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FUZZY STABILITY FOR THE FUNCTIONAL EQUATION STEMMING FROM QUADRATIC-ADDITIVE MAPPING

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Abstract: In this work, we establish a fuzzy version of stability for the functional equation of additive and quadratic type

$$f(x-y)-f(-x+y)-4f(x)+f(2x)-f(-y)+f(y)=0$$

in the sense of Mirmostafaee and Moslehian.

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1. Introduction and Preliminaries

A classical question in the theory of functional equations is "*when is it true that a mapping, which approximately satisfies a functional equation, must be*

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somehow close to an exact solution of the equation?" Such a problem, called a *stability problem of the functional equation*, was formulated by Ulam [22]. Hyers [6] gave a partial solution of Ulam's problem for the case of approximate additive mappings. His result was generalized by Aoki [1] for additive mappings, and by Rassias [20] for linear mappings to considering an unbounded Cauchy differences. Thereafter, the stability problems of functional equations have been extensively investigated (see, for example, [4], [5], [7], [9], [10], [12]-[16], [21]).

Katsaras [8] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, some authors have introduced several types of fuzzy norm in different points of view. In particular, Bag and Samanta [2], following Cheng and Mordeson [3], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11]. Recently, Mirmostafaee and Moslehian [17, 18] dealt with a fuzzy stability for the additive functional equation and for the quadratic functional equation.

Now we take account the functional equation

$$f(x-y)-f(-x+y)-4f(x)+f(2x)-f(-y)+f(y)=0. \qquad (1.1)$$

Throughout this work, we promise that the equation (1.1) is called the functional equation of *additive and quadratic type* and every solution of the equation (1.1) is said to be a *quadratic-additive mapping*. Quite recently, Park [19] obtained a stability of the functional equation (1.1) by handling the odd part and the even part of the given mapping, respectively.

The main goal of this work offer a general stability result of the functional equation (1.1) in the fuzzy normed linear space in the manner of Mirmostafaee and Moslehian [17]. So as to accomplish it, we present a Cauchy sequence, starting from a given mapping, which converges to the desired mapping in the fuzzy sense. Unlike Park's attempt, based on our motion in this work, we can take the desired approximate solution at only one time. For this reason, the idea is a refinement with respect to the simplicity of the proof.

For explicitly later use, we state the following definition of a fuzzy normed space given in [2] and the fundamental concepts:

Definition 1.1. (see [2]) Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ (the so-called *fuzzy subset*) is said to be *a fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x,c)=0$ for $c \leq 0$;

(N2) $x=0$ if and only if $N(x,c)=1$ for all $c>0$;

(N3) $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$;

(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;

(N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$.

In this case, the pair (X, N) is called a *fuzzy normed linear space.*

The examples of fuzzy norms and the properties of fuzzy normed linear spaces are given in [17, 18].

Definition 1.2. Let (X, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.3. Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for each $\varepsilon > 0$ and each $t > 0$, there exists n_0 such that, for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is known that every convergent sequence in a fuzzy normed space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called a *fuzzy Banach space.*

2. Fuzzy Stability of Equation (1.1)

From now on, let (X, N) be a fuzzy normed space and (Y, N') a fuzzy Banach space, respectively. Given a mapping $f : X \to Y$, we set

$$Df(x, y) := f(x - y) - f(-x + y) - 4f(x) + f(2x) - f(-y) + f(y)$$

for all $x, y \in X$. Given $q > 0$, a mapping f is said to be a *fuzzy q-almost quadratic-additive mapping* if

$$N'(Df(x, y), t + s) \geq \min\{N(x, s^q), N(y, t^q)\} \tag{2.1}$$

for all $x, y \in X \setminus \{0\}$ and all $s, t \in (0, \infty)$. Now we get the general stability in the fuzzy normed linear space.

Theorem 2.1. *Let $q \in (0, \infty) \setminus \{\frac{1}{2}, 1\}$ and let f be a fuzzy q-almost quadratic-additive mapping from a fuzzy normed space (X, N) into a fuzzy Banach space (Y, N') with $f(0) = 0$. Then there is a unique quadratic-additive mapping $F : X \to Y$ such that*

$$N'(F(x)-f(x),t) \\ \geq \begin{cases} \sup_{0<s<t} N\left(x,[\frac{1}{2}(2-2^p)s]^q\right) & \textit{if} \quad q>1, \\ \sup_{0<s<t} N\left(x,[\frac{1}{4}(4-2^p)(2^p-2)s]^q\right) & \textit{if} \quad \frac{1}{2}<q<1, \\ \sup_{0<s<t} N\left(x,[\frac{1}{2}(2^p-4)s]^q\right) & \textit{if} \quad 0<q<\frac{1}{2} \end{cases} \tag{2.2}$$

for each $x \in X$ *and all* $t>0$, *where* $p=\frac{1}{q}$.

