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SPQ-INJECTIVE MODULES AND SQP-INJECTIVE MODULES

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Abstract: Let R be a ring. A right R-module M is called *simple principally quasi -injective* (briefly *SPQ-injective*) if, every R-homomorphism from a principal submodule of M to M with simple image extends to an endomorphism of M. A right R-module M is called *simple quasi-principally injective* (briefly *SQP-injective*) if, every R-homomorphism from an M-cyclic submodule of M to M with simple image extends to an endomorphism of M. SPQ-injective modules and SQP-injective modules with some Kasch conditions are investigated, and minimal quasi-injective modules are also investigated, some results on right principally injective rings obtained by Weimin Xue are improved.

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Throughout this paper, R is an associative ring with identity, and all modules are unitary. As usual, $J(R)$ and $Soc(_RR)(Soc(R_R))$ denote respectively the Jacobson radical, and the left (right) socle of R. We denote the socle and the

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Jacobson radical of a module N by $Soc(N)$ and $Rad(N)$ respectively. The Goldie dimension and the length of a module N are denoted by $G(N)$ and $c(N)$ respectively. Let M be a right $R-$module with $S = End(M_R)$, $X \subseteq M$, $A \subseteq S$ and $B \subseteq R$. Then we write $l_S(X) = \{s \in S \mid sx = 0, \forall x \in X\}$, $r_M(A) = \{m \in M \mid am = 0, \forall a \in A\}$, $l_M(B) = \{m \in M \mid mb = 0, \forall b \in B\}$ and $r_R(X) = \{r \in R \mid xr = 0, \forall x \in X\}$. The left annihilators of a subset X of R is denoted by $l_R(X)$ (or $l(X)$ for short), and the right annihilators of a subset X of R is denoted by $r_R(X)$ (or $r(X)$ for short).

Recall that R is said to be *right P-injective* [4] if every right R-homomorphism f from a principally right ideal I to R extends to R_R, or equivalently, if such a right R-homomorphism f is given by a left multiplication by some element $a \in R$. This concept has generalized to module in two ways by Nicholson W. K and Nguyen V. S ect. in paper [2] and paper [7] respectively. Following [2], A right R-module M is called *principally quasi -injective* (briefly *PQ-injective*) if, every R-homomorphism from a principal submodule of M to M extends to an endomorphism of M. And following [7], a right R-module M is called *quasi-principally injective* (briefly *QP-injective*) if, every R-homomorphism from an M-cyclic submodule of M to M extends to an endomorphism of M, where a module N is called M-cyclic if N is a homomorphic image of M. Recall that a ring R is called *right Kasch* [8] if every simple right R-module embeds in R, or equivalently, $l_R(T) \neq 0$ for every maximal right ideal T of R. The concept of right Kasch rings were generalized to modules in paper [1]. Following [1], a module M_R is said to be *Kasch* provided that every simple module in $\sigma[M]$ embeds in M, where $\sigma[M]$ is the category consisting of all M-subgenerated right R-modules. And according to [12], a module M_R is called *strongly Kasch* if every simple right R-module embeds in M. In this note, we generalize the concepts of PQ-injective modules and QP-injective modules to SPQ-injective modules and SQP-injective modules respectively, some properties of them are given, especially, SPQ-injective modules and SQP-injective modules with some Kasch conditions are investigated, and minimal quasi-injective modules are also investigated, some results on right principally injective rings obtained by Weimin Xue in paper [9] are improved.

We start with the following definition.

Definition 1. Let R be a ring and M be a right R-module. R is called *right simple P-injective* if every homomorphism from a principally right ideal of R to R with simple image extends to R. M is called *simple principally quasi-injective* (briefly *SPQ-injective*) if, every R-homomorphism from a principal submodule of M to M with simple image extends to an endomorphism of M. M is called *simple quasi-principally injective* (briefly *SQP-injective*) if, every

R-homomorphism from an M-cyclic submodule of M to M with simple image extends to an endomorphism of M.

Clearly, a ring R is right simple P-injective if and only if R_R is SPQ-injective if and only if R_R is SQP-injective. It is easy to see that PQ-injective modules are SPQ-injective, and QP-injective modules are SQP-injective. Our following Example 2 shows that SPQ-injective modules need not be PQ-injective and SQP-injective modules need not be QP-injective.

