International Journal of Pure and Applied Mathematics
Volume 77 No. 5 2012, 639-647
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

A STUDY ON QUASI-IDEALS IN TERNARY SEMIGROUPS

Parinyawat Choosuwan[1], Ronnason Chinram[2 §]
[1,2]Department of Mathematics and Statistics
Faculty of Science
Prince of Songkla University
Hat Yai, Songkhla 90110, THAILAND
[2]Centre of Excellence in Mathematics
CHE, Si Ayuthaya Road, Bangkok 10400, THAILAND

Abstract: A ternary semigroup is a nonempty set together with a ternary multiplication which is associative. Any semigroup can be reduced to a ternary semigroup but a ternary semigroup does not necessarily reduce to a semigroup. In this paper, we give some characterizations of minimal and maximal quasi-ideals in ternary semigroups.

AMS Subject Classification: 20M12, 20N10
Key Words: quasi-ideals, ternary semigroups

1. Introduction

Algebraic structures play a prominent role in mathematics with wide applications in many disciplines such as computer sciences, information sciences, engineering, physics etc. The theory of ternary algebraic system was introduced by Lehmer [6] in 1932, but earlier such structures was studied by Kasner [5] who give the idea of n-ary algebras. Lehmer investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. Ternary semigroups are universal algebras with one associative ternary operation. The notion of ternary semigroup was known to Banach (cf. Los [7]) who is credited with example of a ternary semigroup which can not reduce to

Received: September 29, 2011

[§]Correspondence author

a semigroup. A nonempty set T with a ternary operation $T \times T \times T \to T$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$ is called a *ternary semigroup* if it satisfies the following associative law holds:

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]] \text{ for any } x_1, x_2, x_3, x_4, x_5 \in T.$$

In this paper, we denote $[x_1x_2x_3]$ by $x_1x_2x_3$. We can see that any semigroup can be reduced to a ternary semigroup. Banach showed that a ternary semigroup does not necessarily reduce to a semigroup by this example.

Example 1.1. $T = \{-i, 0, i\}$ is a ternary semigroup while T is not a semigroup under the multiplication.

Example 1.2. *$T = \mathbb{Z}^-$ is a ternary semigroup while T is not a semigroup under the multiplication.*

However, Los [7] proved that every ternary semigroup can be embedded in a semigroup.

Let T be a ternary semigroup. For nonempty subsets A, B and C of T, let $ABC := \{abc \mid a \in A, b \in B \text{ and } c \in C\}$. A nonempty subset S of T is called a *ternary subsemigroup* if $SSS \subseteq S$. Sioson [8] studied ideal theory in ternary semigroups. A nonempty subset A of a ternary semigroup T is called a *left ideal* of T if $TTA \subseteq A$, a *right ideal* of T if $ATT \subseteq A$ and a *lateral ideal* of T if $TAT \subseteq A$. If A is a left, right and lateral ideal of T, A is called an *ideal* of T. A ternary subsemigroup B of T is called a *bi-ideal* of T if $BTBTB \subseteq B$. A ternary subsemigroup Q of T is called a *quasi-ideal* of T if $QTT \cap (TTQTT \cup TQT) \cap TTQ \subseteq Q$. Every right, left and lateral ideal of T is a quasi-ideal of T. But the converse is not true in general. Moreover, the intersection of left ideal, right ideal and lateral ideal of T is a quasi-ideal of T and every quasi-ideal of T can be obtained in this way. In [1], Dixit and Dewan proved that every quasi-ideal of T is a bi-ideal of T but the converse is not true in general by giving example. Moreover, Dixit and Dewan studied minimal quasi-ideals in ternary semigroups in [2].

In this paper, we give some characterizations of minimal and maximal quasi-ideals in ternary semigroups that are studied analogous to the characterizations of minimal and maximal quasi-ideals in semigroups and the characterizations of minimal and maximal later ideals in ternary semigroups considered by Iampan (see [3] and [4]).

The following proposition is easy to verify.

Proposition 1.1. *Let A be a nonempty subset of a ternary semigroup. The following statements hold.*

(1) TTA is a left ideal of T.

