, -1, -2, ...$ and the formulae (48) and (49) we have,

$$\begin{gathered}\lim_{\alpha\to n-mn-2km} \Gamma(\tfrac{\alpha-\mu}{2m}+1) = \Gamma(\tfrac{n-mn-2km}{2m} - \tfrac{\mu}{2m} + 1)\\ = \Gamma(\tfrac{n}{2m} - \tfrac{n}{2} - k - \tfrac{\mu}{2m} + 1) = \Gamma(\tfrac{n}{2m} - \tfrac{n}{2} - k - \tfrac{2ms}{2m} + 1)\\ = \Gamma(\tfrac{n}{2m} - \tfrac{n}{2} - k - s + 1) = \infty\end{gathered} \tag{52}$$

if $\mu = 2ms, s = 0, 1, 2, ...$ and

$$\lim_{\alpha\to n-mn-2km} \Gamma(\tfrac{\mu-\alpha}{2m}) = \Gamma(s - \tfrac{n}{2m} + \tfrac{n}{2} + k) \neq \infty \tag{53}$$

if $\mu = 2ms, s = 0, 1, 2, ...$ and $m = 1, 2, ..$

On the other hand, from (40) and (50) we have,

$$\begin{gathered}\lim_{\alpha\to -(mn+2km-n)} \frac{(V\pm i0)^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha}{2m})} = \frac{\operatorname*{Res}_{\alpha=-mn-2km+n} (V\pm i0)^{\frac{\alpha-n}{2m}}}{\operatorname*{Res}_{\alpha=-mn-2km+n} \Gamma(\frac{\alpha}{2m})}\\ = \frac{\pi^{\frac{n}{2}} L_a^k\{\delta(x)\}}{2^{2k}k!\Gamma(\frac{n}{2}+k)k!e^{\pm\frac{v\pi i}{2}}\sqrt{|\Delta|}}.\frac{(-1)^{\frac{n}{2}+k-\frac{n}{2m}}\Gamma(k-\frac{n}{2m}+1+\frac{n}{2})}{2m},\end{gathered} \tag{54}$$

where L_a is defined by(24) and Δ is the determinant of the coefficients of $V(x_1, ...x_n)$.From (46), and taking into account (52), (53) and (54) we conclude that $B_{n-mn-2km}(V)$ diverge for the case $\mu = 2ms, s = 0, 1, 2, ..$ and $m = 1, 2, ...$

Let $B^*_{\alpha+mn-n}(V)$ be the modified Kernel of $B_{\alpha+mn-n}(V)$ defined by

$$B^*_{\alpha+mn-n}(V) = \begin{cases} \frac{B_{\alpha+mn-n}(V)}{d_{\alpha,m,n}} \text{ if } \mu = 2ms \text{ and } v = 2mt + m,\\ s = 0, 1, 2, .., t = 1, 2, ...\text{and } m = 1, 2, ...\\ \text{and}\\ 0 \text{ for all others cases}\end{cases} \tag{55}$$

where

$$d_{\alpha,m,n} = \pi^{-1}\Gamma(\frac{\alpha}{2m})\Gamma(1 - \frac{\alpha}{2m}), \tag{56}$$

$\mu + v = n$ dimension of the space and $B_{\alpha+mn-n}(V)$ is defined by (3).

In order to study $B_{\alpha+mn-n}(V)$, we shall consider four cases:

i) $\mu = 2ms + m$ and $v = 2mt$;

ii) $\mu = 2ms + m$ and $v = 2mt + m$;

iii) $\mu = 2ms$ and $v = 2mt + m$ and

iv) $\mu = 2ms$ and $v = 2mt$.