Example 2. Let $\mathbb{Z}_{2^\propto} = \{\frac{m}{2^i} + \mathbb{Z} \mid m \in \mathbb{Z}, i \in \mathbb{Z}^+\}$ be the prüfer group of type $2^\propto$. Then by [9, Example 4], the trivial extension $T(\mathbb{Z}, \mathbb{Z}_{2^\propto})$ of $\mathbb{Z}$ by $\mathbb{Z}_{2^\propto}$ is a commutative simple-injective ring, of course, it is simple P-injective, but it is not P-injective.

Recall that a right R-module M is said to be *simple quasi-injective* if every homomorphism from a submodule of M to M with simple image extends to an endomorphism of M. Clearly, simple quasi-injective modules are both SPQ-injective and SQP-injective. The next example 3 shows that SPQ-injective modules and SQP-injective modules need not be simple quasi-injective.

Example 3. Let K be a field and L be a proper subfield of K such that $\rho : K \to L$ is an isomorphism, and let $K[\rho; x]$ be the ring of twisted left polynomials over K where $xk = \rho(k)x$ for all $k \in K$. Set $R = K[\rho; x]/(x^2)$. Then R is right P-injective, but R is not right simple- injective.

Proof. By Rutter [6, Example 1], R is a right P-injective, left artinian local ring with only two idempotents 0 and 1 but R is not QF. Hence R is right minfull. By [5, Theorem 3.7(1)], R is right Kasch. If R is right simple-injective, then by [5, Proposition 6.14], R is left P-injective, and thus R is left and right mininjective and left artinian. It follows that R is QF by [5, Corollary 4.8], a contradiction. □

Next we give a characterization of SPQ-injective modules and SQP-injective modules.

Theorem 4. *Let M be a right R-module with $S = End(M_R)$, then:*

(1) M_R is SPQ-injective if and only if an R-homomorphism $\gamma : T \to M$ extends to an endomorphism of M whenever T is a principal submodule of M and $\gamma(T)$ is semisimple.

(2) M_R is SQP-injective if and only if an R-homomorphism $\gamma : s(M) \to M$ extends to an endomorphism of M whenever $s \in S$ and $\gamma(s(M))$ is semisimple and finitely generated.

Proof. (1) We need only to prove the necessity. Assume that M_R is SPQ-injective. If $\gamma(T) = 0$ then $\gamma = 0$. Otherwise, let $\gamma(T) = K_1 \oplus \cdots \oplus K_n$, where the K_i are simple submodules. If $\pi_i : \gamma(T) \to K_i$ is the projection, then $\pi_i\gamma = s_i\cdot$ for some $s_i \in S$ by hypothesis. It is routine to verify that $\gamma = (s_1 + \cdots + s_n)\cdot$, as required.

The proof of (2) is similar to (1). □

Theorem 5. *Let M_R be a SPQ-injective Kasch module with $S = End(M_R)$, then:*

(1) *If mR is simple, then Sm is simple. In particular, $Soc(M_R) \subseteq Soc({}_SM)$.*

(2) $l_M(J(R)) \trianglelefteq_S M$.

(3) *If M_R is cyclic, then $l_S(Rad(M)) \trianglelefteq_S S$.*

Proof. (1) If mR is simple. Then if $0 \neq sm \in Sm$, define $\gamma : mR \to smR$ by $\gamma(x) = sx$. Then γ is a right R-isomorphism, and hence γ^{-1} extends to an endomorphism of M. Thus, $m = \gamma^{-1}(sm) = \alpha(sm)$ for some $\alpha \in S$, and so Sm is simple.

(2) Let $0 \neq m \in M$. Suppose that T is a maximal submodule of mR. By the Kasch hypothesis, let $\sigma : mR/T \to M$ be monic, and define $f : mR \to M$ by $f(x) = \sigma(x+T)$, then $im(f) = im(\sigma)$ is simple. Since M_R is SPQ-injective, $f = s\cdot$ for some $s \in S$, and then $sm = f(m) = \sigma(m + T) \neq 0$. But $smJ(R) = f(m)J(R)) = \sigma(m+T)J(R) \subseteq Soc(M_R)J(R) = 0$, so $0 \neq sm \in Sm \cap l_M(J(R))$. Therefore, $l_M(J(R)) \trianglelefteq {}_SM$.

