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THE STABILITY OF PEXIDERIZED CAUCHY FUNCTIONAL EQUATION

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Abstract: In this paper, we prove the Hyers-Ulam-Rassias stability of the pexiderized Cauchy functional equation $f(x+y+z)-g(x)-h(y)-j(z)=0$ in Banach spaces. We also use the definition of fuzzy normed spaces to establish the fuzzy version of the Hyers-Ulam-Rassias stability.

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1. Introduction

The first problem on the stability of group homomorphism was given by S.M. Ulam [9] in 1940. He discussed the number of unsolved problems before the Mathematics club of the university of wiscosin. In the next year Hyer [2] gave the first affirmative answer of the Ulam's problem for Banach spaces. A generalized version of the theorem of Hyers was given by Th.M. Rassias [10] in 1978 which allows Cauchy difference to be unbounded. The generalization given by T.M. Rassias is called the Hyers-Ulam-Rassias stability.

In 1994, P. Gavruta [6] provided a further generalization of Th.M.Rassias theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general function

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$\phi(x, y)$ for the existence of unique linear mapping. The Hyers-Ulam-Rassias stability of various functional equations have been extensively introduced by a number of Mathematicians. In 1999, K.W. Jun, D.S. Shin and B.D. Kim [5] proved the stability of functional equation $f(x+y) - g(x) - h(y) = 0$, which is called pexider functional equation. Later on, in 2000 this result was generalized by Y.H. Lee and K.W. Jun [7]. The functional equation

$$f(x + y + z) = f(x) + f(y) + f(z) \tag{1.1}$$

is called the Cauchy functional equation in 3-variable.Since f is a solution of it is said to be additive or satisfies the Cauchy functional equation. The Hyers-Ulam-Rassias stability of this equation was introduced by J.R. Lee and C. Park [3] in 2009 on Banach Algebra. The functional equation

$$f(x + y + z) = g(x) + h(y) + j(z) \tag{1.2}$$

for all $x, y, z \in X$ is called the pexiderized Cauchy functional equation since it is satisfied by $f, g, h,$ and j. In the first section, we prove the stability problem in the sense of Hyers-Ulam-Rassias and P.Gavruta for the Pexiderized Cauchy functional equation (1.2). In other section we extend this result for the fuzzy stability by using Direct and Fixed point approach.

2. Hyers-Ulam-Rassias Stability of (1.2)

In this section, we prove the Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation (1.2). Throughout this section, let $(X, +)$ be and abelian group, $(Y, *)$ be a Banach space and $\phi : X \times X \times X \to [0, \infty)$ a mapping such that

$$\varepsilon(x) = \sum_{j=1}^{\infty} \frac{\begin{pmatrix} \phi(0, 0, 3^{j-1}x) + \phi(3^{j-1}x, 0, 0) \\ +\phi(0, 3^{j-1}x, 0) + \phi(3^{j-1}x, 3^{j-1}x, 3^{j-1}x) \end{pmatrix}}{3^j}$$

and

$$\lim_{n\to\infty} \frac{\phi(3^n x, 3^n y, 3^n z)}{3^n} = 0 \quad \text{for all } x, y, z \in X.$$

Theorem 2.1. *Let $f, g, h, j : X \to Y$ be mapping satisfying the inequality*

$$\|f(x + y + z) - g(x) - h(y) - j(z)\| \leq \phi(x, y, z), \tag{2.1}$$

for all $x, y, z \in X$. *Then there exists a unique additive mapping* $T : X \to Y$ *such that*

$$\|f(x) - T(x)\| \leq 2\|g(0)\| + 2\|h(0)\| + 2\|j(0)\| + \varepsilon(x) \tag{2.2}$$
$$\|g(x) - T(x)\| \leq 2\|g(0)\| + 3\|h(0)\| + 3\|j(0)\| + \phi(x, 0, 0) + \varepsilon(x) \tag{2.3}$$
$$\|h(x) - T(x)\| \leq 3\|g(0)\| + 2\|h(0)\| + 3\|j(0)\| + \phi(0, x, 0) + \varepsilon(x) \tag{2.4}$$
$$\|j(x) - T(x)\| \leq 3\|g(0)\| + 3\|h(0)\| + 2\|j(0)\| + \phi(0, 0, x) + \varepsilon(x) \tag{2.5}$$

for all $x, y, z \in X$.