Proof. We will prove the theorem in three different cases for $q>1$, $\frac{1}{2}<q<1$, and $0<q<\frac{1}{2}$:

Case 1. Let $q>1$ and let $J_n f : X \to Y$ be a mapping defined by

$$J_n f(x) := \frac{1}{2}\left(4^{-n}\left[f(2^n x)+f(-2^n x)\right]+2^{-n}\left[f(2^n x)-f(-2^n x)\right]\right)$$

for all $x \in X$, where $n \in \mathbb{N}\cup\{0\}$. Then $J_0 f(x)=f(x)$, $J_j f(0)=0$, and

$$J_j f(x)-J_{j+1}f(x) \\ = -\frac{2^{j+1}+1}{2\cdot 4^{j+1}}Df(2^j x,2^j x)+\frac{2^{j+1}-1}{2\cdot 4^{j+1}}Df(-2^j x,-2^j x), \tag{2.3}$$

for all $x \in X\setminus\{0\}$ and all $j\geq 0$. This equation, together with (N3), (N4) and (2.1), implies that, if $m\geq 0$ and $n>0$, then

$$\begin{aligned} N'&\left(J_m f(x)-J_{n+m}f(x),\sum_{j=m}^{n+m-1}\left(\frac{2^p}{2}\right)^j t\right) \\ &\geq N'\left(\sum_{j=m}^{n+m-1}\left[J_j f(x)-J_{j+1}f(x)\right],\sum_{j=m}^{n+m-1}\left(\frac{2^p}{2}\right)^j t\right) \\ &\geq \min_{m\leq j\leq n+m-1} N'\left(J_j f(x)-J_{j+1}f(x),\left(\frac{2^p}{2}\right)^j t\right) \\ &\geq \min_{(\sigma,j)\in\{1,-1\}\times\{m,m+1,\cdots,n+m-1\}} N'\left(\sigma\frac{(2^{j+1}-\sigma)Df(-\sigma 2^j x,-\sigma 2^j x)}{2\cdot 4^{j+1}},\right. \\ &\qquad \left.\frac{(2^{j+1}-\sigma)2^{jp}t}{2\cdot 4^{j+1}}\right) \\ &\geq \min_{m\leq j\leq n+m-1} N\left(2^j x,2^j t^q\right) \\ &= N(x,t^q) \end{aligned}$$

for all $x \in X \setminus \{0\}$ and all $t > 0$, where $0 < s < t$. Hence we have the following inequality

$$N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{2^p}{2}\right)^j t\right) \geq N(x, t^q) \tag{2.4}$$

for all $x \in X \setminus \{0\}$, all $t > 0$, all $m \geq 0$ and all $n > 0$. Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} N(x, t^q) = 1$, there is $t_0 > 0$ such that

$$N(x, t_0^q) \geq 1 - \varepsilon.$$

We observe that for some $\tilde{t} > t_0$, the series $\sum_{j=0}^{\infty} \left(\frac{2^p}{2}\right)^j \tilde{t}$ converges for $p = \frac{1}{q} < 1$. It guarantees that for an arbitrary given $c > 0$, there exists $n_0 \geq 0$ such that $\sum_{j=m}^{n+m-1} \left(\frac{2^p}{2}\right)^j \tilde{t} < c$ for each $m \geq n_0$ and each $n > 0$. By virtue of (N5) and (2.4), we feel that

$$\begin{aligned} N'(J_m f(x) - J_{n+m} f(x), c) &\geq N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{2^p}{2}\right)^j \tilde{t}\right) \\ &\geq N(x, \tilde{t}^q) \geq N(x, t_0^q) \geq 1 - \varepsilon \end{aligned}$$

for all $x \in X \setminus \{0\}$, all $m \geq 0$ and all $n > 0$. Thus $\{J_n f(x)\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N') for all $x \in X$, and so we can define a mapping $F : X \to Y$ by

$$F(x) := N'\text{-}\lim_{n\to\infty} J_n f(x).$$

In addition, if we put $m = 0$ in (2.4), we have that

$$N'(f(x) - J_n f(x), t) \geq N\left(x, \frac{t^q}{\left[\sum_{j=0}^{n-1} \left(\frac{2^p}{2}\right)^j\right]^q}\right) \tag{2.5}$$

for all $x \in X \setminus \{0\}$, all $t > 0$ and all $n \in \mathbb{N}$.