(3) If $0 \neq a \in S$, choose a maximal submodule T of the right R-module aM. Since M is Kasch, there exists a monomorphism $f : aM/T \to M$. Define $g : aM \to M$ by $g(x) = f(x+T)$, then $im(g)$ is simple. Since M is SPQ-injective and aM is cyclic, $g = s\cdot$ for some $s \in S$. Take $y \in M$ such that $ay \notin T$, then $say = g(ay) = f(ay + T) \neq 0$, and hence $sa \neq 0$. If $a(Rad(M)) \nsubseteq T$, then $a(Rad(M)) + T = aM$. But $a(Rad(M)) << aM$ because M is cyclic and hence finitely generated, so $T = aM$, a contradiction. Hence $a(Rad(M)) \subseteq T$, and then $(sa)(Rad(M)) = g(a(Rad(M))) = f(0) = 0$, whence $0 \neq sa \in Sa \cap l_S(Rad(M))$. This shows that $l_S(Rad(M)) \trianglelefteq_S S$. □

Recall that a module M_R is called *minimal quasi-injective* [12] if every homomorphism from a simple submodule of M to M extends to an endomorphism of M. Clearly, SPQ-injective modules are minimal quasi-injective.

Theorem 6. *Let M_R be a SPQ-injective strongly Kasch module with $S = End(M_R)$. Then the following statements are equivalent:*

(1) *R is semilocal.*

(2) ${}_SM$ *is finite cogenerated.*

(3) ${}_SM$ *is finite dimensional.*

In this case,

$$Soc(M_R) = Soc({}_SM) = l_M(J(R)),$$

and

$$G({}_SM) = c({}_SSoc(M_R)) = c((R/J(R))_R).$$

Proof. (1) ⇒ (2) Since M_R is strongly Kasch, it is Kasch. Note that R is semilocal, by Theorem 5, $Soc({}_SM) = Soc(M_R) = l_M(J(R)) \trianglelefteq_S M$. Let $J(R) = T_1 \cap \cdots \cap T_n$, where each T_i is a maximal right ideal of R. Assume n is minimal so that no T_i contains the intersection of any of the others, then $T_k + \cap_{i=k+1}^{n} T_i = R, k = 1, 2, \cdots, n-1$, and so $Soc(M_R) = l_M(T_1) \oplus \cdots \oplus l_M(T_n)$. Hence $Soc(M_R)$ is n- generated and semisimple as a left S-module by [12, Theorem 2.2] and [10, Lemma 1] because M_R is minimal quasi-injective and strongly Kasch, and then ${}_SM$ is finite dimensional. In this case, $Soc(M_R) = Soc({}_SM) = l_M(J(R))$, and $G({}_SM) = c({}_SSoc(M_R)) = c((R/J(R))_R)$.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) See [10, Proposition 4]. □

Our following Corollary 7 improves [9, Theorem 1].

Corollary 7. *Let R be a right simple P-injective right Kasch ring. Then the following conditions are equivalent:*

(1) *R is semilocal.*

(2) *R is left finitely cogenerated.*

(3) *R is left finite dimensional.*

In this case,

$$Soc({}_RR) = l_R(J(R)),$$

and

$$G({}_RR) = c({}_RSoc({}_RR)) = c((R/J(R))_R).$$

Theorem 8. *Let R be a semilocal ring and M_R be a minimal quasi-injective module with $S = End(M_R)$. Then the following statements are equivalent:*

(1) *M_R is strongly Kasch.*

(2) $c({}_SSoc(M_R)) = c((R/J(R))_R)$.

Proof. (1) $\Rightarrow$ (2) Since M_R is minimal injective and strongly Kasch, by [13, Theorem 6], $Hom_R(K_R, {}_SM_R)$ is a simple left S-module for every simple right R-module K_R. Since R is semilocal, the right R-module $R/J(R)$ is semisimple. Let $R/J(R) = \oplus_{i=1}^n K_i$, where each K_i is a simple right R-module. Then $Soc(M_R) = l_M(J(R)) \cong Hom(R/J(R), {}_S M_R) \cong \oplus_{i=1}^n Hom(K_i, M)$. Therefore, $c({}_SSoc(M_R)) = n = c((R/J(R))_R)$.

