International Journal of Pure and Applied Mathematics
Volume 77 No. 5 2012, 595-604
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

REMARKS ON HYPERSURFACES OF REVOLUTION IN EUCLIDEAN SPACE $\mathbb{R}^{n+1}$

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Abstract: We investigate hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$. Two main results are obtained, one concerns the Gauss-Kronecker curvature of a graph defined by a radially symmetric function on $\mathbb{R}^n$, and the other gives a sufficient and necessary condition for a hypersurfaces to be hypersurfaces of revolution.

AMS Subject Classification: 53A07, 53C45, 53C40
Key Words: hypersurface of revolution, Gauss-Kronecker curvature, convex hypersurface, sectional curvature

1. Introduction

Hypersurfaces of revolution form an important class of hypersurfaces in Euclidean space. Here by a hypersurface of revolution in Euclidean space $\mathbb{R}^{n+1}$ we mean a hypersurface M of $\mathbb{R}^{n+1}$ such that M is invariant by the action of $SO_l(n+1)$, the subgroup of the special orthogonal group $SO(n+1)$ that fixes a given straight line l (cf. [5]). In recent years, hypersurfaces of revolution have been investigated by many mathematicians from different perspectives. Dorfmeister and Kenmotsu [2] studied hypersurfaces of revolution with periodic mean curvature. Podestà and Spiro [8], Mercuri et al [5] and [6] gave several sufficient conditions for a hypersurface to be a hypersurface of revolution.

Received: September 5, 2011

In this paper, we continue this study, but we do it from a different angle. We have two main results in this paper, one concerns the Gauss-Kronecker curvature of a graph defined by a radially symmetric function on $\mathbb{R}^n$ (it is of course a hypersurface of revolution), and the other gives a sufficient and necessary condition for a hypersurfaces in $\mathbb{R}^{n+1}$ to be a hypersurface of revolution, in which we observe the "meridians" instead of, as usual, the "circles of latitude". Other mathematicians also have studied Gauss-Kronecker curvature in recent years, but they do it with different aspects (cf. [1] and [11]).

Throughout this paper, by a hypersurface of $\mathbb{R}^{n+1}$ we always mean an n-dimensional C^∞ imbedded submanifold (the image of an imbedding) of $\mathbb{R}^{n+1}$ without boundary.

The main results of this paper are the following.

Theorem 1.1. *Let $\Omega = B(o, R)$ be the open ball in $\mathbb{R}^n$ with center $o = (0, \ldots, 0)$ and radius R ($n \geq 2$). Let $\varphi \in C^\infty(\Omega)$ and $\Gamma_\varphi = \{(x, \varphi(x)) \mid x \in \Omega\}$ be the graph defined by φ in Euclidean space $\mathbb{R}^{n+1}$. Denote by K the Gauss-Kronecker curvature of Γ_φ with respect to the upward normal vector field of Γ_φ ("upward" means the $(n+1)$-th component of the normal vector is positive). Suppose that φ is radially symmetric around o, that is, there exists a function $\Phi(r)$ defined on $[0, R)$ such that $\varphi(x) = \Phi(r(x))$ for every $x \in \Omega$, where $r = r(x)$ denotes the distance from o to x. Then we have:*

(1) If n is even and $K \leq 0$, then φ is constant on Ω and hence $K \equiv 0$. It follows that, if n is even and if φ is not constant near point o, then $K(P_o) \geq 0$, where $P_o = (o, \varphi(o)) \in \Gamma_\varphi$.

(2) If n is odd, suppose either that $K \leq 0$ and Φ is increasing on $[0, R)$ (not necessarily strictly) , or that $K \geq 0$ and Φ is decreasing on $[0, R)$ (not necessarily strictly), then φ is constant on Ω and hence $K \equiv 0$.