(2) $TAT \cup TTATT$ is a lateral ideal of T.

(3) ATT is a right ideal of T.

Now we recall some two results from [1].

Proposition 1.2. *(see [1]) The intersection of arbitrary set of quasi-ideals of T is either empty or a quasi-ideal of T.*

Let T be a ternary semigroup and A a nonempty subset of T. A quasi-ideal $(A)_q$ denote the quasi-ideal of T generated by A, that is $(A)_q$ is the smallest quasi-ideal of T containing A.

Proposition 1.3. *(see [1]) Let T be a ternary semigroup and A a nonempty subset of T. Then $(A)_q = A \cup (TTA \cap (TAT \cup TTATT) \cap ATT)$. So for any $a \in T, (a)_q = \{a\} \cup (TTa \cap (TaT \cup TTaTT) \cap aTT)$.*

2. Minimality of Quasi-Ideals in Ternary Semigroups

A ternary semigroup T is called *quasi-simple* if T has no proper quasi-ideals.

Proposition 2.1. *Let T be a ternary semigroup. The following statements are equivalent.*

(1) T is quasi-simple.

(2) $TTa \cap (TaT \cup TTaTT) \cap aTT = T$ for all $a \in T$.

(3) $(a)_q = T$ for all $a \in T$.

Proof. Assume T is quasi-simple and let $a \in T$. Since $TTa \cap (TaT \cup TTaTT) \cap aTT$ is a quasi-ideal of T, $TTa \cap (TaT \cup TTaTT) \cap aTT = T$. Hence (1) implies (2). Since $TTa \cap (TaT \cup TTaTT) \cap aTT \subseteq (a)_q$, thus (2) implies (3). Next, assume (3) holds. Let Q be a quasi-ideal of T and let $a \in Q$. Hence $T = (a)_q \subseteq Q$, this implies $Q = T$. So (3) implies (1). □

An element z of T is called a *zero* if $zxy = xzy = xyz = z$ for all $x, y \in T$ and is denoted by 0. Let T be a ternary semigroup with zero, $T^3 \neq \{0\}$ and $|T| > 1$. T is called *0-quasi-simple* if T has no nonzero proper quasi-ideals.

Proposition 2.2. *Let T be a ternary semigroup with zero, $T^3 \neq \{0\}$ and $|T| > 1$. Then T is 0-quasi-simple if and only if $(a)_q = T$ for all $a \in T \setminus \{0\}$.*

Proof. Assume that T is 0-quasi-simple and let $a \in T \setminus \{0\}$. Thus $(a)_q \neq \{0\}$. This implies $(a)_q = T$.

Conversely, let Q be a nonzero quasi-ideal of T and $a \in Q \setminus \{0\}$. Then $T = (a)_q \subseteq Q \subseteq T$. Therefore T is 0-quasi-simple. □

Example 2.1. (1) The ternary semigroup $\{-i, i\}$ under the multiplication is quasi-simple.

(2) The ternary semigroup $\{-i, 0, i\}$ under the multiplication is 0-quasi-simple.

A quasi-ideal Q of a ternary semigroup T is called a *minimal quasi-ideal* of T if Q does not properly contain any quasi-ideal of T. The following proposition is well-known.

Proposition 2.3. *(see [8]) Let T be a ternary semigroup. Then a quasi-ideal Q of T is minimal if and only if it is the intersection of a minimal left ideal, a minimal right ideal and a minimal lateral ideal of T.*

A nonzero quasi-ideal Q of a ternary semigroup T with zero is called a *0-minimal quasi-ideal* of T if Q does not properly contain any nonzero quasi-ideal of T.

Theorem 2.4. *Let T be a ternary semigroup with zero. Then the intersection of a 0-minimal left ideal, a 0-minimal right ideal and a 0-minimal lateral ideal of T is either $\{0\}$ or a 0-minimal quasi-ideal of T.*

Proof. The proof of this theorem is similar to the proof of Proposition 2.3. □

The following theorem shows the relationship between minimal quasi-ideals and quasi-simple.