Next, we are in the position to show that F is the desired quadratic additive mapping. Using (N4), we arrive at the inequality

$$\begin{aligned} N'(DF(x, y), t) \geq \min\Big\{ &N'\left((F - J_n f)(x - y), \frac{t}{12}\right), \\ &N'\left((J_n f - F)(y - x), \frac{t}{12}\right), \end{aligned} \tag{2.6}$$

$$N'\left(4(J_nf-F)(x),\frac{t}{12}\right),N'\left((F-J_nf)(2x),\frac{t}{12}\right),$$
$$N'\left((J_nf-F)(-y),\frac{t}{12}\right),N'\left((F-J_nf)(y),\frac{t}{12}\right),$$
$$N'\left(DJ_nf(x,y),\frac{t}{2}\right)\Bigg\}, \tag{2.7}$$

for all $x, y \in X \setminus \{0\}$, all $t > 0$ and all $n \in \mathbb{N}$. The first six terms on the right hand side of (2.6) goes to 1 as $n \to \infty$ by the definition of F and (N2), and the last term on the right hand side of (2.6) satisfies

$$N'(DJ_nf(x,y),\frac{t}{2}) \geq \min\left\{N'\left(\frac{Df(2^nx,2^ny)}{2\cdot 4^n},\frac{t}{8}\right),N'\left(\frac{Df(-2^nx,-2^ny)}{2\cdot 4^n},\frac{t}{8}\right),\right.$$
$$\left.N'\left(\frac{Df(2^nx,2^ny)}{2\cdot 2^n},\frac{t}{8}\right),N'\left(\frac{Df(-2^nx,-2^ny)}{2\cdot 2^n},\frac{t}{8}\right)\right\}$$

for all $x, y \in X \setminus \{0\}$, all $t > 0$ and all $n \in \mathbb{N}$. By (N3) and (2.1), we see that

$$\begin{aligned}N'\left(\frac{Df(\pm 2^nx,\pm 2^ny)}{2\cdot 4^n},\frac{t}{8}\right) &= N'\left(Df(\pm 2^nx,\pm 2^ny),\frac{4^nt}{4}\right)\\ &\geq \min\left\{N\left(2^nx,\left(\frac{4^nt}{8}\right)^q\right),N\left(2^ny,\left(\frac{4^nt}{8}\right)^q\right)\right\}\\ &\geq \min\left\{N\left(x,\frac{2^{(2q-1)n}}{2^{3q}}t^q\right),N\left(y,\frac{2^{(2q-1)n}}{2^{3q}}t^q\right)\right\}\end{aligned}$$

and

$$N'\left(\frac{Df(\pm 2^nx,\pm 2^ny)}{2\cdot 2^n},\frac{t}{8}\right) \geq \min\left\{N\left(x,\frac{2^{(q-1)n}}{2^{3q}}t^q\right),N\left(y,\frac{2^{(q-1)n}}{2^{3q}}t^q\right)\right\}$$

for all $x, y \in X \setminus \{0\}$, all $t > 0$ and all $n \in \mathbb{N}$. Together with (N5), since $q > 1$, we can deduce that the last term on the right hand side of (2.6) also tends to 1 as $n \to \infty$. It follows by (2.6) that $N'(DF(x,y),t) = 1$ for each $x, y \in X \setminus \{0\}$ and each $t > 0$. Due to (N2), this means that $DF(x,y) = 0$ for all $x, y \in X \setminus \{0\}$. Since $f(0) = 0$, we get $f(x,0) = Df(x,x) = 0$ for all $x \in X \setminus \{0\}$ and $Df(0,y) = 0$ for all $y \in X$. This means that F is a quadratic additive mapping.

Now, we approximate the difference between f and F in a fuzzy sense. For an arbitrary fixed $x \in X \setminus \{0\}$ and an arbitrary fixed $t > 0$, choose $\varepsilon \in (0,1)$

and $s \in (0,t)$. Since $F(x)$ is the limit of $\{J_n f(x)\}$, we find that there is $n \in \mathbb{N}$ such that

$$N'\left(F(x) - J_n f(x), t-s\right) \geq 1-\varepsilon.$$

By (2.5), we have

$$\begin{aligned} N'(F(x)-f(x),t) &\geq \min\left\{N'\left(F(x)-J_nf(x),t-s\right), N'\left(J_nf(x)-f(x),s\right)\right\} \\ &\geq \min\left\{1-\varepsilon, N\left(x, \frac{s^q}{\left[\sum_{j=0}^{n-1}\left(\frac{2^p}{2}\right)^j\right]^q}\right)\right\} \\ &\geq \min\left\{1-\varepsilon, N\left(x, \frac{(2-2^p)^q s^q}{2^q}\right)\right\}. \end{aligned}$$

Because $\varepsilon \in (0,1)$ is arbitrary, we get the inequality (2.2) in this case.

