

OPTIMAL CONTROL PROBLEM FOR A HOST-PARASITOID MODEL

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Abstract: An optimal control model for a host-parasitoid interaction is considered. The host-parasitoid interaction is described by a coupled pair of partial differential equation (with initial and boundary values) for the host; and a delay ordinary differential equation (with initial value) for the parasitoid. The parasitoid population is assumed to be a biological control agent against the host which is assumed to be a pest population. A time dependent control measure is practiced to make sure that the parasitoid is effective in diminishing the host population. An objective function is defined to study the possibility of minimizing the cost of control measures while maximizing the effectiveness of the parasitoid. Existence of a unique solution is proved to the resulting host-parasitoid state equations. Other results include establishing a unique optimal solution of the optimal control model.

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Key Words: host-parasitoid, biological control, optimal solution

1. Introduction

Host-parasitoid dynamics brings about significant changes in a given environment. Indeed, it is the result of interactions between an organism (parasitoid) which depends on another organism (host) for food and to complete its life cycle. The principal side effects of a parasitoid is to suppress its host or to de-

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rive it to extinction. However, the outcome of host-parasitoid dynamics could be perceived as either positive or negative. The negative side effect of a parasitoid population is felt through the socioeconomic and environmental damages which it causes. This happens when the parasitoids attack economically and environmentally vital biological organisms. Contrary to this, parasitoids could be needed as biological control agents if their host has to be either eliminated or substantially reduced in a given environment ([9], [11], [13], [18]). Take for example, *Entomopathogenic fungi*, which has proved to be a good biological control agent as it is a natural enemy of Varroa mite. Varroa mite causes honey bee colonies to decline in North Carolina [9]. Another example of a biological control agent are the leaf-mining flies in the genus *Hydrellia* (*Hydrellia Balciunasi*, *Hydrellia pakistanae*) which are native to the United States or imported. These flies have been used to control the aquatic weed Hydrilla, a plant population invading the swamps and lakes of the southern parts of America causing environmental damage by out-competing local plants [1]. The Kariba weed in Zimbabwe is another aquatic weed which is controlled by a biological agent (*Cyrtobagous salviniae*) [10].

However, there are certain conditions to consider while implementing biological control. For example, if two or more parasitoids are known to use the same host, assessment should be done to see which parasitoid effectively attacks a specific host [27]. Related results on competition among parasitoids for the same host and outcomes that follow are given in Taylor [28] and references therein. Moreover, if a parasitoid is imported from another environment, then monitoring the newly introduced parasitoid is vital to foster the biological control effort [29]. Monitoring activities include, protecting the parasitoid from possible obstacles such as, disease or natural enemies that hinder its adjustment to the new environment [11].

All of the controlling activities mentioned here come with overhead costs which should be minimized. In other words, the biological control practice should be cost-effective, (see for example studies in [14] and [21]). The implementation of a parasitoid-based biological control to suppress a host population causing economical damage has been studied by Mills and Getz [22] related to case studies. Host-parasitoid dynamics has been studied using several mathematical models ([8], [12], [23]). Most of these models stem from the well known Nicholson-Baily discrete model [24]. Some variations of the Nicholson-Baily models (discrete as well as continuous in time) have been applied in studies involving the roles of spatial refuge [23] and the effects of a common parasitoid for two hosts [8]. Among continuous mathematical models of host-parasitoids problems we find the work of Henson and Cushing [15] where a size-structured

host-parasitoid model was developed. An age-structured version of this model, where the birth rate in the host population is density dependent was latter studied by Blayneh ([6] and [7]). Our study in this paper extends the age-structure host-parasitoid model considered in [6] by adding optimal control terms.

Optimal control models have been used in biological problems, notably in population interactions and diseases (see for example [4], [5], [16] and [19]). The models range from discrete to systems of ordinary differential equations and partial differential equations, with variable applications such as HIV [17], vector-borne diseases [4, 5] and pest control [21]. Although some work on optimal control has been done for some spatial models involving partial differential equation, to the authors' knowledge, optimal control model for a host-parasitoid interaction between an age-structured host and a parasitoid population has not been studied. Our goal is to investigate the impact of a time dependent control function on the outcome of implementing a biological control agent (in particular a parasitoid) to reduce a pest (which we label as host: plant or animal) population. We then employ analytical method to prove the existence of a unique optimal control function where the host species is suppressed while the running cost is a minimal.