Proof. Let $x = y = z$ in the inequality (2.1), we have

$$\|f(3x) - g(x) - h(x) - j(x)\| \leq \phi(x, x, x), \tag{2.6}$$

for all $x \in X$. Substituting $y = z = 0, z = x = 0$ and $x = y = 0$ in inequality (2.1) respectively, we get

$$\|f(x) - g(x)\| \leq \|h(0)\| + \|j(0)\| + \phi(x, 0, 0), \text{ for all } x \in X \tag{2.7}$$
$$\|f(y) - h(y)\| \leq \|g(0)\| + \|j(0)\| + \phi(0, y, 0), \text{ for all } y \in X \tag{2.8}$$

and

$$\|f(z) - j(z)\| \leq \|g(0)\| + \|h(0)\| + \phi(0, 0, z), \text{ for all } z \in X. \tag{2.9}$$

Using the inequality (2.6), (2.7), (2.8) and (2.9) we get

$$\begin{aligned}
&\|f(3x) - 3f(x)\| \\
&\leq \|f(3x) + g(x) - g(x) + h(x) - h(x) + j(x) - j(x) - 3f(x)\| \\
&\leq \|f(3x) - g(x) - h(x) - j(x)\| + \|g(x) + h(x) + j(x) - 3f(x)\| \\
&\leq \|f(3x) - g(x) - h(x) - j(x)\| + \|g(x) - f(x)\| + \|h(x) - f(x)\| \\
&\qquad + \|j(x) - f(x)\| \\
&\leq 2\|g(0)\| + 2\|h(0)\| + 2\|j(0)\| + \phi(0, 0, x) + \phi(x, 0, 0) \\
&\qquad + \phi(0, 0, x) + \phi(x, x, x) \\
&\leq r(x)
\end{aligned} \tag{2.10}$$

where $r(x) = \{2\|g(0)\| + 2\|h(0)\| + 2\|j(0)\| + \phi(0, 0, x) + \phi(x, 0, 0) + \phi(0, 0, x) + \phi(x, x, x)$ for all $x \in X$ and replacing x by $3x$ in inequality (2.10), we get

$$\|f(3^2x) - 3f(3x)\| \leq r(3x) \tag{2.11}$$

Now using the inequality (2.11) it follows that

$$\begin{aligned}\|f(3^2x)-3^2f(x)\| &\le \|f(3^2x)-3f(3x)+3f(3x)-3^2f(x)\|\\ &\le \|f(3^2x)-3f(3x)\|+\|3f(3x)-3^2f(x)\|\\ &\le \|f(3^2x)-3f(3x)\|+3\|f(3x)-3f(x)\|\\ &\le r(3x)+3r(x)\end{aligned}$$

for all $x \in X$.

Applying induction on 'n', we have

$$\|f(3^nx)-3^nf(x)\| \le \sum_{j=1}^{n} 3^{j-1}r(3^{n-j}x) \tag{2.12}$$

Now we claim that the above inequality (2.12) holds for $n+1$. Indeed, substituting $3x$ for x in above equation

$$\|f(3^{n+1}x)-3^nf(3x)\| \le \sum_{j=1}^{n} 3^{j-1}r(3^{n+1-j}x) \tag{2.13}$$

for all $x \in X$. Hence

$$\begin{aligned}&\|f(3^{n+1}x)-3^{n+1}f(x)\|\\ &\le \|f(3^{n+1}x)-3^nf(3x)+3^nf(3x)-3^{n+1}f(x)\|\\ &\le \|f(3^{n+1}x)-3^nf(3x)\|+\|3^nf(3x)-3^{n+1}f(x)\|\\ &\le \|f(3^{n+1}x)-3^nf(3x)\|+3^n\|f(3x)-3f(x)\|\\ &\le \sum_{j=1}^{n+1} 3^{j-1}r(3^{n-j}x)\end{aligned} \tag{2.14}$$

for all $x \in X$. Then from (2.12), we get

$$\|3^{-n}f(3^nx)-f(x)\| \le \sum_{j=1}^{n} 3^{j-n-1}r(3^{n-j}x) \tag{2.15}$$

for all $x \in X$. Now , to prove that $\{3^{-n}f(3^nx)\}$ is a Cauchy sequence in Y. For $m < n$ we have.