Theorem 1.2. *Let M be a complete noncompact connected C^∞ hypersurface of $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with positive sectional curvature. Then M is a hypersurface of revolution if and only if there exists a point $P \in M$ such that all geodesics of M passing through P are plane curves (that is, belonging to a 2-dimensional plane of $\mathbb{R}^{n+1}$). Furthermore, in this case, the axis of revolution must be the normal line l of M at P, M is unbounded along one direction of l, and the n-dimensional volume of the geodesic sphere S_r of M around P with radius r is strictly increasing with respect to r.*

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Denote by $(x_1, x_2, \ldots, x_n)$ the coordinates of points in $\mathbb{R}^n$. The Gauss-Kronecker curvature of Γ_φ is given by the following formula (cf. [10, p.93]):

$$K = \det\left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right) \Bigg/ \left[1 + \sum_{i=1}^{n}\left(\frac{\partial\varphi}{\partial x_i}\right)^2\right]^{(n/2)+1} . \tag{2.1}$$

So K has the same sign as $\det\left(\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right)$ at any point. First we calculate the later. For the sake of simplicity, let

$$\varphi_i = \frac{\partial\varphi}{\partial x_i},\ \varphi_{ij} = \frac{\partial^2\varphi}{\partial x_i\partial x_j};\ r_i = \frac{\partial r}{\partial x_i},\ r_{ij} = \frac{\partial^2 r}{\partial x_i \partial x_j}.$$

Note that

$$r = r(x_1, x_2, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

A straight calculation gives

$$r_i = \frac{x_i}{r},\ r_{ij} = \frac{1}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right), \tag{2.2}$$

where δ_{ij} denotes the Kronecker delta, that is,

$$\delta_{ij} = \begin{cases} 1, & if\ i = j\,, \\ 0, & if\ i \neq j\,. \end{cases}$$

Then by (2.2) we have

$$\begin{aligned} \varphi_i &= \Phi' r_i, \\ \varphi_{ij} &= \Phi'' r_i r_j + \Phi'\, r_{ij} \\ &= \Phi'' \frac{x_i x_j}{r^2} + \Phi'\, \frac{1}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right). \end{aligned}$$

Hence if $i = j$, then

$$\begin{aligned} \varphi_{ii} &= \frac{1}{r^2}\left[\Phi'' x_i^2 + \Phi'\,\frac{1}{r}\left(r^2 - x_i^2\right)\right] \\ &= \frac{1}{r^2}\left[x_i^2\left(\Phi'' - \frac{\Phi'}{r}\right) + r\Phi'\right] \end{aligned} \tag{2.3}$$

and if $i \neq j$, then

$$\begin{aligned}\varphi_{ij} &= \Phi'' \frac{x_i x_j}{r^2} - \frac{x_i x_j}{r^3}\Phi' \\ &= \frac{x_i x_j}{r^2}\left(\Phi'' - \frac{\Phi'}{r}\right). \qquad (2.4)\end{aligned}$$

Let $a = \Phi'' - (\Phi'/r)$, $b = r\Phi'$, then we obtain

$$\varphi_{ii} = \frac{1}{r^2}\left(ax_i^2 + b\right), \qquad (2.5)$$

$$\varphi_{ij} = \frac{1}{r^2}\, a\, x_i x_j, \qquad (i \neq j). \qquad (2.6)$$

Now we have

$$\det\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right) = \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix}$$

$$= \frac{1}{r^{2n}} \begin{vmatrix} ax_1^2 + b & a\,x_1x_2 & \cdots & a\,x_1x_n \\ a\,x_2x_1 & ax_2^2 + b & \cdots & a\,x_2x_n \\ \cdots & \cdots & \cdots & \cdots \\ a\,x_nx_1 & a\,x_nx_2 & \cdots & ax_n^2 + b \end{vmatrix}$$

$$= \frac{1}{r^{2n}} \left\{ \begin{vmatrix} ax_1^2 & 0 & \cdots & 0 \\ a\,x_2x_1 & b & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a\,x_nx_1 & 0 & \cdots & b \end{vmatrix} + \begin{vmatrix} b & a\,x_1x_2 & \cdots & 0 \\ 0 & ax_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a\,x_nx_2 & \cdots & b \end{vmatrix} + \right.$$

$$\left. + \cdots + \begin{vmatrix} b & 0 & \cdots & a\,x_1x_n \\ 0 & b & \cdots & a\,x_2x_n \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & ax_n^2 \end{vmatrix} + \begin{vmatrix} b & 0 & \cdots & 0 \\ 0 & b & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b \end{vmatrix} \right\}$$