Case 1. $\mu = 2ms + m$ and $v = 2mt$ $(n = \mu + v = 2ms + m + 2mt)$. In this case $(\frac{n}{2} + k - \frac{n}{2m})$ is non-negative integer if m is odd, consequently the formula (50) is valid for m odd, therefore from (45) and considering (47) we obtain the following formula

$$B^*_{-2km}(V) = B_{-2km}(V) = \frac{(-1)^{\frac{\mu-1}{2}}(-1)^{\frac{n}{2}-\frac{n}{2m}}\Gamma(k - \frac{n}{2m} + 1 + \frac{n}{2})}{(-1)^{\frac{\mu-m}{2m}} 2^{\frac{n}{m}-n-1} k! \sqrt{|\Delta|} 2m} L^k_a\{\delta(x)\} \tag{57}$$

if m is odd, $a_1, ..., a_\mu > 0$ and $a_{\mu+1}, ..., a_{\mu+v} < 0$.

In fact,from(46),(51)and(54), we have,

$$B^*_{-2km}(V) = B_{-2km}(V) = \lim_{\alpha \to -(mn+2km-n)} B_{\alpha+mn-n}(x) =$$
$$= \frac{\Gamma(\frac{n}{2}+k)(-1)^k(-1)^{\frac{n-k}{2}}}{2^{\frac{n}{m}-n-2k-1}\pi^{\frac{n-2}{2}}} \frac{1}{(-1)^{\frac{\mu-m}{2m}}} \frac{\pi^{\frac{n}{2}} L^k_a\{\delta(x)\}(-1)^{\frac{*****}{2}}(-1)^{\frac{n}{2}+k-\frac{n}{2m}}\Gamma(k-\frac{n}{2m}+1+\frac{n}{2})}{2^{2k}\Gamma(\frac{n}{2}+k)k!2m}$$
$$= \frac{(-1)^{\frac{\mu-1}{2}}(-1)^{\frac{n}{2}-\frac{n}{2m}}\Gamma(k-\frac{n}{2m}+1+\frac{n}{2})}{(-1)^{\frac{\mu-m}{2m}} 2^{\frac{n}{m}-n-1} k! \sqrt{|\Delta|} 2m} L^k_a\{\delta(x)\}. \tag{58}$$

The formula(58) is a generalization of the formula

$$R_{-2j}\{u\} = L^j\{\delta(x)\}, \ j = 0, 1, 2, ... \tag{59}$$

which appears (see [2], p. 9, formula (III.9) and [7], p. 140, formula (2.30)), where $R_\alpha(u)$ is the Marcel Riesz Kernel defined by (6). In fact, by putting $m = 1, a_1 = a_2 = ... = a_\mu = 1$ and $a_{\mu+1} = ... = a_{\mu+v} = -1$ in(58)and considering(6) we have,

$$R_{-2k}\{u\} = B_{-2km}(V)\mid_{m=1} = B_{-2k}(V) = \frac{\Gamma(1+k)(-1)^{\frac{\mu-1}{2}}}{(-1)^{\frac{\mu-1}{2}} k!} L^k\{\delta(x)\} = L^k\{\delta(x)\} \tag{60}$$

where

$$L = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_\mu^2} - \frac{\partial^2}{\partial x_{\mu+1}^2} - ... - \frac{\partial^2}{\partial x_{\mu+v}^2} \tag{61}$$

and $u = u(x_1 ... x_n)$ is defined by (7).

On the other hand, by letting $k = 0$ in(57)and considering that

$$L_a^0\{\delta(x)\} = \delta(x) \tag{62}$$

we have,

$$B_0(V) = \frac{\Gamma(1 + \frac{n}{2} - \frac{n}{2m})(-1)^{\frac{\mu-1}{2}}}{(-1)^{\frac{\mu-m}{2m}} 2^{\frac{n}{m}-m-1}\sqrt{|\Delta|}2m}\delta(x_1 ... x_n) \tag{63}$$

if $\mu = 2ms + m, v = 2mt$, $a_1, ..., a_\mu > 0$, $a_{\mu+1}, ..., a_{\mu+v} < 0$,and m is odd.