Finally, to prove the uniqueness of F, let $F' : X \to Y$ be another quadratic-additive mapping satisfying (2.2). Then we have by (2.3)

$$\begin{cases} F(x) - J_nF(x) = \sum_{j=0}^{n-1}\left[J_jF(x) - J_{j+1}F(x)\right] = 0 \\ F'(x) - J_nF'(x) = \sum_{j=0}^{n-1}\left[J_jF'(x) - J_{j+1}F'(x)\right] = 0 \end{cases} \tag{2.8}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. This equation with (N4) and (2.2) gives that

$$\begin{aligned} &N'(F(x)-F'(x),t) \\ &\quad= N'(J_nF(x)-J_nF'(x),t) \\ &\quad\geq \min\left\{N'\left(J_nF(x)-J_nf(x), \frac{t}{2}\right), N'\left(J_nf(x)-J_nF'(x), \frac{t}{2}\right)\right\} \\ &\quad\geq \min\left\{N'\left(\frac{(F-f)(2^nx)}{2\cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(2^nx)}{2\cdot 4^n}, \frac{t}{8}\right),\right. \\ &\qquad\quad N'\left(\frac{(F-f)(-2^nx)}{2\cdot 4^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(-2^nx)}{2\cdot 4^n}, \frac{t}{8}\right), \\ &\qquad\quad N'\left(\frac{(F-f)(2^nx)}{2\cdot 2^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(2^nx)}{2\cdot 2^n}, \frac{t}{8}\right), \\ &\qquad\quad \left. N'\left(\frac{(F-f)(-2^nx)}{2\cdot 2^n}, \frac{t}{8}\right), N'\left(\frac{(f-F')(-2^nx)}{2\cdot 2^n}, \frac{t}{8}\right)\right\} \\ &\quad\geq \sup_{0<s<t} N\left(x, 2^{-n}[2^{n-3}(2-2^p)s]^q\right) \end{aligned}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. Note that for $q = \frac{1}{p} > 1$, the last term of the previous inequality tends to 1 as $n \to \infty$ by (N5). This implies that

$F(x) = F'(x)$ for all $x \in X \setminus \{0\}$ by (N2) and so we conclude that $F(x) = F'(x)$ for all $x \in X$ by the fact $F(0) = 0 = F'(0)$.

Case 2. Let $\frac{1}{2} < q < 1$ and let $J_n f : X \to Y$ be a mapping defined by

$$J_n f(x) := \frac{1}{2}\left(4^{-n}\big[f(2^n x) + f(-2^n x)\big] + 2^n\left[f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right]\right)$$

for all $x \in X$, where $n \in \mathbb{N} \cup \{0\}$. Then we have $J_0 f(x) = f(x)$ and

$$\begin{aligned} J_j f(x) - J_{j+1} f(x) = &-\frac{1}{2 \cdot 4^{j+1}} Df(2^j x, 2^j x) - \frac{1}{2 \cdot 4^{j+1}} Df(-2^j x, -2^j x) \\ &+ 2^{j-1} Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) - 2^{j-1} Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right) \end{aligned}$$

for all $x \in X \setminus \{0\}$ and $j \geq 0$. If $n + m > m \geq 0$, then we have

$$\begin{aligned} &N'\left(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1}\left[\frac{1}{2}\left(\frac{2^p}{4}\right)^j + \left(\frac{2}{2^p}\right)^{j+1}\right] t\right) \\ &\geq \min_{m \leq j \leq n+m-1} \min\left\{N'\left(-\frac{Df(2^j x, 2^j x)}{2 \cdot 4^{j+1}}, \frac{2^{jp} t}{4^{j+1}}\right),\right. \\ &\qquad N'\left(-\frac{Df(-2^j x, -2^j x)}{2 \cdot 4^{j+1}}, \frac{2^{jp} t}{4^{j+1}}\right), \\ &\qquad N'\left(2^{j-1} Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), \frac{2^j t}{2^{(j+1)p}}\right), \\ &\qquad \left. N'\left(-2^{j-1} Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right), \frac{2^j t}{2^{(j+1)p}}\right)\right\} \\ &\geq \min_{m \leq j \leq n+m-1} \min\left\{N(2^j x, 2^j t^q), N\left(\frac{x}{2^{j+1}}, \frac{t^q}{2^{j+1}}\right)\right\} \\ &= N(x, s^q) \end{aligned}$$