This paper is organized as follows. The basic model for host-parasitoid interactions which is studied in [6] is presented along with descriptions of model parameters and functions in Section 2. This model is then extended to consider an optimal control function in Section 3. The resulting model is augmented by the cost (objective) function to complete the optimal control model formulation. Furthermore, in the same section, a theorem stating that the state system is well-posed is proved and the baseline for the existence and uniqueness of solution to the optimal control model is presented. The existence of an optimal control solution is proved in Section 4 followed by concluding remarks in Section 5.

2. Model

In this study, we consider an optimal control model, where the state equation describes a host-parasitoid interaction. This equation consists of a coupled pair of partial and ordinary differential equations (with initial and boundary conditions). In particular, the dynamics of the host population is described by a partial differential equation of the density of the host as a function of age and time, whereas the adult parasitoid is modeled by an initial valued integro-differential equation as a function of time.

To start with, for the sake of completeness, we first review the base models

in [6]. Let $\rho(t, a)$ be the density of the host population in the range $[a, a + \Delta)$ and let $y(t)$ be the density of the adult parasitoid population at time t . As parasitoids live and reproduces on the host, their reproduction and success depend on the host. On the other hand, the parasitoid has effect on the death rate of the host. A host member which is attacked by parasitoids is assumed to be weak to reproduce that reproduction is limited to adult hosts which are free from parasitoids. It is worthwhile mentioning that similar assumptions are shared with other researchers in biology and mathematics alike (see Barclay [2], Henson and Cushing [15]).

The total adult host population is $\int_A^\infty \rho(t, a)da$, where A is the age at which a host individual becomes adult. The birth rate in the host is $\beta(a, P)$, where P is the total adult host population size. The initial host population is designated by $\phi(t)$ which is assumed to be in $L^1(0, \infty)$.

Model Parameters	
Parameter/ Functions	Description (rates are per day)
A	the age of maturity in the host population
ν	the time it takes for egg of a parasitoid to become adult
$I(\nu)$	the probability that a parasitoid egg produces an emergent parasitoid in ν time period
δ	parasitoid natural death rate
$\varphi(a)$	the probability that un matured host of age a is parasitized
$\phi(a)$	the total number of hosts of age a initially
β	the per capita adult host reproduction rate
λ	the per capita host death rate
$D(a)$	the number of eggs laid in a young host of age a by adult parasitoids
k_0	carrying capacity of the environment

Table 1: Model parameters/functions and their descriptions. A young host is a host a host not at adult stage

The dynamics of the host population with density $\rho(t, a)$ is given by the following initial-boundary value partial differential equation.

$$\frac{\partial \rho(t, a)}{\partial t} + \frac{\partial \rho(t, a)}{\partial a} = -(\lambda(a, P) + \varphi(a)E(y)) \rho(t, a), \tag{2.1}$$

where $E(y)$ is host-parasitoid encounter rate while the adult parasitoid is at level y . The term $\phi(a)E(y)$ describes the per capita removal rate due to par-

asitization. A variety of host-parasitoid encounter rate functions have been considered in the literature. They consider search efficiency, egg-limited dynamics and density-dependent effects (see [26] and references therein). Our model which is a variation of what is studied in [6], considers density-dependent effects. The boundary condition (the total number of newborn individuals) is density dependent as the birth rate $\beta(a, P)$ depends on the total size of the adult host population.

$$\rho(t, 0) = \int_A^\infty \beta(a, P)\rho(t, a)da, \quad t > 0 \quad (2.2)$$

$$\rho(0, a) = \phi(a) \quad (2.3)$$

$$P(t) = \int_0^\infty \rho(t, a)da. \quad (2.4)$$

The dynamic of the adult parasitoid population level is given by a delay differential equation

$$\begin{cases} y'(t) = I(v)E(y(t-v)) \int_0^A \varphi(a)D(a)\rho(t-v, a)da - \delta y, & t > 0 \\ y(t) = y_0, & t \in [-v, 0]. \end{cases} \quad (2.5)$$

In this sub model, $I(v)$ is the probability that a parasitoid egg produces an emergent parasitoid in ν time period, whereas, δ is the per capita natural death rate of parasitoids. Analytical studies have been done on the model given by (2.1) - (2.5) (see [6] and [7]).