It's clear that by putting $m = 1, a_1 = ... = a_\mu = 1, a_{\mu+1} = ... = a_{\mu+v} = -1$ in(63)and using(6) we have,

$$R_0(u) = B_0(V) = \frac{\Gamma(1)(-1)^{\frac{\mu-1}{2}}}{(-1)^{\frac{\mu-1}{2}}}\delta(x_1 ... x_n) = \delta(x_1 ... x_n) \text{ if } \mu \text{ is odd and } \nu \text{ is even.} \tag{64}$$

The formula(64)appears in [2], p. 40.

Case 2. $\mu = 2ms$ and $v = 2mt+m$ ($n = \mu+v = 2ms+2mt+m = 2m(s+t)+m$). In this case $(\frac{n}{2}+k-\frac{n}{2m})$ is non-negative integer if m is odd, consequently the formula (50) is valid for m odd, therefore from(55), using(16),(47),(54)and(56) we have,

$$\begin{aligned} B^*_{-2km} &= \lim\nolimits_{\alpha\to-(mn+2km-n)} B^*_{\alpha+mn-n}(V) = \\ &= \lim\nolimits_{\alpha\to-(mn+2km-n)} \sin(\tfrac{\alpha}{2m})\pi B_{\alpha+mn-n}(V) = \frac{\Gamma(\frac{n}{2}+k)}{2\pi i 2^{\frac{n}{m}-n-2k-1}\pi^{\frac{n-2}{2}}}\cdot \\ &. \lim\nolimits_{\alpha\to-(mn+2km-n)} \frac{\pi\cos\frac{\alpha}{2m}\pi}{(-1)(-1)^{\frac{\mu}{2m}}}\left[e^{(\frac{n}{2}+k)\pi i}\frac{(V+i0)^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha}{2m})} - e^{-(\frac{n}{2}+k)\pi i}\frac{(V-i0)^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha}{2m})}\right] \\ &= \frac{\Gamma(\frac{n}{2}+k)\pi(-1)^{-\frac{h}{2m}+\frac{n}{2}+k}}{2\pi i 2^{\frac{n}{m}-n-2k-1}\pi^{\frac{n-2}{2}}(-1)(-1)^{\frac{\mu}{2m}}}\cdot\frac{(-1)^k\pi^{\frac{n}{2}}L_a^k\delta(-1)^{\frac{n}{2}+k-\frac{n}{2m}}.\Gamma(k-\frac{n}{2m}+1+\frac{n}{2})}{2^{2k}\Gamma(\frac{n}{2}+k)k!\sqrt{|\Delta|2m}}\cdot \\ &\cdot\left[\frac{e^{\frac{n}{2}\pi i}}{e^{v\frac{\pi}{2}i}} - \frac{e^{-\frac{n}{2}\pi i}}{e^{-v\frac{\pi}{2}i}}\right] \\ &= \frac{[(-1)^{ms}-(-1)^{ms}](-1)^k\Gamma(k-\frac{n}{2m}+1+\frac{n}{2})L_a^k\delta}{i2^{\frac{n}{m}-n}.(-1)(-1)^{\frac{\mu}{2m}}k!\sqrt{|\Delta|}m.} = 0. \end{aligned} \tag{65}$$

By letting $m = 1$, $a_1 = a_2 = ...a_\mu = 1, a_{\mu+1} = a_{\mu+2} = ... = a_{\mu+v} = -1$ in(65) and considering the formulae (5)and(6) the formula65) is a generalization of formula

$$B^*_{-2k}(V) = R^*_{-2k}(V) = 0 \text{ if } \mu = 2s \text{ and } v = 2t + 1 \tag{66}$$

due to M.A. Aguirre (see [6], p. 141, formula (2.36)), where $R^*_\alpha(u)$ is defined by

$$R^*_\alpha(u) = \begin{cases} \frac{R_\alpha(u)}{d_{\alpha,1,n}} \text{ if } \mu \text{ is even and } v \text{ odd} \\ R_\alpha(u) \text{ for all others cases} \end{cases} \tag{67}$$

