VERTEX COVERING AND INDEPENDENT NUMBER ON DIFFERENCE GRAPHS

Thanin Sitthiwirattham
Department of Mathematics
Faculty of Applied Science
King Mongkut’s University of Technology North Bangkok
Bangkok, 10800, THAILAND

and

Centre of Excellence in Mathematics, CHE
Sri Ayutthaya Road, Bangkok 10400, THAILAND

Abstract: Let $\alpha(G)$ and $\beta(G)$ be the independent number and vertex covering number of $G$, respectively. The difference graph, $G_1 \triangle G_2$, of connected graphs $G_1$ and $G_2$ has vertex set $V(G_1 \triangle G_2) = V(G_1) = V(G_2)$ and edge set $E(G_1 \triangle G_2) = [E(G_1) - E(G_2)] \cup [E(G_2) - E(G_1)]$. In this paper, we determine generalizations of some graph parameters: independent number and vertex covering number on difference graph.

AMS Subject Classification: 05C69, 05C70, 05C76

Key Words: difference graph, independent number, vertex covering number

1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let $G_1$ and $G_2$ be connected graphs. The Difference graph of $G_1$ and $G_2$, denote by $G_1 \triangle G_2$, be the graph that $V(G_1 \triangle G_2) = V(G_1) = V(G_2)$ and $E(G_1 \triangle G_2) = [E(G_1) - E(G_2)] \cup [E(G_2) - E(G_1)]$. 
Next, we give the definitions about some graph parameters. A subset $U$ of the vertex set $V(G)$ of $G$ is said to be an independent set of $G$ if the induced subgraph $G[U]$ is a trivial graph. An independent set of $G$ with maximum number of vertices is called a maximum independent set of $G$. The number of vertices of a maximum independent set of $G$ is called the independent number of $G$, denoted by $\alpha(G)$.

A vertex of graph $G$ is said to cover the edges incident with it, and a vertex cover of a graph $G$ is a set of vertices covering all the edge of $G$. The minimum cardinality of a vertex cover of a graph $G$ is called the vertex covering number of $G$, denoted by $\beta(G)$.

Next we give some properties about difference graph

**Proposition 1.** Any difference graphs of connected graphs are always connected.

**Proposition 2.** Let $G_1$ and $G_2$ be connected graphs, the difference graph $G_1 \triangle G_2 \cong [G_1 - E(H)] \cup [G_2 - E(H)] = \overline{H_1} \cup \overline{H_2}$ where $G_1 \cap G_2 = H$

**Example.**

![Graphs](image)

Figure 1: The graph of $G_1 \triangle G_2$
2. Vertex Covering Number of the Difference Graph $G_1 \triangle G_2$

We now prove lemma before stating our main results. We begin this section by giving the lemma 3 which show character of vertex covering number of each $H_i$.

**Lemma 3.** Let $G_1 \cap G_2 = H$ and $G_1 \triangle G_2 = [G_1 - E(H)] \cup [G_2 - E(H)] = \overline{H_1} \cup \overline{H_2}$. For each $\overline{H_i}$, then

$$\beta(\overline{H_i}) = \beta(G_i) - \max\{Q_i\}$$

where $\max\{Q_i\}$ be maximum cardinal number of $Q_i = \{u \in V(H) | N_{\overline{H_i}}(u) - C_i = \emptyset\}$; $C_i$ are the minimum vertex covering set of $G_i, i = 1, 2$.

**Proof.** For each $\overline{H_i} = G_i - E(H)$, we get $\beta(\overline{H_i}) = \beta(G_i - E(H)) \leq \beta(G_i)$.

Let $Q_i = \{u \in V(H) | N_{\overline{H_i}}(u) - C_i = \emptyset\}$, we choose the minimum vertex covering set $C_i$ of $G_i$ such that $Q_i$ have the maximum cardinal number $= \max\{Q_i\}$.