(2) $\Rightarrow$ (1) Let K be a simple right R-module, then it is a simple $R/J(R)$-module. But the ring $R/J(R)$ is semisimple, there is an isomorphism of right $R/J(R)$-modules $R/J(R) \cong K \oplus H$, and hence $R/J(R) \cong K \oplus H$ as right R-modules. Thus, $c((R/J(R))_R) = c({}_SSoc(M_R)) = l_M(J(R)) = c(Hom_R(K \oplus H, {}_S M_R)) = c(Hom_R(K, {}_S M_R)) \oplus c(Hom_R(H, {}_S M_R)) \leq 1 + c(H) = c(K \oplus H) = c((R/J(R))_R)$. Thus, $c(Hom_R(K, {}_S M_R)) = 1$, and whence K embeds in M_R, as required. □

Our following Corollary 8 improves [9, Proposition 5]

Corollary 9. *Let R be a right minimal injective semilocal ring. Then the following statements are equivalent:*

(1) *R is right Kasch.*

(2) $c({}_RSoc(R_R)) = c((R/J(R))_R)$.

Definition 10. *A module M_R is said to be Weakly Kasch provided that every simple M-cyclic module embeds in M.*

Clearly, a ring R is right Kasch if and only if R_R is Weakly Kasch.

Theorem 11. *For a right R-module M_R with $S = End(M_R)$, the following are equivalent:*

(1) *M_R is Weakly Kasch.*

(2) *$l_S(T) \neq 0$ for every maximal submodule T of M_R.*

(3) *For every maximal submodule T of M_R, there exists a $0 \neq s \in S$ such that $T = Ker(s)$.*

(4) *$T = r_M l_S(T)$ for every maximal submodule T of M_R.*

Proof. (1)$\Rightarrow$(2) Since M_R is Weakly Kasch, there exists a monomorphism $\varphi : M/T \to M$. Define $s : M \to M$ by $m \mapsto \varphi(m + T)$. Then $0 \neq s \in S, sT = \varphi(0) = 0$, and so $l_S(T) \neq 0$.

(2)$\Rightarrow$(3) Let $0 \neq s \in l_S(T)$, then $T \subseteq Ker(s) \neq M$, and so $T = Ker(s)$ by the maximality of T in M.

(3)$\Rightarrow$(4) Let T be a maximal submodule of M_R. By (3), there exists a $s \in S$ such that $T = Ker(s)$, and then $r_M l_S(T) = r_M l_S(Ker(s)) = Ker(s) = T$.

(4)$\Rightarrow$(2) Obvious.

(3)$\Rightarrow$(1). Let N be an M-cyclic simple module. Then $N \cong M/T$ for some maximal submodule T of M. By (3), $T = Ker(s)$ for some $0 \neq s \in S$. Let $\varphi : M/T \to M; m + T \mapsto sm$, then φ is a monomorphism, so M/T embeds in M, and hence N embeds in M. □

Theorem 12. *Let M_R be a finitely generated SQP-injective weakly Kasch module with $S = End(M_R)$. Then $l_S(Rad(M)) \trianglelefteq_S S$.*

Proof. If $0 \neq a \in S$, choose a maximal submodule T of the right R-module aM. Since M is weakly Kasch, there exists a monomorphism $f : aM/T \to M$. Define $g : aM \to M$ by $g(x) = f(x + T)$, then $im(g)$ is simple. Since M is SQP-injective, $g = s\cdot$ for some $s \in S$. Take $y \in M$ such that $ay \notin T$, then $say = g(ay) = f(ay + T) \neq 0$, and hence $sa \neq 0$. If $a(Rad(M)) \nsubseteq T$, then $a(Rad(M)) + T = aM$. But $a(Rad(M)) << aM$ because M is finitely generated, so $T = aM$, a contradiction. Hence $a(Rad(M)) \subseteq T$, and then $(sa)(Rad(M)) = g(a(Rad(M))) = f(0) = 0$, whence $0 \neq sa \in Sa \cap l_S(Rad(M))$. This shows that $l_S(Rad(M)) \trianglelefteq_S S$. □

Theorem 13. *Let M_R be a finitely generated nonzero weakly Kasch module with $S = end(M_R)$. If S is left finite dimensional, then $M/RadM$ is semisimple.*