$$= \frac{1}{r^{2n}}\left(b^{n-1}ax_1^2 + b^{n-1}ax_2^2 + \ldots + b^{n-1}ax_n^2 + b^n\right)$$

$$= \frac{b^{n-1}}{r^{2n}}\left(ar^2 + b\right) = \frac{(r\Phi')^{n-1}}{r^{2n}}\left[r^2\left(\Phi'' - \frac{\Phi'}{r}\right) + r\Phi'\right]$$

$$= \frac{(\Phi')^{n-2}}{r^{n-1}}\,\Phi'\Phi''. \qquad (2.7)$$

From (2.7) we see that if n is even and $K \leq 0$, then

$$[(\Phi')^2]' = 2\Phi'\Phi'' \leq 0, \qquad (2.8)$$

which implies that $(\Phi')^2$ is decreasing on $[0, R)$ (not necessarily strictly). But we know that $\Phi'(0) = 0$ by the radially symmetry of φ, then Φ, and hence φ, must be constant. The rest part of assertion (1) is obvious by the continuity of Gauss-Kronecker curvature K. This proves the assertion (1) of the theorem.

If n is odd and either that $K \leq 0$ and Φ is increasing, or that $K \geq 0$ and Φ is decreasing, then from (2.7) and (2.8) we also have $[(\Phi')^2]' \leq 0$, the same argument as above shows also that φ is constant on Ω and hence $K \equiv 0$. This proves the second part of the theorem. The proof of Theorem 1.1 is finished. □

3. Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2. At first we prove two lemmas. We recall some notations. A regular curve C in $\mathbb{R}^2$ is said to be strictly convex if for every $p \in C$, C lies entirely on one side of l, the tangent line of C at P, and p is the only point which lies on l. The definition of strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ is similar (see [4]).

Lemma 3.1. *Let C be a smooth non-closed strictly convex plane curve in $\mathbb{R}^2$ with infinite length defined by*

$$\begin{cases} x = \phi(s), \\ y = \psi(s), \end{cases} \quad -\infty < s < +\infty,$$

where s is the arc length parameter. Suppose $(\phi(0), \psi(0)) = (0, 0)$ and $y(s) \geq 0$ for $\forall\ s \in (-\infty, +\infty)$. Then

$$\lim_{s\to+\infty} \psi(s) = \lim_{s\to-\infty} \psi(s) = +\infty.$$

Proof. Let $\mathbf{t}(s) = (\phi'(s), \psi'(s))$ be the unit tangent vector to C and $\mathbf{n}(s)$ the unit normal vector to C such that $\{\mathbf{t}(s), \mathbf{n}(s)\}$ gives a right handed orthonormal basis of $\mathbb{R}^2$ for each s. Let S^1 be the unit circle in $\mathbb{R}^2$ with center $(0, 0)$. Define maps $\tau : C \to S^1$, $\nu : C \to S^1$ such that $\tau(s) = \mathbf{t}(s)$ and $\nu(s) = \mathbf{n}(s)$. It is a well-known fact (the proof is elementary) that, for a smooth non-closed strictly convex plane curve C with infinite length, $\nu(C)$ (and hence $\tau(C)$) is a connected open subset of S^1 and lies in a semicircle of S^1. Furthermore, let $p_0 = (\phi(s_0), \psi(s_0)) \in C$ such that $\tau(p_0)$ to be the center point of the set $\tau(C)$, then $\{p_0; \mathbf{t}(s_0), \mathbf{n}(s_0)\}$ forms a new coordinate system of $\mathbb{R}^2$ such that C is the graph of a nonnegative function in this coordinate system. It is obvious that this function is unbounded along both the positive and negative directions of its independent variable. Now we turn back to the original coordinates of $\mathbb{R}^2$. Note that the tangent vector of C at $(0,0)$ is parallel to the x-axis. Thus the function $y=\psi(s)$ is necessarily increasing on $[0,+\infty)$ and decreasing on $(-\infty,0]$, otherwise we can find another point with parameter $s\neq 0$ and at which the tangent vector is also parallel to the x-axis, which is impossible. Let $\lim_{s\to+\infty}\psi(s)=A$, $\lim_{s\to-\infty}\psi(s)=B$. We need to prove both A and B are $+\infty$. We only give the proof of $A=+\infty$, one can prove $B=+\infty$ similarly. We argue by contradiction. If A is a finite number, then $A>0$ and when $s\geq 0$ the curve C lies between the x-axis and the line $y=A$. Thus the set $\{\phi(s):s\in[0,+\infty)\}\subset\mathbb{R}$ is unbounded (otherwise the set $\{(\phi(s),\psi(s)): s\in[0,+\infty)\}\subset\mathbb{R}^2$ is bounded, which is impossible). Choose a sequence $0< s_1<s_2<\ldots<s_n<\cdots$ such that $\lim_{i\to+\infty}x(s_i)=\infty$ (either $+\infty$ or $-\infty$). Hence for every $k=2,3,\cdots$, there exists $\widetilde{s}_k\in(s_1,s_k)$ such that the tangent line of C at $\widetilde{P}_k=(\phi(\widetilde{s}_k),\psi(\widetilde{s}_k))$ is parallel to the straight line $\overline{P_1P_k}$, where $P_k=(\phi(s_k),\psi(s_k))$. Note that the slope of $\overline{P_1P_k}\to 0$ when $k\to+\infty$, and so we have $\lim_{k\to+\infty}\dfrac{dy}{dx}(\widetilde{s}_k)=0$, which contradicts the fact that $\tau(C)$ is a connected open subset of S^1 and lies in a semicircle of S^1, since we already have $\dfrac{dy}{dx}(0)=0$. Hence we have $A=+\infty$. $\square$