Hence, the minimum vertex covering of $\overline{H_i}$ have

$$\beta(\overline{H_i}) = \beta(G_i) - \max\{Q_i\}; \quad i = 1, 2. \quad \square$$

**Example.**

![Vertex Covering Set Diagrams](image-url)

Figure 2: The minimum vertex covering set of $\overline{H_1}, \overline{H_2}$ and $G_1 \triangle G_2$

Next, we establish Theorem 4 for a vertex covering number of $G_1 \triangle G_2$. 

**Theorem 4.** Let $G_1 \cap G_2 = H$ and $G_1 \triangle G_2 = [G_1 - E(H)] \cup [G_2 - E(H)] = \overline{H_1} \cup \overline{H_2}$, then

$$\beta(G_1 \triangle G_2) = \beta(G_1) \cup \beta(G_2) - (|Q_1| + |Q_2| + |R|),$$
where \( R = \{ u \mid u \text{ belong to the minimum vertex covering set of } \overline{H_1} \text{ and } \overline{H_2} \} \), 
\( \max\{Q_i\} \) be maximum cardinal number of \( Q_i = \{ u \in V(H) \mid N_{\overline{H_i}}(u) - C_i = \emptyset \} \); 
\( C_i \) are the minimum vertex covering set of \( G_i, i = 1, 2 \).

Proof. Since \( G_1 \triangle G_2 = [G_1 - E(H)] \cup [G_2 - E(H)] = \overline{H_1} \cup \overline{H_2} \), we get 
\( \beta(G_1 \triangle G_2) = \beta(\overline{H_1} \cup \overline{H_2}) \leq \beta(\overline{H_1}) + \beta(\overline{H_2}) \).

Let \( R = \{ u \mid u \text{ belong to the minimum vertex covering set of } \overline{H_1} \text{ and } \overline{H_2} \} \), 
by definition of difference graph, we have \( E(\overline{H_1}) \cap E(\overline{H_2}) = \emptyset \), so 
\( \beta(G_1 \triangle G_2) = \beta(\overline{H_1}) + \beta(\overline{H_2}) - |R| \).

By lemma 3, imply that 
\[
\beta(G_1 \triangle G_2) = \beta(\overline{H_1}) + \beta(\overline{H_2}) - |R| \\
= \beta(G_1) \cup \beta(G_2) - (|Q_1| + |Q_2| + |R|).
\]

![Diagram of vertex coverings and independent sets](image-url)

Figure 3: The region of \( R \) of \( G_1 \triangle G_2 \)

3. Independent Number of \( G_1 \triangle G_2 \)

We begin this section by giving the lemma 5 that shows a relation of vertex covering number and independent number.
**Lemma 5.** (see [1]) Let $G$ be a simple graph with order $n$. Then $\alpha(G) + \beta(G) = n$.

Next, we establish Theorem 6 for a independent number of $G_1 \triangle G_2$.

**Theorem 6.** Let $G_1 \cap G_2 = H$ and $G_1 \triangle G_2 = (G_1 - E(H)) \cup (G_2 - E(H)) = H_1 \cup H_2$. For each $H_i$, then

$$\alpha(G_1 \triangle G_2) = \alpha(H_1) + \alpha(H_2) - |S|$$

where $S = \{u|u \text{ belong to the maximum independent set of } H_1 \text{ and } H_2\}$

**Proof.** By Theorem 4 and Lemma 5, we can also show that

$$\alpha(G_1 \triangle G_2) + \beta(G_1 \triangle G_2) = n$$

$$\alpha(G_1 \triangle G_2) = n - \beta(H_1) - \beta(H_2) + |R|$$

$$= n - (n - \alpha(H_1)) - (n - \alpha(H_2)) + |R|$$

$$= \alpha(H_1) + \alpha(H_2) - (n - |R|)$$

$$= \alpha(H_1) + \alpha(H_2) - |S|.$$

\[ \square \]

**Acknowledgments**

This research is supported by King Mongkut’s University of Technology North Bangkok, Thailand.

**References**