The functions and parameters of this model are described in Table (1) and satisfy the following technical assumptions. Note that $R_+ = [0, \infty)$.

H1 $\phi, \lambda, \phi, \beta$, and D are in $C^+[0, \infty)$ and bounded on $[0, \infty)$. $I \in C(R_+, [0, 1])$.

H2 $E \in C^1[0, \infty)$, $E' > 0$, $E(0) = 0$, $\lim_{y \rightarrow \infty} E(y) = E_\infty < \infty$. E satisfies Lipschitz condition,
 $\|E(x_1) - E(x_2)\| \leq K\|x_1 - x_2\|$, $K > 0$ is a constant and $x_1, x_2 \in [0, \infty)$.

H3 $\phi \in L_1(R_+)$.

H4 $\lambda(a, P)$ satisfies Lipschitz condition in the second variable uniformly for all $a > 0$.

3. Optimal Control Model Formulation

In Section 2, the basic model is given without control applied to enhance the effect of the parasitoid on the host population. The objective of this section is to formulate the control model which extends system (2.1) - (2.5). To this end, we assume that a time dependent control is practiced to monitor the condition of the parasitoid. The effect of the control measures on the host is indirect: it goes through the response of the parasitoid to the control measure. Accordingly, we have made no modification to the host model given by (2.1) - (2.4) in Section 2.

We formulate the model for the adult parasitoid population by including a time dependent control function. In general, this control function simply describes the overall measures considered so that the biological control strategy is effective. With a control function included, the dynamic of the adult parasitoid becomes

$$\begin{cases} y'(t) = I(v)E(y(t-v)) \int_0^A \varphi(a)(1+u(t))D(a)\rho(t-v,a)da - \delta y, & t > 0 \\ y(t) = y_0, & t \in [-v, 0], \end{cases} \tag{3.1}$$

where $u(t)$ in in the control set defined by

$$\mathcal{U} = \{u(t) : u(t) \text{ is piecewise continuous on } [0, T] \text{ and } 0 \leq u \leq 1\}. \tag{3.2}$$

Experiments and follow up of parasitoid progress could be practiced for some-time continuously, but in most cases, this practice could be interrupted from time to time. This means, the practice could jump up or drop to a lower value as needed. In order to address this practice, we assume that the control function is piecewise continuous on $[0, T]$, where T is the duration of control practice. To formulate the optimal control model we define the objective function

$$\begin{aligned} J(u) = & \frac{1}{2} \int_0^T \int_0^A [(\rho(t,a) - \bar{\rho}_0(a,t))^2 da] dt + \frac{\alpha_1}{2} \int_0^T (y(t) - k_0)^2 dt \\ & + \frac{\alpha_2}{2} \int_0^T (u(t) - u_0(t))^2 dt, \end{aligned} \tag{3.3}$$

subject to the state system given by the coupled pair of equations (2.1) - (2.4) and (3.1). We assume that $\rho, y \in L^2([0, T])$ and the control $u \in \mathcal{U}$, where \mathcal{U} is

the non-empty closed bounded convex subset of $L^2([0, T])$ which is defined by (3.2).