Lemma 3.2. *Let Σ be a noncompact connected strictly convex surface of $\mathbb{R}^3$ and $P\in\Sigma$. If all non-constant geodesics ("non-constant" means "being not a single point") of Σ passing through P are non-closed plane curves and can be extended infinitely (in both directions of the curves), then:*

(1) Σ is a surface of revolution with axis of revolution to be the normal line of Σ at P.

(2) Σ is unbounded along one direction of the normal line of Σ at P.

(3) The length of the geodesic circle C_s is strictly increasing with respect to s, where $C_s=exp_{P}(\partial D(s))$ ($D(s)$ is the open disc in the tangent plane of Σ at P with center P and radius s).

Proof. In this proof of Lemma 3.2 bellow, all geodesics means non-constant geodesics.

(1). Since Σ is connected and strictly convex, so all the geodesics passing through P are strictly convex in the plane containing it.

Let l be the normal line of Σ at P. First we claim that every plane which contains a non-constant geodesic passing through P is a normal plane of Σ at P (by a normal plane of Σ at P we means a plane containing l). In fact, let

$\alpha = \alpha(s)$ be a geodesic passing through P with arc length parameter such that $\alpha(0) = P$, and π_α the plane containing α. Then $\alpha''(s) \parallel \eta(s)$ for all s (the characteristic property of geodesics, cf. [7]), here η denotes the unit normal vector field of Σ. Since $\alpha''(s) \parallel \pi_\alpha$ for all s, if $\alpha''(0) \neq 0$, then the assertion is true; if $\alpha''(0) = 0$, by the strictly convexity of α, there exists a sequence s_k such that $\lim_{k\to\infty} s_k = 0$ and $\alpha''(s_k) \neq 0$ (note that $\alpha''(s_k) \equiv 0$ on a interval means α being a straight line on the interval). Thus the normal lines of Σ at $\alpha(s_k)$ lie in π_α, then l lies in π_α by taking limit. This prove the claim.

Denote by (x, y, z) the coordinate of a point in $\mathbb{R}^3$. Without loss of generality, let $P = (0, 0, 0)$, l be the z-axis and Σ lies in the upper-half space. For any $c > 0$, Let ρ_c be the plane $z = c$. By the strictly convexity of Σ and Lemma 3.1, for any $c > 0$, the intersection of Σ with the plane ρ_c must be a smooth strictly convex curve homeomorphic to the circle. Let $\beta_c(t)$ be a parametrization of the above intersection curve with parameter t being in some interval ($\beta_c(t)$ denotes the position vector of a point on the intersection curve). Let $\eta_c(t)$ and η_0 be the unit normal vectors of Σ at $\beta_c(t)$ and P respectively. For every t, let π_t be the normal plane of Σ at P containing the point $\beta_c(t)$, and $\alpha(s)$ be the geodesic of Σ lying in π_t. Then $\beta_c(t) \parallel \pi_t$ and $\eta_0 \parallel \pi_t$. Since $\alpha(s)$ is a geodesic, by a similar argument as in the above analysis, we also have $\eta_c(t) \parallel \pi_t$. Then, for $\forall$ t, the mixed scalar product of $\beta_c(t), \eta_0$ and $\eta_c(t)$ must be zero, that is,