Proof. Let $\Omega = \{I \mid 0 \neq I = l_S(X)$ for some $X \subseteq M\}$. By Theorem 11, Ω is not an empty set. Since S is left finite dimensional, there exist some minimal members $I_1, I_2, \cdots, I_n$ in Ω such that $I = \oplus_{i=1}^n I_i$ is a maximal direct sum of minimal members in Ω. The proof is completed by establishing the following claims:

Claim 1. *$r_M(I_i)$ is a maximal submodule of M for each i.*

Proof. Since M is finitely generated and weakly Kasch, by Theorem 11, $r_M(I_i) \subseteq T_i = r_M l_S(T_i)$ for some maximal submodule T_i. By [11, Lemma 5] and Theorem 11, $I_i \supseteq l_S r_M l_S(T_i) = l_S(T_i) \neq 0$, and so $I_i = l_S(T_i)$ by the minimality of I_i in Ω. Now we choose $0 \neq a_i \in l_S(T_i)$. Then $T_i = r_M(a_i)$, and thus $r_M(I_i) = r_M l_S(T_i) = r_M l_S r_M(a_i) = r_M(a_i) = T_i$.

Claim 2. *$RadM = \cap_{i=1}^n r_M(I_i)$.*

Proof. Clearly, $RadM \subseteq \cap_{i=1}^n r_M(I_i)$. If T is a maximal submodule of M, then $l_S(T)$ is minimal in Ω. In fact, if $l_S(T) \supseteq l_S(X) \neq 0$, where $X \subseteq M$, then $T \subseteq r_M l_S(X) \neq M$. So $T = r_M l_S(X)$, and hence $l_S(T) = l_S(X)$. Thus $l_S(T) \cap I \neq 0$. Taking some $0 \neq b \in l_S(T) \cap I$, we have $T = r_M(b) \supseteq \cap_{i=1}^n r_M(I_i)$. This gives that $\cap_{i=1}^n r_M(I_i) \subseteq RadM$, and the claim follows. □

Recall that a right R-module M is called mininjective [5] if every R-homomorphism from an minimal right ideal of R to M extends to a homomorphism of R to M.

Definition 14. A right R-module M is called quasi-mininjective if every R-homomorphism from an M-cyclic simple submodule of M to M extends to an endomorphism of M.

Clearly, SQP-injective modules are quasi-mininjective.

Theorem 15. *Let M be a right R-module with $S = End(M_R)$. Then the following statements are equivaent:*

(1) *M_R is quasi-mininjective.*

(2) *For any $s \in S$, if sM is simple then the left S-module $Hom_R(sM,_S M_R)$ is simple.*

(3) *$Hom_R(M/T, M)$ is simple or 0 as a left S-module for every maximal submodule T of M.*

(4) *The annihilator left ideal $l_S(T)$ is simple or 0 for every maximal submodule T of M.*

(5) *For any $s \in S$, if sM is simple then $l_S(Ker(s)) = Ss$*

(6) *For any $s \in S$, if sM is simple then Ss is simple and $Ss \trianglelefteq l_S(Ker(s))$.*

Proof. (1)⇒(2) Let $f, g \in Hom_R(s(M),_S M_R)$ and assume that $f \neq 0$. Then $f : sM \to f(sM)$ is an isomorphism and $gf^{-1} : f(s(M)) \to M$ is a hommorphism. Since f(sM) is a simple M-cyclic submodule of M, gf^{-1} can be extended to an endmormorphism h of M. Thus $g = hf$, proving (2).

(2)⇒(3) Assume (2). Then if $Hom_R(M/T, M) = 0$, there is nothing to prove. If $Hom_R(M/T, M) \neq 0$. Take $0 \neq \varphi \in Hom(M/T,_S M_R)$, let $s = \varphi\pi$, where π is the natural epimorphism of M to M/T, then $s \in S$ and $sM \cong M/T$ is simple. And so $Hom(M/T,_S M_R) \cong Hom(sM,_S M_R)$ is a simple left S-module.

(3)⇒(4) Let π be the natural epimorphism of M to M/T, then the mapping σ: $Hom(M/T,_S M_R) \to l_S(T); f \mapsto f\pi$ is an isomorphism, and so (4) follows.

(4)⇒(5) If sM is simple, then Ker(s) is a maximal submodule of M and $0 \neq Ss \subseteq l_S(Ker(s))$. By (4), $l_S(Ker(s))$ is simple, and hence $l_S(Ker(s)) = Ss$.