$$\|3^{-n}f(3^nx)-3^{-m}f(3^mx)\| \le \sum_{j=m}^{n-1} \|3^{-j}f(3^jx)-3^{-(j+1)}f(3^{j+1}x)\|$$

$$\leq \sum_{j=m}^{n-1} \frac{r(3^j x)}{3^{j+1}} \tag{2.16}$$

Taking the limit as $m \to \infty$, we get

$$\lim_{m\to\infty} \|3^{-n} f(3^n x) - 3^{-m} f(3^m x)\| = 0$$

for all $x \in X$. Since X is a Banach space it follows that the sequence $\{3^{-n} f(3^n x)\}$ is convergent. Now we define mapping $T : X \to Y$ by

$$T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n} \tag{2.17}$$

Now, we prove that T satisfies the inequality (2.1), replacing x, y and z by $3^n x, 3^n y$ and $3^n z$ respectively and dividing by 3^n in inequality (2.1), we get

$$\left\| \frac{f(3^n x + 3^n y + 3^n z)}{3^n} - \frac{g(3^n x)}{3^n} - \frac{h(3^n y)}{3^n} - \frac{j(3^n z)}{3^n} \right\| \leq \frac{\phi(3^n x, 3^n y, 3^n z)}{3^n}, \tag{2.18}$$

for all $x, y, z \in G$.

Also from equation (2.7), (2.8) and (2.9) we get

$$\left\| \frac{g(3^n x)}{3^n} - \frac{f(3^n x)}{3^n} \right\| \leq \frac{\|h(0)\| + \|j(0)\| + \phi(3^n x, 0, 0)}{3^n}, \tag{2.19}$$

$$\left\| \frac{h(3^n x)}{3^n} - \frac{f(3^n x)}{3^n} \right\| \leq \frac{\|g(0)\| + \|j(0)\| + \phi(0, 3^n x, 0)}{3^n}, \tag{2.20}$$

$$\left\| \frac{j(3^n x)}{3^n} - \frac{f(3^n x)}{3^n} \right\| \leq \frac{\|g(0)\| + \|h(0)\| + \phi(0, 0, 3^n x)}{3^n}, \tag{2.21}$$

Taking limit as $n \to \infty$, inequality (2.19), (2.20) and (2.21) implies that

$$\lim_{n\to\infty} \frac{g(3^n x)}{3^n} = \lim_{n\to\infty} \frac{f(3^n x)}{3^n} \tag{2.22}$$

$$\lim_{n\to\infty} \frac{h(3^n x)}{3^n} = \lim_{n\to\infty} \frac{f(3^n x)}{3^n} \tag{2.23}$$

$$\lim_{n\to\infty} \frac{j(3^n x)}{3^n} = \lim_{n\to\infty} \frac{f(3^n x)}{3^n} \tag{2.24}$$

Using (2.22), (2.23) and (2.24) inequality (2.18) implies that

$$\left\| \frac{f(3^n x + 3^n y + 3^n z)}{3^n} - \frac{g(3^n x)}{3^n} - \frac{h(3^n y)}{3^n} - \frac{j(3^n z)}{3^n} \right\|$$

$$= \|T(x+y+z) - T(x) - T(y) - T(z)\|, \tag{2.25}$$

for a all $x, y, z \in X$. To prove inequality (2.2) taking the limit $n \to \infty$ in (2.15), we have

$$\begin{aligned} \|T(x) - f(x)\| &\leq \lim_{n\to\infty} \sum_{j=1}^{n} 3^{j-n-1} r(3^{n-j}x) \\ &\leq \lim_{n\to\infty} \left\{ \left(1 - \frac{1}{3^n}\right)(2\|g(0)\| + 2\|h(0)\| + 2\|j(0)\|) \right. \\ &\quad \left. + \sum_{j=1}^{n} \frac{\begin{pmatrix} \phi(0,0,3^{j-1}x) + \phi(3^{j-1}x,0,0) \\ +\phi(0,3^{j-1}x,0) + \phi(3^{j-1}x,3^{j-1}x,3^{j-1}x) \end{pmatrix}}{3^j} \right\} \\ &= 2\|g(0)\| + 2\|h(0)\| + 2\|j(0)\| + \varepsilon(x) \end{aligned} \tag{2.26}$$

Similarly using the inequalities (2.7), (2.8), (2.9) and (2.26) the inequality (2.3), (2.4) and (2.5) also holds.