Theorem 2.5. *Let T be a ternary semigroup and Q a quasi-ideal of T. Q is a minimal quasi-ideal of T if and only if Q is quasi-simple.*

Proof. Assume Q is a minimal quasi-ideal of T and let A be a quasi-ideal of Q. Then $QQA \cap AQQ \cap (QAQ \cup QQAQQ) \subseteq Q$. It is easy to verify that $QQA \cap AQQ \cap (QAQ \cup QQAQQ)$ is a quasi-ideal of T. Since Q is minimal and $QQA \cap AQQ \cap (QAQ \cup QQAQQ) \subseteq A \subseteq Q$, $QQA \cap AQQ \cap (QAQ \cup QQAQQ) = A = Q$. Hence Q is quasi-simple. Conversely, assume Q is quasi-simple. Let A be a quasi-ideal of T such that $A \subseteq Q$. So A is a quasi-ideal of Q, this implies $A = Q$. Hence Q is a minimal quasi-ideal of T. □

Theorem 2.6. *Let T be a ternary semigroup with zero and Q a nonzero quasi-ideal of T. The following statements hold.*

(1) If Q is 0-quasi-simple, then Q is a 0-minimal quasi-ideal of T.

(2) If Q is a 0-minimal quasi-ideal of T and $QQA \cap AQQ \cap (QAQ \cup QQAQQ) \neq \{0\}$ for all a nonzero quasi-ideal A of Q, then Q is 0-quasi-simple.

Proof. (1) Assume Q is 0-quasi-simple. Let A be a nonzero quasi-ideal of T such that $A \subseteq Q$. So A is a nonzero quasi-ideal of Q, this implies $A = Q$. Hence Q is a 0-minimal quasi-ideal of T.

(2) Assume Q is a 0-minimal quasi-ideal of T and let A be a nonzero quasi-ideal of Q. Thus $QQA \cap AQQ \cap (QAQ \cup QQAQQ) \subseteq A$. Similar to the proof of Theorem 2.5, hence Q is 0-quasi-simple. □

Example 2.2. (1) In $\mathbb{Z}_{18}$, consider the ternary semigroup $T = \{\overline{1}, \overline{3}, \overline{9}\}$ under the usual multiplication. It is easy to see that $\{\overline{9}\}$ is a minimal quasi-ideal of T. By Theorem 2.5, the ternary semigroup $\{\overline{9}\}$ is quasi-simple.

(2) In $\mathbb{Z}_{18}$, consider the ternary semigroup $T = \{\overline{0}, \overline{1}, \overline{3}, \overline{9}\}$ under the usual multiplication. It is easy to see that $Q = \{\overline{0}, \overline{9}\}$ is 0-quasi-simple and Q is a quasi-ideal of T, by Theorem 2.6(1), Q is a 0-minimal quasi-ideal of T.

(3) The converse of Theorem 2.6(1) is not true in general. In $\mathbb{Z}_{81}$, consider the ternary semigroup $T = \{\overline{0}, \overline{3}, \overline{27}\}$ under the usual multiplication. It is easy to see that $Q = \{\overline{0}, \overline{27}\}$ is a 0-minimal quasi-ideal of T but Q is not 0-quasi-simple.

Theorem 2.7. *Let T be a ternary semigroup having proper quasi-ideals. Then every proper quasi-ideal of T is minimal if and only if the intersection of any two distinct proper quasi-ideals is empty.*

Proof. Let Q_1 and Q_2 be two distinct proper quasi-ideals of T. Then Q_1 and Q_2 are minimal. If $Q_1 \cap Q_2 \neq \emptyset$, then by Proposition 1.2, $Q_1 \cap Q_2$ is a quasi-ideal of T. Since $Q_1 \cap Q_2$ is a proper subset of Q_1, a contradiction. Hence $Q_1 \cap Q_2 = \emptyset$. The converse is obvious. □

Theorem 2.8. *Let T be a ternary semigroup with zero having nonzero proper quasi-ideals. Then every nonzero proper quasi-ideal of T is minimal if and only if the intersection of any two distinct proper quasi-ideals is $\{0\}$.*

Proof. Using the same proof of Theorem 2.7. □

3. Maximality of Quasi-Ideals of Ternary Semigroups

In this section we characterize maximal quasi-ideals of ternary semigroups.