(see [6], p. 139, formula (2.25)), $R_\alpha(u)$ is defined by (6) and

$$d_{\alpha,1,n} = \pi^{-1}\Gamma(\frac{\alpha}{2})\Gamma(1-\frac{\alpha}{2}) \tag{68}$$

Case 3. $\mu = 2ms$ and $v = 2mt$ $(n = 2ms + 2mt = 2m(s+t))$. In this case $(\frac{n}{2} + k - \frac{n}{2m})$ is non-negative integer for all $m = 1, 2, \ldots$ consequently the formula (50) is valid for all $m \geq 1$.

On the other hand, from [7] we know that the following formulae are valid

$$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k} V_+^\lambda = \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)}\delta_1^{(\frac{n}{2}+k-1)}(V) + \frac{(-1)^{\frac{v}{2}}\pi^{\frac{n}{2}}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}L_a^k\{\delta(x)\} \tag{69}$$
if μ and v are both even,$a_1, \ldots, a_\mu > 0$ and $a_{\mu+1}, \ldots, a_{\mu+v} < 0$

and

$$\operatorname*{Res}_{\lambda=-\frac{n}{2}-k} V_+^\lambda = \frac{(-1)(-1)^{\frac{v}{2}}\pi^{\frac{n}{2}}}{2^{2k}\Gamma(\frac{n}{2}+k)k!}L_a^k\{\delta(x)\} \tag{70}$$
if μ and v are both even,$a_1, \ldots, a_n > 0$ and $a_{\mu+1}, \ldots, a_{\mu+v} < 0$.

From (55), using (4), (9), (16), (17), and (50) we have:

$$B^*_{-2km} = \lim_{\alpha\to-(mn+2km-n)} B^*_{\alpha+mn-n}(V)$$
$$= \lim_{\alpha\to-(mn+2km-n)} B_{\alpha+mn-n}(V)$$
$$= \lim_{\alpha\to-(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}\Gamma(\frac{\alpha-\mu}{2m}+1)\Gamma(\frac{\mu-\alpha}{2m})}{\pi^{\frac{n-2}{2}}\Gamma(\frac{\alpha-n}{2m}+1)\Gamma(\frac{1}{2}-\frac{\alpha}{2m})\Gamma(\frac{\alpha}{2m})}$$
$$= \lim_{\alpha\to-(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}\Gamma(\frac{\alpha-\mu}{2m}+1)\Gamma(\frac{\mu-\alpha}{2m})}{d_{\alpha,m,n}\Gamma(\frac{1}{2}-\frac{\alpha}{2m})\Gamma(\frac{\alpha}{2m})} = \frac{1}{2^{\frac{\alpha}{m}-1}\pi^{-\frac{1}{2}}\Gamma(\frac{\alpha}{2m})\Gamma(\frac{1}{2}+\frac{\alpha}{2m})}$$
$$= \lim_{\alpha\to-(mn+2km-n)} \frac{\pi\cos(\frac{\alpha}{2m})}{\sin(\frac{\mu-\alpha}{2m})\pi}\cdot\frac{V_+^{\frac{\alpha-n}{2m}}}{d_{\alpha,m,n}\pi^{\frac{n-2}{2}}\Gamma(\frac{\alpha-n}{2m}+1)2^{\frac{\alpha}{m}-1}\Gamma(\frac{\alpha}{2m})} =$$
$$\lim_{\alpha\to-(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha-n}{2m}+1)}\cdot\frac{\cos(\frac{\alpha}{2m}\pi)}{[\sin\frac{\mu}{2m}\cos\frac{\alpha}{2m} - \cos\frac{\mu}{2m}\pi\sin\frac{\alpha}{2m}\pi]}\cdot\frac{1}{\pi^{\frac{n-2}{2}}2^{\frac{\alpha}{m}-1}\Gamma(\frac{\alpha}{2m})}$$
$$= \lim_{\alpha\to-(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha-n}{2m}+1)}\cdot\frac{\cos(\frac{\alpha}{2m}\pi)}{(-1)(-1)^{\frac{\mu}{2m}}\sin\frac{\alpha}{2m}\Gamma(\frac{\alpha}{2m})\pi^{\frac{n-2}{2}}2^{\frac{\alpha}{m}-1}}$$