Now, to prove that T is unique mapping. Let us consider another mapping $U : X \to Y$ such that

$$U(x+y+z) = U(x) + U(y) + U(z)$$

So that

$$\begin{aligned} \|T(x) - U(x)\| &= \|3^{-n}T(3^n x) - 3^{-n}U(3^n x)\| \\ &\leq \|3^{-n}T(3^n x) - 3^{-n}f(3^n x) + 3^{-n}f(3^n x) - 3^{-n}U(3^n x)\| \end{aligned}$$

$$\begin{aligned} &\leq \|3^{-n}T(3^n x) - 3^{-n}f(3^n x)\| + \|3^{-n}f(3^n x) - 3^{-n}U(3^n x)\| \\ &\leq 3^{-n}(2\|g(0)\| + 2\|h(0)\| + 2\|j(0)\| + \varepsilon(x)) \\ &\quad + 3^{-n}(2\|g(0)\| + 2\|h(0)\| + 2\|j(0)\| + \varepsilon(3^n x)) \end{aligned} \tag{2.27}$$

for all $x \in X$. Taking the limit in (2.27) as $n \to \infty$, we get

$$T(x) = U(x)$$

for all $x \in X$. Hence proved the desired result. □

Corollary 2.2. *Let $f, g, h, j : X \to Y$ be such that $g(0) = 0, h(0) = 0, j(0) = 0$ and*

$$\|f(x+y+z) - g(x) - h(y) - j(z)\| \leq \phi(x, y, z), \tag{2.28}$$

for all $x, y, z \in X$. *Then there exists a unique additive mapping* $T : X \to Y$ *such that*

$$T(x+y+z) = T(x) + T(y) + T(z),$$
$$\|f(x) - T(x)\| \le \varepsilon(x), \tag{2.29}$$
$$\|g(x) - T(x)\| \le \phi(x, 0, 0) + \varepsilon(x), \tag{2.30}$$
$$\|h(x) - T(x)\| \le \phi(0, x, 0) + \varepsilon(x), \tag{2.31}$$

and

$$\|j(x) - T(x)\| \le \phi(0, 0, x) + \varepsilon(x) \tag{2.32}$$

for all $x, y, z \in X$.

Corollary 2.3. *Let* $f, g, h, j : X \to Y$ *be mapping and* $\delta > 0$ *such that* $g(0) = 0, h(0) = 0, j(0) = 0$,

$$\|f(x+y+z) - g(x) - h(y) - j(z)\| \le \delta, \tag{2.33}$$

for all $x, y, z \in X$. *Then there exists a unique additive mapping* $T : X \to Y$ *such that*

$$T(x+y+z) = T(x) + T(y) + T(z),$$
$$\|f(x) - T(x)\| \le 2\delta, \tag{2.34}$$
$$\|g(x) - T(x)\| \le 3\delta, \tag{2.35}$$
$$\|h(x) - T(x)\| \le 3\delta, \tag{2.36}$$

and

$$\|j(x) - T(x)\| \le 3\delta \tag{2.37}$$

for all $x, y, z \in X$.

Corollary 2.4. *Let* B_1, B_2 *be the two Banach spaces and* $f, g, h, j : B_1 \to B_2$ *be mappings. Let* $\theta \ge 0$ *and* $p \in [0, 1)$ *such that*

$$\|f(x+y+z) - g(x) - h(y) - j(z)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p), \tag{2.38}$$

for all $x, y, z \in B_1$. *Then there exists a unique additive mapping* $T : B_1 \to B_2$ *such that*

$$\|f(x) - T(x)\| \le 2\|g(0)\| + 2\|h(0)\| + 2\|j(0)\| + \frac{6\theta}{3 - 3^p}\|x\|^p \tag{2.39}$$

$$\|g(x) - T(x)\| \leq 2\|g(0)\| + 3\|h(0)\| + 3\|j(0)\| + \frac{9-3^p}{3-3^p}\theta\|x\|^p \tag{2.40}$$

$$\|h(x) - T(x)\| \leq 3\|g(0)\| + 2\|h(0)\| + 3\|j(0)\| + \frac{9-3^p}{3-3^p}\theta\|x\|^p \tag{2.41}$$

$$\|j(x) - T(x)\| \leq 3\|g(0)\| + 3\|h(0)\| + 2\|j(0)\| + \frac{9-3^p}{3-3^p}\theta\|x\|^p \tag{2.42}$$

for all $x, y, z \in B_1$.