Theorem 3.1. *Let Q be a quasi-ideal of T. If either*

(1) $T \setminus Q = \{a\}$ for some $a \in T$ or

(2) $T \setminus Q \subseteq TTb \cap bTT \cap (TbT \cup TTbTT)$ for all $b \in T \setminus Q$,

then Q is a maximal quasi-ideal of T.

Proof. Let A be a quasi-ideal of T such that Q is a proper subset of A.

Case 1: $T \setminus Q = \{a\}$ for some $a \in T$.

Since Q is a proper subset of A, $A \setminus Q \subseteq T \setminus Q = \{a\}$. Then $A \setminus Q = \{a\}$. So $A = Q \cup \{a\} = T$.

Case 2: $T \setminus Q \subseteq TTb \cap bTT \cap (TbT \cup TTbTT)$ for all $b \in T \setminus Q$.

Consider $b \in A \setminus Q$. So $T \setminus Q \subseteq TTb \cap bTT \cap (TbT \cup TTbTT) \subseteq TTA \cap ATT \cap (TAT \cup TTATT) \subseteq A$. Thus $T \setminus Q \subseteq A$. Hence $T = Q \cup (T \setminus Q) \subseteq Q \cup A = A \subseteq T$, so $A = T$.

Therefore Q is a maximal quasi-ideal of T. □

Example 3.1. (1) Consider a ternary semigroup $\mathbb{Z}^-$ under the multiplication. Let $Q = \mathbb{Z}^- \setminus \{-1\}$. It is easy to verify that Q is a quasi-ideal of $\mathbb{Z}^-$. By Theorem 3.1(1), Q is a maximal quasi-ideal of $\mathbb{Z}^-$.

(2) Consider a ternary semigroup $T = \{-i, 0, i\}$ under the multiplication. Let $Q = \{0\}$. Clearly, Q is a quasi-ideal of T. Since $TTi \cap iTT \cap (TiT \cup TTiTT) = T$ and $TT(-i) \cap (-i)TT \cap (T(-i)T \cup TT(-i)TT) = T$, by Theorem 3.1(2), Q is a maximal quasi-ideal of T.

Theorem 3.2. *If Q is a maximal quasi-ideal of a ternary semigroup T and $Q \cup (a)_q$ is a quasi-ideal of T for all $a \in T \setminus Q$, then either*

(1) $T \setminus Q = \{a\}$ and $a^3 \in Q$ for some $a \in T \setminus Q$, and $TTb \cap bTT \cap (TbT \cup TTbTT) \subseteq Q$ for all $b \in T \setminus Q$ or

(2) $T \setminus Q \subseteq (a)_q$ for all $a \in T \setminus Q$.

Proof. Let Q be a maximal quasi-ideal of T and assume $Q \cup (a)_q$ is a quasi-ideal of T for all $a \in T \setminus Q$.

Case 1: $TTa \cap aTT \cap (TaT \cup TTaTT) \subseteq Q$ for some $a \in T \setminus Q$.

Then $a^3 \in TTa \cap aTT \cap (TaT \cup TTaTT) \subseteq Q$, so $a^3 \in Q$. Since $Q \cup \{a\} = (Q \cup TTa \cap aTT \cap (TaT \cup TTaTT)) \cup \{a\} = Q \cup (\{a\} \cup TTa \cap aTT \cap (TaT \cup

$TTaTT)) = Q \cup (a)_q$, by assumption, $Q \cup \{a\}$ is a quasi-ideal of T. Since $a \in T \setminus Q, Q$ is a proper subset of $Q \cup \{a\}$. This implies $Q \cup \{a\} = T$ because Q is a maximal quasi-ideal of T. Thus $T \setminus Q \subseteq \{a\}$. Let $b \in T \setminus Q$. So $b = a$. Then $TTb \cap bTT \cap (TbT \cup TTbTT) = TTa \cap aTT \cap (TaT \cup TTaTT) \subseteq Q$. Hence $TTb \cap bTT \cap (TbT \cup TTbTT) \subseteq Q$ for all $b \in T \setminus Q$.