$$= \lim_{\alpha\to-(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha-n}{2m}+1)} . \frac{\cos(\frac{\alpha}{2m}\pi)\Gamma(1-\frac{\alpha}{2m})}{(-1)(-1)^{\frac{\mu}{2m}}\pi}.$$

$$= \lim_{\alpha\to-(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha-n}{2m}+1)} .(-1)^k(-1)^{\frac{n}{2m}}(-1)^{\frac{n}{2}}$$

$$= \lim_{\alpha\to-(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha-n}{2m}+1)} . \lim_{\alpha\to-(mn+2km-n)} \frac{\cos(\frac{\alpha}{2m}\pi)\Gamma(1-\frac{\alpha}{2m})}{(-1)(-1)^{\frac{\mu}{2m}}\pi\pi^{\frac{n-2}{2}}2^{\frac{\alpha}{m}-1}}$$

$$= \lim_{\alpha\to-(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha-n}{2m}+1)} \frac{(-1)^k(-1)^{\frac{n}{2m}}(-1)^{\frac{n}{2}}\Gamma(1-\frac{n}{2m}+\frac{n}{2}+k)}{(-1)(-1)^{\frac{\mu}{2m}}\pi\pi^{\frac{n-2}{2}}2^{\frac{n}{m}-n-2k-1}} \quad (71)$$

Now taking into account the formula

$$\underset{\alpha=-(mn+2km-n)}{\operatorname{Res}} \Gamma(\frac{\alpha-n}{2m}+1)$$
$$= \underset{\lambda=-\frac{n}{2}-k}{\operatorname{Res}} \Gamma(\lambda+1) = \frac{(-1)^{\frac{n}{2}+k-1}}{(\frac{n}{2}+k-1)!} = \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)}, \quad (72)$$

we have

$$\begin{aligned} \lim_{\alpha\to-(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha-n}{2m}+1)} &= \frac{\underset{\alpha=-(mn+2km-n)}{\operatorname{Res}} V_+^{\frac{\alpha-n}{2m}}}{\underset{\alpha=-(mn+2km-n)}{\operatorname{Res}} \Gamma(\frac{\alpha-n}{2m}+1)} \\ = \frac{\Gamma(\frac{n}{2}+k)}{(-1)^{\frac{n}{2}+k-1}}\{&\frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)}\delta_1^{(\frac{n}{2}+k-1)}(V) + \frac{(-1)^{\frac{v}{2}}\pi^{\frac{n}{2}}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}L_a^k\{\delta(x)\}\} \\ &= \delta_1^{(\frac{n}{2}+k-1)}(V) + \frac{(-1)^{\frac{v}{2}}\pi^{\frac{n}{2}}}{(-1)^{\frac{n}{2}+k-1}2^{2k}k!}L_a^k\{\delta(x)\}. \end{aligned} \quad (73)$$

From(71)and(73), we arrive at the following formula

$$\begin{aligned} B^*_{-2km}(V) = B_{-2km}(V) = \frac{(-1)^k(-1)^{\frac{n}{2m}}(-1)^{\frac{n}{2}}\Gamma(1-\frac{n}{2m}+\frac{n}{2}+k)}{(-1)(-1)^{\frac{\mu}{2m}}\pi^{\frac{n}{2}}2^{\frac{n}{m}-n-2k-1}}. \\ [\delta_1^{(\frac{n}{2}+k-1)}(V) + \frac{(-1)^{\frac{v}{2}}\pi^{\frac{n}{2}}}{(-1)^{\frac{n}{2}+k-1}k!2^{2k}}L_a^k\{\delta(x)\}] \end{aligned} \quad (74)$$

if $a_1, ..., a_\mu > 0$ and $a_{\mu+1}, ..., a_{\mu+v} < 0$.