Corollary 2.5. *Let* $0 \leq p+q+r < 1$*, where* p, q *and* r *are the positive real numbers and let* $f, g, h, j : X \to Y$ *be mappings such that* $g(0) = 0, h(0) = 0, j(0) = 0$ *and*

$$\|f(x+y+z) - g(x) - h(y) - j(z)\| \leq \theta\|x\|^p\|y\|^q\|z\|^r, \tag{2.43}$$

for all $x, y, z \in X$*. Then there exists a unique additive mapping* $T : X \to Y$ *such that*

$$T(x+y+z) = T(x) + T(y) + T(z),$$

$$\|f(x) - T(x)\| \leq \frac{\theta}{3-3^{p+q+r}}\|x\|^{p+q+r} \tag{2.44}$$

$$\|g(x) - T(x)\| \leq \frac{\theta}{3-3^{p+q+r}}\|x\|^{p+q+r} \tag{2.45}$$

$$\|h(x) - T(x)\| \leq \frac{\theta}{3-3^{p+q+r}}\|x\|^{p+q+r} \tag{2.46}$$

$$\|j(x) - T(x)\| \leq \frac{\theta}{3-3^{p+q+r}}\|x\|^{p+q+r} \tag{2.47}$$

for all $x, y, z \in X$.

3. Fuzzy Hyers-Ulam-Rassias Stability of (1.2)

In this section, first we shall adopt, the usual terminology, Notations and Conventions of the theory of Fuzzy normed spaces. (see in [1,8]). After that we prove the Fuzzy stability of equation (1.2) in the sense of Hyers-Ulam-Rassias and P.Gavruta by using direct and fixed point approach.

Definition 3.1. Let X be a linear space. A fuzzy subset N of $X \times R$ into [0,1] is called a fuzzy norm on X if for every $x, y \in X$ and $s, t \in R$

(N_1) $N(x, t) = 0$ for $t \leq 0$,

(N_2) $N(x,t) = 1$ if and only if $x = 0$ for all $t > 0$,

(N_3) $N(cx,t) = N(x, t/|c|)$ if $c \neq 0$,

(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$,

(N_5) $N(x,.)$ is non-decreasing function on R and $\lim_{t\to\infty} N(x,t) = 1$.

Example 3.2. Let $(X, \|.\|)$ be a normed space. Define $a * b = ab$ and $a * b = \min(a,b)$ such that

$$N(x,t) = \begin{cases} 1 - \dfrac{1}{\exp(t/\|x\|)}, & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Then $(X, N, *)$ is a fuzzy normed space.

Definition 3.3. A sequence $\{x_n\}$ in a fuzzy normed space (X,N) is called Cauchy sequence if for every $t > 0$ and $\epsilon > 0$, there exists a positive integer k such that $N(x_m - x_n, t) > 1 - \varepsilon$ whenever $n, m \geq k$.

Definition 3.4. A sequence $\{x_n\}$ in a fuzzy normed space (X,N) converges to $x \in X$ if for every $t > 0$ and $\epsilon > 0$, there exists a positive integer k such that for all $n \geq k$, we have $N(x_n - x, t) > 1 - \varepsilon$. When x is the limit of the sequence $\{x_n\}$ then we write $N - \lim_{n\to\infty} = x$

Remark 3.5. Every convergent sequence in a fuzzy normed space is Cauchy sequence. The fuzzy normed space is called a Banach space if it is complete also, (i. e every Cauchy sequence in X is convergent to a point in X).

Theorem 3.6. *(Fixed point alternative). Let (X,d) be a complete generalized metric space and a contractive mapping $J : X \to X$, with the Lipschitz constant L. Then, for each given element $x \in X$, either*

(A_1) *$d(J^n x, J^{n+1}x) = +\infty$ for all $n \geq 0$,*
Or

(A_2) *There exists a natural n_0 such that:*

(A_{20}) *$d(J^n x, J^{n+1}x) = +\infty$ for all $n \geq n_0$,*

(A_{21}) *The sequence $(J^n x)$ is convergent to a fixed point y^* of J;*

(A_{22}) *y^* is the unique fixed point of J in the set $Y = \{y \in X,\ d(J^{n_o}x, y) < +\infty\}$;*