Although the parasitoid is used as a biological control agent, its number should be maintained at the level that the environment can sustain. To monitor this, the term

$$\frac{\alpha_1}{2} \int_0^T (y(t) - k_0)^2 dt$$

is considered to be part of the objective function. This term measures the variation of the parasitoid population level relative to the carrying capacity k_0 of the environment. Furthermore, we assume that the host density, which is observed before the control practice started is $\bar{\rho}_0$. This could be obtained experimentally at a given location (assumed to be closed to immigration of hosts). The regularization or penalty term

$$\frac{\alpha_2}{2} \int_0^T (u(t) - u_0(t))^2 dt$$

represents the cost of implementing the control mechanism with weight constant $\alpha_2 \geq 0$, where $u_0(t)$ is an estimation of the control variable u . Therefore, $\int_0^T (u(t) - u_0(t))^2 dt$ describes the minimal cost provided that the optimal control $u(t)$ is close enough to $u_0(t)$. Note that $0 \leq u_0(t) \leq 1$. The state system for the host and the parasitoid, with control function added, is well-posed. This is established by the following theorem.

Theorem 1. *Suppose assumptions (H1)-(H4) hold and that \mathcal{U} is defined by (3.2). For each control $u \in \mathcal{U}$, the state equation given by (2.1) - (2.4) and (3.1) has a unique solution pair $(\rho_u(t, a), y_u(t))$, for $t \in [0, T]$.*

Proof. The existence and uniqueness of the solution of (2.1)-(2.5) is proved in [6]. Indeed, in [6], the proof is provided for $y \in C^+[0, T]$. This model is extended to the modified system given by (2.1)-(2.4) and (3.1). This extension is done by introducing a control function $u(t)$, which is piecewise continuous on $[0, T]$. Clearly $u(t)$ has finite discontinuity points t_1, \dots, t_n in $[0, T]$. Consider $[0, t_1]$, where $u(t)$ is continuous and $0 < t_1 < T$ and apply the result in [6] to prove that (2.1) - (2.4) and (3.1) has unique solution pair $(\rho_u(t, a), y_u(t))$ on $[0, t_1]$. Note that this process can be repeated on each of the subintervals $[t_1, t_2], \dots, [t_{n-1}, T]$. At each end point t_i , consider the boundary and initial values (2.2) and (2.3) to be $\rho(t_i, 0) = \int_A^\infty \beta(a, P(t_i))\rho(t_i, a)da$ and $\rho(t_i, a) = \phi_i(a)$, in that order. From this proof, it is clear that the solution of the pair of the state equations (2.1) - (2.4) and (3.1) is piecewise continuous on $[0, T]$. \diamond

The solution of the coupled PDE (2.1) - (2.4), ρ and the ODE (3.1), y depend on the control parameter u and will be denoted by $\rho(u)$ and $y(u)$, respectively.

The optimal control problem is formulated as follows: Determine $y^*(t)$, $u^*(t)$, and $\rho^*(t, a)$ such that

$$(O) \left\{ \begin{array}{l} J(\rho^*, y^*, u^*) = \text{Inf } J(\rho, y, u) \\ y'(t) = I(v)E(y(t-v)) \int_0^A \varphi(a)(1+u(t))D(a)\rho(t-v, a) da - \delta y, \quad t > 0 \\ y(t) = y_0, \quad t \in [-v, 0] \\ \frac{\partial \rho(t, a)}{\partial t} + \frac{\partial \rho(t, a)}{\partial a} = -(\lambda(a, P) + \varphi(a)E(y))\rho(t, a) \\ \rho(t, 0) = \int_A^\infty \beta(a, P)\rho(t, a)da, \quad t > 0 \\ \rho(0, a) = \phi(a) \\ P(t) = \int_0^\infty \rho(t, a)da. \end{array} \right. \tag{3.4}$$

Theorem (2) establishes the existence of optimal control for (3.4). But, before that, we define the Gâteaux derivative (see [20])

Definition 1. The Gâteaux derivative (directional derivative) of X , $\widehat{X}(\tilde{u})$, is defined as follows:

$$\widehat{X}(u, \tilde{u}) = \widehat{X}(\tilde{u}) = \lim_{\alpha \rightarrow 0} \frac{X(u + \alpha\tilde{u}) - X(u)}{\alpha}, \tag{3.5}$$

(where \tilde{u} is the perturbation of u)