We observe that by putting $m = 1$, $a_1 = a_2 = ... = a_\mu = 1, a_{\mu+1} = a_{\mu+2} = ... a_{\mu+v} = -1$in(74) and considering(6)and(7) we obtain

$$\begin{aligned} B^*_{-2k}(V) = B_{-2k}(V) = R^*_{-2k}(u) = R_{-2k}(u) = \\ = \frac{(-1)^k k!}{(-1)(-1)^{\frac{\mu}{2}}2^{-2k-1}\pi^{\frac{n}{2}}}[\delta_1^{(\frac{n}{2}+k-1)}(u) + \frac{(-1)^{\frac{v}{2}}\pi^{\frac{n}{2}}}{(-1)^{\frac{n}{2}+k-1}k!}L^k\{\delta(x)\}], \end{aligned} \quad (75)$$

where L is the operator defined by(61), $R^*_\alpha(u)$ is defined by(67)and $\delta_1^{(l)}(u)$ is understood in the sense of regularization of $\delta^{(l)}(u)$ (see [9], p. 249) if $l \geq \frac{n}{2} - 1$.

The formula(75) is due to Manuel A. Aguirre and appears in (see [6], p. 143, formula (2.41)).

On the other hand,from(71) and using(69),the formula(74) can be rewritten in the following form

$$\begin{aligned} B^*_{n-mn-2km} = B_{n-mn-2km} &= \frac{(-1)^k(-1)^{\frac{n}{2m}}(-1)^{\frac{n}{2}}\Gamma(1-\frac{n}{2m}+\frac{n}{2}+k)}{(-1)(-1)^{\frac{\mu}{2m}}\pi^{\frac{n}{2}}2^{\frac{n}{m}-n-2k-1}} \cdot \\ &\cdot \frac{(-1)(-1)^{\frac{v}{2}}\pi^{\frac{n}{2}}L_a^k\{\delta(x)\}}{2^{2k}\Gamma(\frac{n}{2}+k)k!} \cdot \frac{\Gamma(\frac{n}{2}+k)}{2m(-1)^{\frac{n}{2}+k-1}} \\ &= \frac{(-1)(-1)^{\frac{n}{2m}}(-1)^{\frac{v}{2}}\Gamma(1-\frac{n}{2m}+\frac{n}{2}+k)}{(-1)^{\frac{\mu}{2m}}2^{\frac{n}{m}-n-1}k!}L_a^k\{\delta(x)\} \\ &= \frac{(-1)(-1)^{\frac{v}{2m}}(-1)^{\frac{v}{2}}\Gamma(1-\frac{n}{2m}+\frac{n}{2}+k)}{2^{\frac{n}{m}-n-1}k!\ 2m}L_a^k\{\delta(x)\}. \end{aligned} \tag{76}$$

if $a_1, ..., a_\mu > 0, a_{\mu+1}, ..., a_{\mu+v} < 0, \mu = 2ms$ and $v = 2mt$.

In particular, by putting $k = 0$ in(76) and considering(62) we have,

$$B_0(V) = \frac{(-1)(-1)^{\frac{v}{2m}}(-1)^{\frac{v}{2}}\Gamma(1-\frac{n}{2m}+\frac{n}{2})}{2^{\frac{n}{m}-n-1}}\delta(x) \text{ if } \mu = 2ms \text{ and } v = 2mt. \tag{77}$$

Case 4. $\mu = 2ms + m$ and $v = 2mt + m$ $(n = 2ms + m + 2mt + m = 2m(s + t + 1))$. In this case $(\frac{n}{2} + k - \frac{n}{2m})$ is a non-negative integer for all $m = 1, 2, ...$, consequently the formula (50) is valid for all $m \geq 1$.