(A_{23}) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Direct Approach

Theorem 3.7. *Let X be linear space and (Z, N^1) be a fuzzy normed space. Let $\phi : X \times X \times X \to Z$ be a mappings such that*

$$\phi(3x, 3y, 3z) = \alpha\phi(x, y, z), \quad \text{for all } x, y, z \in X \text{ and } t > 0 \tag{3.1}$$

for some real number $0 < |\alpha| < 3$. Let (Y, N) be a fuzzy Banach space and let f, g, h and j be mappings from X to Y such that

$$N(f(x + y + z) - g(x) - h(y) - j(z), t) \geq N^1(\phi(x, y, z), t), \tag{3.2}$$

for all $x, y, z \in X$ and $t > 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$N(T(x) - f(x), t) \geq M_1\left(x, \frac{3 - |\alpha|}{9}t\right) \tag{3.3}$$

$$N(T(x) - g(x), t) \geq M_1\left(x, \frac{9 - 3|\alpha|}{30 - |\alpha|}t\right) \tag{3.4}$$

$$N(T(x) - h(x), t) \geq M_1\left(x, \frac{9 - 3|\alpha|}{30 - |\alpha|}t\right) \tag{3.5}$$

$$N(T(x) - j(x), t) \geq M_1\left(x, \frac{9 - 3|\alpha|}{30 - |\alpha|}t\right) \tag{3.6}$$

where

$$M_1(x, t) = \min\{N^1(\phi(x, y, z), t), N^1(\phi(x, 0, 0), t), N^1(\phi(0, y, 0), t),\ N^1(\phi(0, 0, z), t)\}.$$

Proof. Taking $x = y = z$ in (3.2), we get

$$N(f(3x) - g(x) - h(x) - j(x), t) \geq N^1(\phi(x, x, x), t) \tag{3.7}$$

Substituting $y = z = 0, x = z = 0$ and $x = y = 0$,in inequality (3.2) respectively, we get

$$N(f(x) - g(x), t) \geq N^1(\phi(x, 0, 0), t), \tag{3.8}$$

$$N(f(y) - h(y), t) \geq N^1(\phi(0, y, 0), t), \tag{3.9}$$

$$N(f(z) - j(z), t) \geq N^1(\phi(0, 0, z), t), \tag{3.10}$$

for all $x, y, z \in X$ and $t > 0$. Using the inequalities (3.7), (3.8), (3.9), and (3.10) we conclude that

$$\begin{aligned} &N(f(x+y+z) - f(x) - f(y) - f(z), t) \\ &\quad \geq \min\{N^1(\phi(x,y,z), t/4), N^1(\phi(x,0,0), t/4), \\ &\qquad\quad N^1(\phi(0,y,0), t/4), N^1(\phi(0,0,z), t/4)\} \\ &\quad \geq M_1(x,t) \end{aligned} \tag{3.11}$$

Taking $x = y = z$ in (3.11), we get

$$N(f(3x) - 3f(x), t) \geq M_1(x,t) \tag{3.12}$$

then,

$$M_1(3^n x, t) = M_1\left(x, \frac{t}{\alpha^n}\right) \tag{3.13}$$

Replace x by $3^n x$ in (3.12), we get

$$N(f(3^{n+1}x) - 3f(3^n x), t) \geq M_1(3^n x, t)$$

So that

$$\begin{aligned} N\left(\frac{f(3^{n+1}x)}{3^{n+1}} - \frac{f(3^n x)}{3^n}, t\right) &= N(f(3^{n+1}x) - 3f(3^n x), 3^{n+1}t) \\ &\geq M_1(3^n x, 3^{n+1}t) \\ &\geq M_1(x, 3^{n+1}t/\alpha^n) \end{aligned}$$

Or

$$N\left(\frac{f(3^{n+1}x)}{3^{n+1}} - \frac{f(3^n x)}{3^n}, \frac{\alpha^n}{3^{n+1}}t\right) \geq M_1(x,t)$$

for all $x \in X$ and $t > 0$. Therefore, for each $n > m \geq 0$, we say that,

$$\begin{aligned} &N\left(\frac{f(3^n x)}{3^n} - \frac{f(3^m x)}{3^m}, \sum_{i=m+1}^{n} \frac{\alpha^{i-1}}{3^i}t\right) \\ &\quad = N\left(\sum_{i=m+1}^{n} \frac{f(3^i x)}{3^i} - \frac{f(3^{i-1}x)}{3^{i-1}}, \sum_{i=m+1}^{n} \frac{\alpha^{i-1}}{3^i}t\right) \\ &\quad \geq \min \bigcup_{i=m+1}^{n} \left[N\left(\frac{f(3^i x)}{3^i} - \frac{f(3^{i-1}x)}{3^{i-1}}, \frac{\alpha^{i-1}}{3^i}t\right)\right] \end{aligned}$$

$$\geq M_1(x,t) \tag{3.14}$$

Given for each $\epsilon > 0$ and $t_0 > 0$, there is some $t_1 > t_0$ such that $M_1(x, t_1) > 1-\varepsilon$. Therefore, by the convergence of series $\sum \frac{\alpha^{i-1}}{3^i} t_1$, we can find some no. such that $\sum \frac{\alpha^{i-1}}{3^i} t_1 < t_0$ for each $n > m > n_0$.