Case 2: $TTa \cap aTT \cap (TaT \cup TTaTT) \nsubseteq Q$ for all $a \in T \setminus Q$.

Let $a \in T \setminus Q$. Then Q is a proper subset of $Q \cup (a)_q$. By assumption and maximality of Q, $Q \cup (a)_q = T$. Hence $T \setminus Q \subseteq (a)_q$. □

Example 3.2. (1) In $\mathbb{Z}_{81}$, consider the ternary semigroup $T = \{\overline{0}, \overline{3}, \overline{27}\}$ under the usual multiplication and a maximal quasi-ideal $Q = \{\overline{0}, \overline{27}\}$. It is easy to verify that Q satisfies (1) of Theorem 3.2.

(2) Consider a ternary semigroup $\mathbb{Z}^-$ under the multiplication and a maximal quasi-ideal $Q = \mathbb{Z}^- \setminus \{-1\}$ of $\mathbb{Z}^-$. It is easy to verify that Q satisfies (2) of Theorem 3.2.

For a ternary semigroup T, let $\mathcal{U}$ denote the union of all proper quasi-ideals of T. Then the following lemma holds.

Lemma 3.3. *$T = \mathcal{U}$ if and only if $(a)_q \neq T$ for all $a \in T$.*

Theorem 3.4. *Let T be a ternary semigroup. One of the following four conditions is satisfied.*

(1) $\mathcal{U}$ is not a quasi-ideal of T.

(2) $(a)_q \neq T$ for all $a \in T$.

(3) There exists $a \in T$ such that $(a)_q = T, \{a\} \nsubseteq TTa \cap aTT \cap (TaT \cup TTaTT)$ and $a^3 \in \mathcal{U}, T$ is not quasi-simple, $T \setminus \mathcal{U} = \{x \in T \mid (x)_q = T\}$ and $\mathcal{U}$ is the unique maximal quasi-ideal of T.

(4) $T \setminus \mathcal{U} \subseteq (a)_q$ for all $a \in T \setminus \mathcal{U}, T$ is not quasi-simple, $T \setminus \mathcal{U} = \{x \in T \mid (x)_q = T\}$ and $\mathcal{U}$ is the unique maximal quasi-ideal of T.

Proof. Assume that $\mathcal{U}$ is a quasi-ideal of T. Now, we consider the following two cases.

Case 1: $\mathcal{U} = T$.

In this case, by Lemma 3.3, the condition (2) holds.

Case 2: $\mathcal{U} \neq T$.

So T is not quasi-simple. Claim that $\mathcal{U}$ is the unique maximal quasi-ideal of T. Let Q be a quasi-ideal of T such that $\mathcal{U}$ is a proper subset of Q. If $Q \neq T$, then $Q \subseteq \mathcal{U}$, this is a contradiction. Hence $\mathcal{U}$ is a maximal ideal of T. Next, let

Q' be a maximal quasi-ideal of T. Thus $Q' \subseteq \mathcal{U}$, this implies $Q' = \mathcal{U}$. Hence $\mathcal{U}$ is the unique maximal quasi-ideal of T. Since $\mathcal{U} \neq T$, clearly, $(a)_q = T$ for all $a \in T \setminus \mathcal{U}$. So $T \setminus \mathcal{U} = \{x \in T \mid (x)_q = T\}$. Then for all $x \in T \setminus \mathcal{U}$, we have that $\mathcal{U} \cup (x)_q = T$ is a quasi-ideal of T. By Theorem 3.2, we have the following two cases.

(i) $T \setminus \mathcal{U} = \{a\}$ and $a^3 \in \mathcal{U}$ for some $a \in T \setminus \mathcal{U}$, and $TTb \cap bTT \cap (TbT \cup TTbTT) \subseteq \mathcal{U}$ for all $b \in T \setminus \mathcal{U}$.

(ii) $T \setminus \mathcal{U} \subseteq (a)_b$ for all $a \in T \setminus \mathcal{U}$.