On the other hand, from [7] we know that the following formula is valid

$$\lim_{\lambda \to -\frac{n}{2}-k}(\lambda + \frac{n}{2} + k)^2 V_+^\lambda = \frac{(-1)^{\frac{v+1}{2}}\pi^{\frac{n-2}{2}}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}L_a^k\{\delta(x)\} \text{ if } \mu \text{ and } v \text{ are both odd.} \tag{78}$$

From (55) and using(14)and(22) we have

$$\begin{aligned} B^*_{-2km}(V) &= \lim_{\alpha \to -(mn+2km-n)} B^*_{\alpha+mn-n}(V) \\ &= \lim_{\alpha \to -(mn+2km-n)} B_{\alpha+mn-n}(V) \\ &= \lim_{\alpha \to -(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{(-1)^{\frac{\mu-m}{2m}}H_{m,n}(\alpha)} = \end{aligned}$$

$$= \lim_{\alpha \to -(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha-n}{2m}+1)\Gamma(\frac{\alpha}{2m})}\frac{1}{(-1)^{\frac{\mu-m}{2m}}} \cdot \lim_{\alpha \to -(mn+2km-n)} \frac{1}{\pi^{\frac{n-2}{2}}2^{\frac{\alpha}{m}-1}} =$$

$$\frac{1}{\pi^{\frac{n-2}{2}}2^{\frac{n}{m}-n-2k-1}(-1)^{\frac{\mu-m}{2m}}} \cdot \lim_{\alpha \to -(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha-n}{2m}+1)\Gamma(\frac{\alpha}{2m})}. \tag{79}$$

On the other hand,using the formula(78)and considering the formulae(50)and(72) we have

$$
\begin{aligned}
&\lim_{\alpha\to-(mn+2km-n)} \frac{V_+^{\frac{\alpha-n}{2m}}}{\Gamma(\frac{\alpha-n}{2m}+1)\Gamma(\frac{\alpha}{2m})} \\
&\lim_{\alpha\to-(mn+2km-n)} \frac{(\alpha+mn+2km-n)^2 V_+^{\frac{\alpha-n}{2m}}}{(\alpha+mn+2km-n)^2\Gamma(\frac{\alpha-n}{2m}+1)\Gamma(\frac{\alpha}{2m})} \\
&= \frac{\lim_{\lambda\to-\frac{n}{2}-k}(\lambda+\frac{n}{2}+k)^2 V_+^{\lambda}}{\left[\lim_{\alpha\to-(mn+2km-n)}(\alpha+mn+2km-n)\Gamma(\frac{\alpha-n}{2m}+1)\right]\left[\lim_{\alpha\to-(mn+2km-n)}(\alpha+mn+2km-n)\Gamma(\frac{\alpha}{2m})\right]} \\
&= \frac{\frac{(-1)^{\frac{v+1}{2}}\pi^{\frac{n-2}{2}}}{2^{2k}k!\Gamma(\frac{n}{2}+k)} L_a^k\{\delta\}}{\frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)}\cdot\frac{2m}{(-1)^{\frac{n}{2}+k-\frac{n}{2m}}\Gamma(k-\frac{n}{2m}+1+\frac{n}{2})}} \\
&= \frac{(-1)^{\frac{v+1}{2}}\pi^{\frac{n-2}{2}}\Gamma(\frac{n}{2}+k+1-\frac{n}{2m})}{2^{2k}k!(-1)^{\frac{n}{2m}-1}2m} L_a^k\{\delta(x)\}.
\end{aligned}
\tag{80}
$$