$$\begin{aligned} N\bigg(\frac{f(3^n x)}{3^n} - \frac{f(3^m x)}{3^m}, t_0\bigg) &\geq N\bigg(\frac{f(3^n x)}{3^n} - \frac{f(3^m x)}{3^m}, \sum_{i=m+1}^{n} \frac{\alpha^{i-1}}{3^i} t_1\bigg) \\ &\geq M_1(x, t_1) > 1 - \varepsilon \end{aligned}$$

It follows that $\left\{\frac{f(3^n x)}{3^n}\right\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N). Since (Y, N) is complete normed space it converges to some points $T(x) \in Y$. Therefore, there exists the mapping $T : X \to Y$ defined by $T(x) = N - \lim_{n\to\infty} \frac{f(3^n x)}{3^n}$. Fix $x, y, z \in X$ and $t > 0$.

Now we prove that T satisfies (3.2), therefore from inequality (3.11)we get

$$\begin{aligned} &N\bigg(\frac{f(3^n(x+y+z))}{3^n} - \frac{f(3^n x)}{3^n} - \frac{f(3^n y)}{3^n} - \frac{f(3^n z)}{3^n}, \frac{t}{5}\bigg) \\ &= N\bigg(f(3^n(x+y+z)) - f(3^n x) - f(3^n y) - f(3^n z), \frac{3^n t}{5}\bigg) \\ &\geq \min\bigg\{N^1\bigg(\phi(3^n x, 3^n y, 3^n z), \frac{3^n t}{20\alpha^n}\bigg), N^1\bigg(\phi(3^n x, 0, 0), \frac{3^n t}{20\alpha^n}\bigg), \\ &\qquad N^1\bigg(\phi(0, 3^n y, 0), \frac{3^n t}{20\alpha^n}\bigg), N^1\bigg(\phi(0, 0, 3^n z), \frac{3^n t}{20\alpha^n}\bigg)\bigg\} \end{aligned}$$

for all $x, y, z \in X$ and $t > 0$. We can say that

$$\begin{aligned} &N(T(x+y+z) - T(x) - T(y) - T(z), t) \\ &\geq \min\bigg\{N\bigg(T(x+y+z) - \frac{f(3^n(x+y+z))}{3^n}, \frac{t}{5}\bigg), \\ &\qquad N\bigg(T(x) - \frac{f(3^n x)}{3^n}, \frac{t}{5}\bigg), N\bigg(T(y) - \frac{f(3^n y)}{3^n}, \frac{t}{5}\bigg), \\ &\qquad N\bigg(T(z) - \frac{f(3^n(z))}{3^n}, \frac{t}{5}\bigg), \\ &\qquad N\bigg(\frac{f(3^n(x+y+z))}{3^n} - \frac{f(3^n x)}{3^n} - \frac{f(3^n y)}{3^n} - \frac{f(3^n z)}{3^n}, \frac{t}{5}\bigg)\bigg\}, \end{aligned}$$

Therefore, taking $n \to \infty$, right hand side tends to 1. Which implies that

$$N(T(x+y+z) - T(x) - T(y) - T(z), t) = 1$$

Hence

$$T(x+y+z) = T(x) + T(y) + T(z)$$

To prove that inequality (3.3) holds, using (3.14) with $m = 0$, we get

$$\begin{aligned}
&N(T(x) - f(x), t)\\
&\quad = N\left(T(x) - \frac{f(3^n x)}{3^n} + \frac{f(3^n x)}{3^n} - f(x), t\right)\\
&\quad \geq \min\left[N\left(T(x) - \frac{f(3^n x)}{3^n}, \frac{t}{2}\right), N\left(\frac{f(3^n x)}{3^n} - f(x), \frac{t}{2}\right)\right]\\
&\quad \geq \min\left[N\left(T(x) - \frac{f(3^n x)}{3^n}, \frac{t}{2}\right), M_1\left(x, \frac{3t}{\sum_{i=1}^{n} (\alpha/3)^{i-1}}\right)\right]\\
&\quad \geq M_1\left(x, \frac{3t}{\sum_{i=0}^{\infty} (\alpha/3)^{i}}\right) \geq M_1\left(x, \frac{3-\alpha}{9}t\right) \qquad (3.15)
\end{aligned}$$