$$[\beta_c(t), \eta_0, \eta_c(t)] = 0. \tag{3.1}$$

In fact, by the arbitrariness of c and t, we have proved that

$$[\mathbf{x}, \eta_0, \eta] = 0 \tag{3.2}$$

at all points on Σ, where $\mathbf{x}$ denotes the position vector of a point on Σ and η denotes the unit normal vector field of Σ.

Give a local parametrization of Σ

$$\mathbf{x} : (u, v) \mapsto \mathbf{x}(u, v), \quad (u, v) \in \mathsf{U} \subset \mathbb{R}^2 \tag{3.3}$$

near any point $Q \neq P$ on Σ such that the coordinate curves $v = \textit{constant}$ to be the geodesics emanating from P and the coordinate curves $u = \textit{constant}$ to be the intersection curves of Σ with the planes $z = \textit{constant}$. Then

$$\eta = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}, \tag{3.4}$$

where $\times$ denotes the cross product. Thus we have

$$[\mathbf{x}, \eta_0, \mathbf{x}_u \times \mathbf{x}_v] = 0,$$

that is,

$$\langle \mathbf{x} \times \eta_0, \mathbf{x}_u \times \mathbf{x}_v \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. By the well-known Lagrange identity we get

$$\langle \mathbf{x}, \mathbf{x}_u \rangle \langle \eta_0, \mathbf{x}_v \rangle - \langle \mathbf{x}, \mathbf{x}_v \rangle \langle \eta_0, \mathbf{x}_u \rangle = \langle \mathbf{x} \times \eta_0, \mathbf{x}_u \times \mathbf{x}_v \rangle = 0.$$

Since $\eta_0 \perp \mathbf{x}_v$, so we have

$$\langle \mathbf{x}, \mathbf{x}_v \rangle \langle \eta_0, \mathbf{x}_u \rangle = 0. \tag{3.5}$$

By the above analysis we see that $\langle \eta_0, \mathbf{x}_u \rangle \neq 0$ at any point $Q \neq P$ on Σ, so from (3.5) we have

$$\langle \mathbf{x}, \mathbf{x}_v \rangle = 0 \tag{3.6}$$

at $Q \neq P$. Applying (3.6) to $\beta_c(t)$ $(c > 0)$ we get

$$\langle \beta_c(t), \beta_c'(t) \rangle = 0,$$

which implies that

$$\|\beta_c(t)\| = C \quad (constant). \tag{3.7}$$

Hence, for any $c > 0$, $\beta_c(t)$ must be a circle with center to be a point on l. Therefor Σ is a surface of revolution with axis of revolution to be the normal line of Σ at P.

(2). This is a straight result of Lemma 3.1.

(3). Let L_s be the length of the geodesic circle C_s. We take a geodesic $\alpha(s)$ passing through P and choose a coordinate system on the plane in which C lies such that $\alpha(s)$ is the graph of function $y = f(x)$, $P = \alpha(0) = (0, 0)$, $f(x) \geq 0$ and $f(-x) = f(x)$. Choose the arc length parameter s such that $x(s) > 0$ for $s > 0$. From the analysis of Lemma 3.1 we know that $\frac{dy}{dx} > 0$ for $x > 0$ and $\frac{dy}{ds} \geq 0$ for $s > 0$. But

$$\frac{dy}{dx} = \frac{dy}{ds}\frac{ds}{dx},$$

thus we have $\frac{dx}{ds} > 0$ for $s > 0$. Therefore $L_s = 2\pi x(s)$ is strictly increasing with respect to s. □

Now we are ready to give the proof of of Theorem 1.2.