From(79)and using(80), we obtain the following formula

$$
\begin{aligned}
B^*_{-2km}(V) = B_{-2km}(V) &= \frac{1}{\pi^{\frac{n-2}{2}} 2^{\frac{n}{m}-2k-1-n}(-1)^{\frac{\mu-m}{2m}}}\cdot \\
&\cdot\frac{(-1)^{\frac{v+1}{2}}\pi^{\frac{n-2}{2}}\Gamma(\frac{n}{2}+k+1-\frac{n}{2m})}{2^{2k}k!(-1)^{\frac{n}{2m}-1}2m} L_a^k\delta = \\
&= \frac{(-1)^{\frac{v+1}{2}}\Gamma(\frac{n}{2}+k+1-\frac{n}{2m})}{2^{\frac{n}{m}-n-1}2m(-1)^{\frac{\mu-m}{2m}}k!(-1)^{\frac{n}{2m}-1}} L_a^k\{\delta(x)\}.
\end{aligned}
\tag{81}
$$

In particular by putting $k=0$ in(81) and considering(62) we have,

$$
B_0(V) = \frac{(-1)^{\frac{v+1}{2}}\Gamma(\frac{n}{2}+1-\frac{n}{2m})}{2^{\frac{n}{m}-n-1}2m(-1)^{\frac{\mu-m}{2m}}(-1)^{\frac{n}{2m}-1}}\delta(x) \text{ if } \mu \text{ and } v \text{ are both odd.} \tag{82}
$$

We observe that by putting $m=1$,$a_1 = a_2 = ... = a_\mu = 1$ and $a_{\mu+1} = ... = a_{\mu+v} = -1$ in(81) and considering(5)and(6) we obtain the following formula,

$$
\begin{aligned}
B^*_{-2k}(V) = B_{-2k}(V) = R^*_{-2k}(u) = R_{-2k}(u) = \\
= \frac{(-1)^{\frac{v+1}{2}}k!}{2^{-1}2k!(-1)^{\frac{\mu-1}{2}}(-1)^{\frac{n}{2}-1}} L^k\delta = (-1)L^k\delta
\end{aligned}
\tag{83}
$$

where $R^*_\alpha(u)$ is defined by(67),$R_\alpha(u)$ in(6)and L is the operator defined by(61).

The formula (83) is due to Manuel A. Aguirre and appear in [6], p. 146, formula (2.52).

Acknowledgments

This work was partially supported by Comisión de Investigaciones Científicas de la provincia de Buenso Aires (C.I.C.), Argentina.

References

[1] Y. Nozaki, On Riemann-Liouville integral of ultra-hyperbolic type, *Kodai Mathematical Seminar Reports*, **6**, No. 2 (1964), 69-87.

[2] S.E. Trione, On Marcel Riesz ultra-hyperbolic kernel, Series I, *Trab. de Matemática*, preprint, No. 116, IAM-CONICET (1987).

[3] M. Riesz, L'integrale de Riemann-Liuoville et le probleme de caucht, *Acta Math.*, **81** (1949), 1-223.

[4] A. Erdelyi, Ed., *Higher Transcendental Functions*, Volumes I and II, McGraw-Hill, New York (1953).

[5] M.A. Aguirre, The distributional convolution product of Marcel Riesz, ultra-hyperbolic kernel, *Revista de la Unión Matemática Argentina*, **39** (1995).

[6] M.A. Aguirre, The distributionalHankel transform of Marcel Riesz, ultra-hyperbolic kernel, *Studies in Applied Mathematics*, **93** (1994), 133-162.

[7] M.A. Aguirre, The Residue of distribution $\left(\sum_{i=1}^{n} a_i x_i^2 \pm i0\right)^{\lambda}$, To Appear.

[8] M.A. Aguirre, The Fourier transform of $\left((\sum_{i=1}^{n} a_i x_i^2)^h \pm i0\right)^{\lambda}$, *International Journal of Pure and Applied Mathematic*, **64**, No. 3, 389-397.

[9] I.M. Gelfand, G.E. Shilov, *Generalized Functions*, Volume I, Academic Press, New York (1964).