(5)⇒(6) Let sM be simple. Then by (5), $l_S(Ker(s)) = Ss$, and whence $Ss \trianglelefteq l_S(Ker(s))$. If $0 \neq t \in Ss$, then $Ker(s) \subseteq Ker(t) \neq M$, and so $Ker(s) = Ker(t)$ and tM is also simple. By (5), $Ss = St$, this shows that Ss is simple.

(6)$\Rightarrow$(1) Let K be an M-cyclic simple submodule of M, then $K = sM$ for some $s \in S$. Let f be any nonzero homomorphism of sM to M, then $fs \in l_S(Ker(s))$. Since sM and fsM are both simple, by (6), Ss and $S(fs)$ are both simple and $Ss \trianglelefteq l_S(Ker(s))$. This implies that $S(fs) = Ss$, and then $fs = us$ for some $u \in S$, which shows that $f = u\cdot$, as required.

Theorem 16. *Let M_R be a finitely generated SQP-injective weakly Kasch module with $S = End(M_R)$. Then the following conditions are equivalent:*

(1) *$M/Rad(M)$ is semisimple.*

(2) *S is left finitely cogenerated.*

(3) *S is left finite dimensional.*

In this case,

$$Soc({}_SS) = l_S(Rad(M)),$$

and

$$G({}_SS) = c({}_SSoc({}_SS)) = c(M/Rad(M)).$$

Proof. (1) $\Rightarrow$ (2). It is trival in case $M = 0$. If $M \neq 0$, then $M/RadM \neq 0$ because M is finitely generated. As $M/RadM$ is finitely generated and semisimple, by [11, Lemma 8], there exist maximal submodules $T_1, T_2, \cdots, T_n$ such that $M/RadM \cong \oplus_{i=1}^{n} M/T_i$. Since M_R is quasi-mininjective and weakly Kasch, by [11, Lemma 7], Theorem 11 and Theorem 15,

$$l_S(RadM) \cong {}_SHom_R(M/RadM, {}_SM_R) \cong {}_SHom_R(\oplus_{i=1}^{n} M/T_i, {}_SM_R) \cong \oplus_{i=1}^{n} l_S(T_i)$$

is a n-generated semisimple left ideal of S. Noting that M_R be a finitely generated SQP-injective weakly Kasch module, by Theorem 12, $l_S(RadM) = Soc({}_SS) \trianglelefteq_S S$, and therefore S is left finitely cogenerated, and $Soc({}_SS) = l_S(Rad(M))$, and $G({}_SS) = c({}_SSoc({}_SS)) = c(M/Rad(M))$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) See Theorem 13. □

We call a right R-module M idempotent faithful if for every idempotent element $e \in R$, $Me = 0$ if and only if $e = 0$.

Lemma 17. *Let R be a semiperfect ring and M_R be an idempotent faithful module with $S = End(M_R)$. If $Soc(M_R) \trianglelefteq_S M$, then M_R is strongly Kasch.*

Proof. Let T be a maximal right ideal of R. Since R is semiperfect, there exists an idempotent element e in R such that $T/J(R) = ((1-e)+J(R))R/J(R)$, and thus $1-e \in T$ and $eR \cap T \subseteq J(R)$. Then $l_M(eR \cap T) \supseteq l_M(J(R)) = Soc(M_R)$, so $l_M(eR \cap T)$ is essential in $_SM$ by hypothesis. In particular, $0 \neq Me \cap l_M(eR \cap T) = l_M((1-e)R \oplus (eR \cap T)) = l_M(T)$. Hence M_R is strongly Kasch. □

Theorem 18. *Let R be a semiperfect ring and M_R be an idempotent faithful SPQ-injective module with $S = End(M_R)$. Then the following conditions are equivalent:*

(1) *M_R is strongly Kasch.*

(2) *M_R is Kasch.*

(3) *$Soc(M_R) \trianglelefteq_S M$.*

Proof. (1)⇒(2) Obvious.

(2) ⇒ (3) Since M_R is a SPQ-injective Kasch module, by Theorem 5(2), We have $l_M(J(R)) \trianglelefteq_S M$. But R is a semiperfect ring, it is semilocal, so $l_M(J(R)) = Soc(M_R)$, and then (3) follows.

(3) ⇒ (1) By Lemma 17.

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