Theorem 2. Assume that \mathcal{U} is a non-empty closed bounded convex subset of $L^2([0, T])$ and that $f, \phi \in L^2([0, T])$. Then, there exists $\tilde{\alpha}$ such that for $\alpha \geq \tilde{\alpha}$, the cost function, J , is strictly convex. Moreover, for $\alpha \geq \tilde{\alpha}$, there exists a unique optimal control $u^* \in \mathcal{U}$ and associated solution pair ρ^* and y^* such that

$$J(u^*) \leq J(u), \quad \forall u \in \mathcal{U}.$$

4. Existence of Optimal Control

We devote this section to give the proof of Theorem 2. But first, we state and prove the following lemma where we will use the Gâteaux derivative given in Definition 1.

Lemma 1. *Let ρ and y be the unique solution of (2.1) - (2.4) and (3.1). The mapping $u \mapsto (\rho(u), y(u))$ has a Gâteaux derivative in every direction $\tilde{u} \in L^2([0, T])$.*

The Gâteaux derivatives $\hat{\rho}$ and \hat{y} are solutions of the systems

$$\left\{ \begin{aligned} \frac{\partial \hat{\rho}(t, a)}{\partial t} + \frac{\partial \hat{\rho}(t, a)}{\partial a} &= -(\lambda(a, P) + \varphi(a)E(y)) \hat{\rho}(t, a) - (\varphi(a)E'(y)\hat{y}) \hat{\rho}(t, a), \\ \hat{\rho}(t, 0) &= \int_A^\infty \beta(a, P) \hat{\rho}(t, a) da, \quad t > 0, \\ \hat{\rho}(0, a) &= 0, \\ P(t) &= \int_0^\infty \hat{\rho}(t, a) da \end{aligned} \right. \tag{4.1}$$

and

$$\left\{ \begin{aligned} \hat{y}'(t) &= I(v)E'(y(t-v)) \hat{y}(t-v) \int_0^A \varphi(a) (1+u(t)) D(a) \rho(t-v, a) da + \\ &\quad I(v)E(y(t-v)) \int_0^A [\varphi(a) (1+u(t)) D(a) \hat{\rho}(t-v, a) + \varphi(a) \tilde{u}(t) \times \\ &\quad D(a) \rho(t-v, a)] da - \delta \hat{y}, \quad t > 0, \\ \hat{y}(t) &= 0, \quad t \in [-v, 0], \end{aligned} \right. \tag{4.2}$$

respectively.

Furthermore, there exists $\tilde{\alpha}$ such that, for all $\alpha \geq \tilde{\alpha}$, the mapping $u \mapsto J(\rho, y, u)$ is strictly convex and lower semicontinuous on \mathcal{U} .

Proof. The existence and characterization of the Gâteaux derivatives are obtained using the Monotone Convergence Theorem and regularity conditions (existence of first derivative) on E' (see [3, 20]).

To prove the convexity of the cost function J , we show that the function $\Upsilon(\zeta) = J(u + \zeta \tilde{u})$, $\zeta \in \mathbb{R}$ is strictly convex near $\zeta = 0$. The derivative $\Upsilon'(\zeta)$ of Υ at $\zeta = 0$ reads:

$$\begin{aligned} \Upsilon'(\zeta) &= \int_0^T \int_0^A [(\rho(t, a) - \bar{\rho}_0(a, t)) \hat{\rho}(t, a) da] dt + \alpha_1 \int_0^T (y(t) - k_0) \hat{y}(t) dt + \\ &\quad \alpha_2 \int_0^T (u(t) - u_0(t)) \tilde{u}(t) dt, \end{aligned}$$

where ρ , y , $\hat{\rho}$, and \hat{y} are solutions of (2.1)-(2.4) and (3.1), (4.1) and (4.2) (in

that order).