for all $x \in X \setminus \{0\}$ and $t > 0$. In the similar way following (2.4) of the previous case, we can define the limit $F(x) := N'\text{-}\lim_{n\to\infty} J_n f(x)$ of the Cauchy sequence $\{J_n f(x)\}$ in the Banach fuzzy space Y. Moreover, by letting $m = 0$ in the above inequality, then we have

$$N'(f(x) - J_n f(x), t) \geq N\left(x, \frac{s^q}{\left(\sum_{j=0}^{n-1}\left[\frac{1}{2}\left(\frac{2^p}{4}\right)^j + \left(\frac{2}{2^p}\right)^{j+1}\right]\right)^q}\right) \tag{2.9}$$

for each $x \in X \setminus \{0\}$ and $t > 0$.

In order to prove that F is a quadratic additive function, it is sufficient to show that the last term of (2.6) in Case 1 tends to 1 as $n \to \infty$. By (N3) and (2.1), we yield that

$$\begin{aligned} & N'\left(DJ_nf(x,y),\frac{t}{2}\right) \\ & \geq \min\left\{N'\left(\frac{Df(2^nx,2^ny)}{2\cdot 4^n},\frac{t}{8}\right),N'\left(\frac{Df(-2^nx,-2^ny)}{2\cdot 4^n},\frac{t}{8}\right),\right. \\ & \qquad \left. N'\left(2^{n-1}Df\left(\frac{x}{2^n},\frac{y}{2^n}\right),\frac{t}{8}\right),N'\left(2^{n-1}Df\left(\frac{-x}{2^n},\frac{-y}{2^n}\right),\frac{t}{8}\right)\right\} \\ & \geq \min\left\{N(x,2^{(2q-1)n-3q}t^q),N(y,2^{(2q-1)n-3q}t^q),\right. \\ & \qquad \left. N(x,2^{(1-q)n-3q}t^q),N(y,2^{(1-q)n-3q}t^q)\right\} \end{aligned}$$

for each $x, y \in X \setminus \{0\}$ and $t > 0$. Observe that all the terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, since $\frac{1}{2} < q < 1$. Employing the similar argument after (2.6), we see that $DF(x,y) = 0$ for all $x, y \in X$. Recall, in Case 1, the inequality (2.2) follows from (2.5). By the same reasoning, we get (2.2) from (2.9) in this case.

Now, to prove the uniqueness of F, let F' be another quadratic additive mapping satisfying (2.2). Then, together with (N4), (2.2), and (2.8), we have that

$$\begin{aligned} & N'(F(x)-F'(x),t) = N'(J_nF(x)-J_nF'(x),t) \\ & \geq \min\left\{N'\left(J_nF(x)-J_nf(x),\frac{t}{2}\right),N'\left(J_nf(x)-J_nF'(x),\frac{t}{2}\right)\right\} \\ & \geq \min\left\{N'\left(\frac{(F-f)(2^nx)}{2\cdot 4^n},\frac{t}{8}\right),\left(\frac{(f-F')(2^nx)}{2\cdot 4^n},\frac{t}{8}\right),\right. \\ & \qquad N'\left(\frac{(F-f)(-2^nx)}{2\cdot 4^n},\frac{t}{8}\right),N'\left(\frac{(f-F')(-2^nx)}{2\cdot 4^n},\frac{t}{8}\right), \\ & \qquad N'\left(2^{n-1}(F-f)\left(\frac{x}{2^n}\right),\frac{t}{8}\right),N'\left(2^{n-1}(f-F')\left(\frac{x}{2^n}\right),\frac{t}{8}\right), \\ & \qquad \left. N'\left(2^{n-1}(F-f)\left(\frac{-x}{2^n}\right),\frac{t}{8}\right),N'\left(2^{n-1}(f-F')\left(\frac{-x}{2^n}\right),\frac{t}{8}\right)\right\} \\ & \geq \min\Big\{\sup_{0<s<t} N\big(x,2^{-n}\left[2^{2n-4}(4-2^p)(2^p-2)s\right]^q\big), \\ & \qquad \sup_{0<s<t} N\big(x,2^{n}\left[2^{-n-4}(4-2^p)(2^p-2)s\right]^q\big)\Big\} \end{aligned}$$

for all $x \in X \setminus \{0\}$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} 2^{(2q-1)n} = \lim_{n\to\infty} 2^{(1-q)n} = \infty$$

in this case, both terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ by (N5). This means that

$$N'(F(x) - F'(x),\ t) = 1$$

for all $x \in X \setminus \{0\}$ and so $F(x) = F'(x)$ for all $x \in X$ by (N2).