Assume (i) holds. If $\{a\} \subseteq TTa \cap aTT \cap (TaT \cup TTaTT)$, then $T = (a)_q = TTa \cap aTT \cap (TaT \cup TTaTT)$. By assumption, $T = TTa \cap aTT \cap (TaT \cup TTaTT) \subseteq \mathcal{U}$ and so $\mathcal{U} = T$, this is a contradiction. Hence $\{a\} \not\subseteq TTa \cap aTT \cap (TaT \cup TTaTT)$. In this case, the condition (3) holds. It is easy to see that case (ii) is the condition (4). □

Example 3.3. (1) Consider the ternary semigroup $T = \{-i, i\}$ under the multiplication. Then T is quasi-simple, this implies $\mathcal{U} = \emptyset$. So T satisfies the condition (1) of Theorem 3.4.

(2) Consider the ternary semigroup $T = \mathbb{Z}^- \setminus \{-1\}$ under the multiplication. It is easy to verify that $(x)_q \neq T$ for all $x \in T$. Hence T satisfies the condition (2) of Theorem 3.4.

(3) In $\mathbb{Z}_{81}$, consider the ternary semigroup $T = \{\overline{0}, \overline{3}, \overline{27}\}$ under the usual multiplication. Thus $\mathcal{U} = \{\overline{0}, \overline{27}\}$. It is easy to verify that T satisfies the condition (3) of Theorem 3.4.

(4) Consider the ternary semigroup $T = \mathbb{Z}^-$ under the multiplication. Thus $\mathcal{U} = \mathbb{Z}^- \setminus \{-1\}$. It is easy to verify that T satisfies the condition (4) of Theorem 3.4.

Next, let T be a ternary semigroup with zero. Let $\mathcal{U}_0$ denote the union of all nonzero proper quasi-ideal of T. Using the same proof of Theorem 3.4, the following theorem holds.

Theorem 3.5. *Let T be a ternary semigroup with zero. One of the following four conditions is satisfied.*

(1) $\mathcal{U}_0$ is not a quasi-ideal of T.

(2) $(a)_q \neq T$ for all $a \in T$.

(3) There exists $a \in T$ such that $(a)_q = T, \{a\} \not\subseteq TTa \cap aTT \cap (TaT \cup TTaTT)$ and $a^3 \in \mathcal{U}_0, T$ is not 0-quasi-simple, $T \setminus \mathcal{U}_0 = \{x \in T \mid (x)_q = T\}$ and $\mathcal{U}_0$ is the unique maximal quasi-ideal of T.

(4) $T \setminus \mathcal{U}_0 \subseteq (a)_q$ *for all* $a \in \mathcal{U}_0, T$ *is not 0-quasi-simple,* $T \setminus \mathcal{U}_0 = \{x \in T \mid (x)_q = T\}$ *and* $\mathcal{U}_0$ *is the unique maximal quasi-ideal of* T*.*

Acknowledgements. This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

References

[1] V.N. Dixit, S. Dewan, A note on quasi and bi-ideals in ternary semigroup, *Int. J. Math. Math. Sci.*, **18** (1995), 501-508.

[2] V.N. Dixit, S. Dewan, Minimal quasi-ideals in ternary semigroup, *Indian J. Pure Appl. Math.*, **28** (1997), 625-632.

[3] A. Iampan, Lateral ideals of ternary semigroups, *Ukrainian Math. Bull.*, **4** (2007), 525-534.

[4] A. Iampan, Quasi-ideals of semigroups, *J. Algebra Number Theory Appl.*, **12** (2008), 93-102.

[5] E. Kasner, An extension of the group concept, *Bull. Amer. Math. Soc.*, **10**(1904), 290-291.

[6] D.H. Lehmer, A ternary analoue of abelian groups, *Amer. J. Math.*, **59** (1932), 329-338.

[7] J. Los, On the extending of model I, *Fund. Math.*, **42** (1955), 38-54.

[8] F.M. Sioson, Ideal theory in ternary semigroups, *Math. Japan*, **10** (1965), 63-84.