The second derivative $\Upsilon''(\zeta)$ of $\Upsilon(\zeta)$ is given by

$$\begin{aligned} \Upsilon''(\zeta) &= \int_0^T \int_0^A [\widehat{\rho}^2(t, a) + (\rho(t, a) - \bar{\rho}_0(a, t))\widehat{\rho}'(t, a) da] dt \\ &+ \alpha_1 \int_0^T [\widehat{y}^2(t) + (y(t) - k_0) \widehat{y}'(t)] dt \\ &+ \alpha_2 \int_0^T [\tilde{u}^2(t) + (u(t) - u_0(t)) \tilde{u}'(t)] dt, \end{aligned}$$

where $\widehat{\rho}(t)$ and $\widehat{\rho}'(t)$ are the second derivative of ρ with respect to u in the directions of \tilde{u} and \tilde{u}' , respectively. Whereas, $\widehat{y}(t)$ and $\widehat{y}'(t)$ are the second derivatives y with respect to u in the directions \tilde{u} and \tilde{u}' , respectively.

It is straightforward to see that $\widehat{\rho}'$ and $\widehat{y}'(t)$ are solutions of

$$\left\{ \begin{aligned} \frac{\partial \widehat{\rho}'(t, a)}{\partial t} + \frac{\partial \widehat{\rho}'}{\partial a} &= -(\lambda(a, P) + \varphi(a)E(y)) \widehat{\rho}'(t, a) - \varphi(a)E'(y)\widehat{y}\rho'(t, a) \\ &\quad - \varphi(a)E''(y)\widehat{y}^2\rho(t, a) - \varphi(a)E'(y)\widehat{y}'\rho(t, a) - \varphi(a)E'(y)\widehat{y}\widehat{\rho}(t, a) \\ \widehat{\rho}'(t, 0) &= \int_A^\infty \beta(a, P)\widehat{\rho}'(t, a)da, \quad t > 0, \\ \widehat{\rho}'(0, a) &= 0, \end{aligned} \right. \tag{4.3}$$

and

$$\left\{ \begin{aligned} \widehat{y}'(t) &= I(v)E'(y(t-v))\widehat{y}(t-v) \int_0^A \varphi(a)(1+u(t))D(a)\rho(t-v, a)da + \\ &\quad I(v)E(y(t-v)) \int_0^A [\varphi(a)(1+u(t))D(a)\widehat{\rho}(t-v, a) - \varphi(a)\tilde{u}(t) \times \\ &\quad D(a)\rho(t-v, a)] da - \delta\widehat{y}, \quad t > 0, \\ \widehat{y}'(t) &= 0, \quad t \in [-v, 0], \end{aligned} \right. \tag{4.4}$$

respectively.

We derive (as in [3]) a priori estimates on $\rho, \widehat{\rho}, \widehat{\rho}', y,$ and $\widehat{y}, \widehat{y}'$ which depend on y_0, ϕ, T . Using these estimates, we can show that, for β large,

$$\Upsilon(0) \geq \gamma|u|_{L^2([0,T])}^2, \quad \forall u \in L^2([0, T]),$$

where γ is a positive constant parameter depending on these parameters, namely (y_0, ϕ, T) . This complete the proof the lemma. \diamond

The strict convexity of the cost function J implies the existence and uniqueness of the minimum of the functional J and therefore the optimal control [20].

5. Concluding Remarks

We study optimal control measures to limit the growth of a host species (non exotic plant or animal) which is invading a local environment. The invading species is known to cause damage by preventing the local animal or plant species from evolving as it should. To prevent the host species from invading the local habitat, a biological control approach is used instead of mechanical and drug-based control measures which could have a lasting negative effect. The biological control agent used in this study is a parasitoid (which depends on the host for survival and hence limits the host's growth). Using a coupled pair of partial and a delay ordinary differential equations to describe the host-parasitoid interaction, we introduced a time dependent control term. The control term describes efforts to facilitate the parasitoid success to evolve in the area and attack the invading species. We proved that the resulting model is well-posed. We also proved that a unique biological control approach could be implemented to fight against the invading species and that the method is also cost-effective. It is part of our future plan to implement analytical as well as computational techniques and see what the outcome of introducing more than one parasitoid (natural enemies) of the invading species. This is a problem worthwhile studying, both from the biological and the cost-effectiveness point of view.

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