Case 3. Finally, we take $0 < q < \frac{1}{2}$ and define $J_n f : X \to Y$ by

$$J_n f(x) := \frac{1}{2}\left(4^n\left[f(2^{-n}x) + f(-2^{-n}x)\right] + 2^n\left[f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right]\right)$$

for all $x \in X$, where $n \in \mathbb{N} \cup \{0\}$. Then we have $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \frac{1}{2}(4^j + 2^j) Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) + \frac{1}{2}(4^j - 2^j) Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right)$$

which implies that if $n + m > m \geq 0$, then

$$\begin{aligned}
&N'\left(J_m f(x) - J_{n+m} f(x), 2\sum_{j=m}^{n+m-1}\left(\frac{4}{2^p}\right)^j \frac{t}{2^p}\right) \\
&\geq \min_{m\leq j\leq n+m-1} \min\left\{N'\left(\frac{1}{2}(4^j + 2^j) Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), \frac{(4^j + 2^j)t}{2^{(j+1)p}}\right),\right. \\
&\qquad \left. N'\left(\frac{1}{2}(4^j - 2^j) Df\left(-\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}}\right), \frac{(4^j - 2^j)t}{2^{(j+1)p}}\right)\right\} \\
&\geq \min_{m\leq j\leq n+m-1} N\left(\frac{x}{2^{j+1}}, \frac{t^q}{2^{j+1}}\right) \\
&= N(x, t^q)
\end{aligned}$$

for all $x \in X \setminus \{0\}$ and all $t > 0$. By similar method to the previous cases, it leads us to define a mapping $F : X \to Y$ by

$$F(x) := N'\text{-}\lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Letting $m = 0$ in the above inequality, we have

$$N'(f(x) - J_n f(x,t) \geq N\left(x, \frac{s^q}{\left[\frac{2}{2^p}\sum_{j=0}^{n-1}\left(\frac{4}{2^p}\right)^j\right]^q}\right) \tag{2.10}$$

for all $x \in X \setminus \{0\}$, all $t > 0$ and all $n \in \mathbb{N}$. Note that

$$\begin{aligned} &N'\left(DJ_nf(x,y),\frac{t}{2}\right) \\ &\quad\geq \min\left\{N'\left(\frac{4^n}{2}Df\left(\frac{x}{2^n},\frac{y}{2^n}\right),\frac{t}{8}\right),N'\left(\frac{4^n}{2}Df\left(-\frac{x}{2^n},-\frac{y}{2^n}\right),\frac{t}{8}\right),\right. \\ &\qquad\qquad \left. N'\left(2^{n-1}Df\left(\frac{x}{2^n},\frac{y}{2^n}\right),\frac{t}{8}\right),N'\left(2^{n-1}Df\left(-\frac{x}{2^n},-\frac{y}{2^n}\right),\frac{t}{8}\right)\right\} \\ &\quad\geq \min\left\{N\big(x,2^{(1-2q)n-3q}t^q\big),N\big(y,2^{(1-2q)n-3q}t^q\big),N\big(x,2^{(1-q)n-3q}t^q\big),\right. \\ &\qquad\qquad \left. N\big(y,2^{(1-q)n-3q}t^q\big)\right\} \end{aligned}$$

for each $x, y \in X \setminus \{0\}$, each $t > 0$ and each $n \in \mathbb{N}$. Since $0 < q < \frac{1}{2}$, both terms on the right hand side tend to 1 as $n \to \infty$ which implies that the last term of (2.6) tends to 1 as $n \to \infty$. Therefore, $DF \equiv 0$. Moreover, using the similar argument after (2.6) in Case 1, we get the inequality (2.2) from (2.10) in this case.

To prove the uniqueness of F, let $F' : X \to Y$ be another quadratic additive function satisfying (2.2). Then by (2.8), we deduce that