Proof of Theorem 1.2. Note that, under the hypotheses of the theorem, M is orientable (cf. [9]). By a theorem of H. Wu (cf. [12, Main theorem]), M is strictly convex and homeomorphic to $\mathbb{R}^n$, and the Gauss map $\gamma : M \to S^n$ is a diffeomorphism on to an open convex subset of S^n. In particular, $\gamma(M)$ lies in an open hemisphere of S^n. We also know that all geodesics of M passing through P are non-closed, because there is no closed geodesic on manifold with positive sectional curvature (see [3, P.248]).

First suppose that $P \in M$ such that all geodesics of M passing through P are plane curves. A similar argument as in the proof of Lemma 3.2 shows that every plane in which a non-constant geodesic passing through P lies must contain l, the normal line of M at P. Let $\alpha = \alpha(s)$ be a non-constant geodesic passing through P. Then it is easy to see that $\gamma(\alpha)$ is just a part of a great circle on S^n. Let M_p be the tangent space of M at P. If $n = 2$, then M is a surface of revolution by Lemma 3.2. If $n > 2$, for every 2-dimensional subspace N of M_p, let E_N be the 3-dimensional plane of $\mathbb{R}^{n+1}$ spanned by N and l, $\Sigma_N = E_N \cap M$. Let $\mathcal{A}$ be the family of geodesics $\alpha : \mathbb{R} \to M$ with arc length parameter such that $\alpha(0) = P$ and $\alpha'(0) \in N$. Then $\Sigma_N = \bigcup_{\alpha \in \mathcal{A}, s \in \mathbb{R}} \alpha(s)$. It is obvious that Σ_N is a smooth noncompact connected strictly convex surface of 3-dimensional Euclidean space E_N. Note that every geodesic in $\mathcal{A}$ is also a geodesic of Σ_N. So all non-constant geodesics of Σ_N passing through P are plane curves. Then by Lemma 3.2 Σ_N is a surface of revolution with axis of revolution to be the normal line of M at P. Note that for all N, the axes of revolution of Σ_N are the same line l. By the arbitrariness of N, then M is necessarily a hypersurface of revolution with the axis of revolution to be the normal line of M at P.

Conversely, if M is a hypersurface of revolution, by the strict convexity of M, then it is easy to see that the axis of revolution intersects M at a point P and the axis of revolution is a normal line of M at P. A simple calculation shows that all geodesics of M passing through P are plane curves.

The rest of the theorem is a straight result of Lemma 3.1 and Lemma 3.2. □

References

[1] A.C. Asperti, R.M.B. Chaves, L.A.M. Sousa, The Gauss-Kronecker curvature of minimal hypersurfaces in four-dimensional space forms, *Math. Z.*, **267** (2011), 523-533.

[2] J. Dorfmeister, K. Kenmotsu, Rotational hypersurfaces of periodic mean curvature, *Diff. Geom. Appl.*, **27** (2009), 702-712.

[3] W. Klingenberg, *Riemannian Geometry*, Walter De Gruyter, Berlin (1995).

[4] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry, Vol. II*, Interscience, New York (1969).

[5] F. Mercuri, F. Podestà, J.A.P. Seixas, R. Tojeiro, Cohomogeneity one hypersurfaces of Euclidean Spaces, *Comment. Math. Helv.*, **81** (2006), 471-479.

[6] F. Mercuri, J.A.P. Seixas, Hypersurfaces of cohomogeneity one and hypersurfaces of revolution, *Diff. Geom. Appl.*, **20** (2004), 225-239.

[7] R.S. Millman, G.D. Parker, *Elements of Differential Geometry*, Prentice-Hall, Englewood Cliffs, NJ (1977).

[8] F. Podestà, A. Spiro, Cohomogeneity one manifolds and hypersurfaces of the Euclidean space, *Ann. Global Anal. Geom.*, **13** (1995), 169-184.

[9] H. Samelson, Orientability of hypersurfaces in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.*, **22** (1969), 301-302.

[10] J.A. Thorpe, *Elementary Topics in Differential Geometry*, Springer, New York (1979).

[11] K. Tso, A direct method approach for the existence of convex hypersurfaces with prescribed Gauss-Kronecker curvature, *Math. Z.*, **209** (1992), 339-344.

[12] H. Wu, The spherical images of convex hypersurfaces, *J. Diff. Geom.*, **9** (1974), 279-290.