$$\begin{aligned} &N'(F(x)-F'(x),t) \\ &\quad\geq \min\left\{N'\left(J_nF(x)-J_nf(x),\frac{t}{2}\right),N'\left(J_nf(x)-J_nF'(x),\frac{t}{2}\right)\right\} \\ &\quad\geq \min\left\{N'\left(\frac{4^n}{2}(F-f)\left(\frac{x}{2^n}\right),\frac{t}{8}\right),N'\left(\frac{4^n}{2}(f-F')\left(\frac{x}{2^n}\right),\frac{t}{8}\right),\right. \\ &\qquad\qquad N'\left(\frac{4^n}{2}(F-f)\left(-\frac{x}{2^n}\right),\frac{t}{8}\right),N'\left(\frac{4^n}{2}(f-F')\left(-\frac{x}{2^n}\right),\frac{t}{8}\right), \\ &\qquad\qquad N'\left(2^{n-1}(F-f)\left(\frac{x}{2^n}\right),\frac{t}{8}\right),N'\left(2^{n-1}(f-F')\left(\frac{x}{2^n}\right),\frac{t}{8}\right), \\ &\qquad\qquad \left. N'\left(2^{n-1}(F-f)\left(-\frac{x}{2^n}\right),\frac{t}{8}\right),N'\left(2^{n-1}(f-F')\left(-\frac{x}{2^n}\right),\frac{t}{8}\right)\right\} \\ &\quad\geq \sup_{0<s<t} N\big(x,2^n[2^{-2n-3}(2^p-4)s]^q\big) \end{aligned}$$

for all $x \in X \setminus \{0\}$, all $t > 0$ and all $n \in \mathbb{N}$. Observe that for $0 < q < \frac{1}{2}$, the last term goes to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ for all $x \in X \setminus \{0\}$ and $F(x) = F'(x)$ for all $x \in X$ by (N2). This completes the proof of the theorem. $\square$

Corollary 2.2. *Let f be a mapping satisfying the inequality*

$$N'(f(x+y)+f(x-y)-2f(x)-2f(y),t+s) \geq \min\{N(x,s^q),N(y,t^q)\}$$

for all $x,y \in X \setminus \{0\}$ and $s,t>0$ with $f(0)=0$. Then there is a unique quadratic mapping $F: X \to Y$ such that

$$N'(F(x)-f(x),t) \geq \begin{cases} \sup_{0<s<t} N\left(x,[\frac{1}{8}(2-2^p)s]^q\right) & \text{if} \quad q>1, \\ \sup_{0<s<t} N\left(x,[\frac{1}{16}(4-2^p)(2^p-2)s]^q\right) & \text{if} \quad \frac{1}{2}<q<1, \\ \sup_{0<s<t} N\left(x,[\frac{1}{8}(2^p-4)s]^q\right) & \text{if} \quad 0<q<\frac{1}{2} \end{cases} \tag{2.11}$$

for all $x \in X$ and all $t>0$, where $p=\frac{1}{q}$.

Proof. Let $Qf(x,y) := f(x+y)+f(x-y)-2f(x)-2f(y)$ for all $x,y \in X\setminus\{0\}$. Note that

$$\begin{aligned} N'(Df(x,y),4t+4s) =& N'\Big(Qf(x,y)-Qf(y,-x)+Qf(x,-x)+\frac{1}{2}Qf(y,-y) \\ & \qquad -\frac{1}{2}Qf(y,y),4t+4s\Big) \\ \geq& \min\Big\{N'(Qf(x,y),s+t),N'(-Qf(y,-x),s+t), \\ & N'(Qf(x,-x),t+s), \\ & N'\Big(\frac{1}{2}Qf(y,-y),s\Big),N'\Big(-\frac{1}{2}Qf(y,y),t\Big)\Big\} \\ \geq& \min\big\{N(x,s^q),N(y,t^q)\big\} \end{aligned}$$

for all $x,y \in X \setminus \{0\}$ and all $s,t>0$. By Theorem 2.1, there is a unique quadratic-additive mapping $F: X \to Y$ satisfying (2.11) for each $x \in X \setminus \{0\}$ and each $t>0$, where $p=\frac{1}{q}$. In particular, $F: X \to Y$ is the mapping defined by

$$F(x) := N'\text{-}\lim_{n\to\infty} J_n f(x)$$

for all $x,y \in X$. By a similar method used in the proof of Theorem 2.1, we obtain

$$F(x+y)+F(x-y)-2F(x)-2F(y)=0$$

for all $x,y \in X \setminus \{0\}$ from the inequality

$$\begin{aligned} N'(QF(x,y),t) \geq \min\Big\{& N'\left((F-J_nf)(x+y),\frac{t}{8}\right),N'\left((J_nf-F)(x-y),\frac{t}{8}\right), \\ & N'\left(2(J_nf-F)(x),\frac{t}{8}\right),N'\left(2(J_nf-F)(y),\frac{t}{8}\right), \end{aligned}$$

$$N'\left(QJ_nf(x,y),\frac{t}{2}\right)\bigg\}$$

for all $x, y \in X \setminus \{0\}$ and all $n \in \mathbb{N}$. Since $F(0) = 0$, we get $QF(x, 0) = 0$ for all $x \in X$ and

$$QF(0,y) = \frac{Q(y,y) - Q(y,-y)}{2} = 0$$

for all $y \in X \setminus \{0\}$. Hence F is a quadratic mapping. The proof of the corollary is complete. □

Corollary 2.3. *Let f be an odd mapping satisfying all of the conditions of Theorem 2.1. Then there is a unique additive mapping $F : X \to Y$ such that*

$$N'(F(x) - f(x), t) \geq \sup_{0<s<t} N\left(x, (2|2 - 2^p|s)^q\right) \tag{2.12}$$

for all $x \in X \setminus \{0\}$ and all $t > 0$, where $p = \frac{1}{q}$.

Proof. Let J_nf be defined as in Theorem 2.1. Since f is an odd mapping, we obtain

$$J_nf(x) = \begin{cases} \frac{f(2^nx)+f(-2^nx)}{2^{n+1}} & \text{if} \quad 0 < q < 1, \\ 2^{n-1}\left[f(2^{-n}x) + f(-2^{-n}x)\right] & \text{if} \quad q > 1 \end{cases}$$

for all $x \in X$. Notice that $J_0f(x) = f(x)$ and

$$J_jf(x) - J_{j+1}f(x) = \begin{cases} \frac{-1}{2^{j+1}}Df(2^jx, 2^jx) & \text{if} \quad 0 < q < 1, \\ 2^jDf\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) & \text{if} \quad q > 1 \end{cases}$$

for all $x \in X \setminus \{0\}$ and all $j \in \mathbb{N} \cup \{0\}$. From these facts, using a similar argument in Theorem 2.1, we get the quadratic-additive mapping F satisfying (2.12). Note that F is also odd, $F(x) = N'\text{-}\lim_{n\to\infty} J_nf(x)$ and $DF(x, y) = 0$ for all $x, y \in X$. Hence we get

$$F(x+y) - F(x) - F(y) = \frac{1}{2}DF(x,y) - \frac{1}{3}DF(y,y) = 0$$

for all $x, y \in X$, which means that F is an additive mapping. This completes the proof of the corollary. □

We can use the theorem 2.1 to get a classical result in the framework of normed spaces. Let $(X, \|\cdot\|)$ be a normed linear space. Then we can define a fuzzy norm N_X on X by

$$N_X(x,t) = \begin{cases} 0, & t \leq \|x\| \\ 1, & t > \|x\| \end{cases}$$

where $x \in X$ and $t \in \mathbb{R}$ (see [17]). Suppose that $f : X \to Y$ is a mapping into a Banach space $(Y, |||\cdot|||)$ such that $|||Df(x,y)||| \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$. Let N_Y be a fuzzy norm on Y. Then we get

$$N_Y(Df(x,y), s+t) = \begin{cases} 0, & s+t \leq |||Df(x,y)||| \\ 1, & s+t > |||Df(x,y)||| \end{cases}$$

for all $x, y \in X$ and $s, t \in \mathbb{R}$. Consider the case $N_Y(Df(x,y), s+t) = 0$. This implies that

$$\|x\|^p + \|y\|^p \geq |||Df(x,y)||| \geq s+t$$

and so either $\|x\|^p \geq s$ or $\|y\|^p \geq t$ in this case. Hence, for $q = \frac{1}{p}$, we have $\min\{N_X(x, s^q), N_X(y, t^q)\} = 0$ for all $x, y \in X$ and all $s, t > 0$. Therefore, in every case, the inequality

$$N_Y(Df(x,y), s+t) \geq \min\{N_X(x, s^q), N_X(y, t^q)\},$$

that is, f is a fuzzy q-almost quadratic additive mapping. By Theorem 2.1, we arrive at the following stability.

Corollary 2.4. *Let $(X, \|\cdot\|)$ be a normed linear space and let $(Y, |||\cdot|||)$ be a Banach space. If*

$$|||Df(x,y)||| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X \setminus \{0\}$, where $p > 0$ and $p \neq 1, 2$, then there is a unique quadratic-additive mapping $F : X \to Y$ such that

$$|||F(x) - f(x)||| \leq \begin{cases} \frac{2||x||^p}{2-2^p} & \text{if} \quad 0 < p < 1, \\ \frac{4||x||^p}{(2^p-2)(4-2^p)} & \text{if} \quad 1 < p < 2, \\ \frac{2||x||^p}{2^p-4} & \text{if} \quad p > 2 \end{cases}$$

for all $x \in X \setminus \{0\}$.

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