Again to prove the inequality (3.4), we conclude that

$$\begin{aligned}
&N\left(T(x) - g(x), \frac{30-\alpha}{27}t\right)\\
&\quad \geq \min\left[N(T(x) - f(x), t), N\left(f(x) - g(x), \frac{3-\alpha}{27}t\right)\right]\\
&\quad \geq \min\left[M_1\left(x, \frac{3-\alpha}{9}t\right), N^1\left(\phi(x,0,0), \frac{3-\alpha}{27}t\right)\right]\\
&\quad \geq M_1\left(x, \frac{3-\alpha}{9}t\right)
\end{aligned}$$

$$N(T(x) - g(x), t) \geq M_1\left(x, \frac{9-3|\alpha|}{30-|\alpha|}t\right) \qquad (3.16)$$

Thus, similarly inequality (3.5) and (3.6) for h and j also holds. To prove uniqueness of T, assume that T^1 be another additive mapping from X into Y, which satisfies (3.2). Since for each $n \in N, T(3^n x) = 3^n T(x)$ and $T^1(3^n x) = 3^n T^1(x)$. Then

$$N(T(x) - T^1(x), t)$$

$$= N(T(3^n x) - T^1(3^n x), 3^n t)$$
$$\geq \min\left[N\left(T(3^n x) - f(3^n x), \frac{3^n t}{2}\right), N\left(f(3^n x) - T^1(3^n x), \frac{3^n t}{2}\right)\right]$$
$$\geq M_1\left(3^n x, \frac{(3-|\alpha|)3^n t}{18}\right)$$

for all $x \in X, t > 0$ and $n \in N$. Therefore, taking $n \to \infty$ it follows that

$$N(T(x) - T^1(x), t) = 1$$

for all $x \in X, t > 0$. Hence proved the desired result. □

Fixed Point Approach

Theorem 3.8. *Let X be a linear space and let (Z, N^1) be a fuzzy normed space. Let $\phi : X \times X \times X \to Z$ be a function such that*

$$\phi(3x, 3y, 3z) = \alpha\phi(x, y, z) \quad \text{for all } x, y, z \in X \text{ and } t > 0$$

for some real number α with $0 < |\alpha| < 3$. Let (Y, N) be a fuzzy Banach space and let f, g, h and j be mappings from X to Y such that

$$N(f(x+y+z) - g(x) - h(y) - j(z)) \geq N^1(\phi(x, y, z), t), \tag{3.17}$$

for all $x, y, z \in X$ and $t > 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$N(T(x) - f(x), t) \geq M_1\left(x, \frac{3-|\alpha|}{3} t\right) \tag{3.18}$$
$$N(T(x) - g(x), t) \geq M_1\left(x, \frac{9-3|\alpha|}{12-|\alpha|} t\right) \tag{3.19}$$
$$N(T(x) - h(x), t) \geq M_1\left(x, \frac{9-3|\alpha|}{12-|\alpha|} t\right) \tag{3.20}$$
$$N(T(x) - j(x), t) \geq M_1\left(x, \frac{9-3|\alpha|}{12-|\alpha|} t\right) \tag{3.21}$$

*where $M_1(x, t) = \min\{N^1(\phi(x, y, z), t), N^1(\phi(x, 0, 0), t),$
$N^1(\phi(0, y, 0), t), N^1(\phi(0, 0, z), t)\}$*

Proof. First consider the set $E = \{\psi : A \to B, \psi(0) = 0\}$ and introduce the generalized metric on E:

$$d_{M_1}(\psi_1, \psi_2) = \inf\{\varepsilon \in R_+ : N(\psi_1(x) - \psi_2(x)) \geq M_1(x, t),$$

$$\forall\, x \in A, t > 0\} \tag{3.22}$$

Then we can prove that d_{M_1} is a complete generalized metric on E (see[4] or [11]). Define a function $J : E \to E$ by

$$J\psi(x) = \frac{\psi(3x)}{3} \quad \text{for all } x \in A \tag{3.23}$$

We claim that J is a strictly contractive mapping with the Lipschitz constant $a/3$. For given $\psi_1, \psi_2 \in E$, let $\epsilon \in (0, \infty)$ be an arbitrary constant with $d_{M_1}(\psi_1, \psi_2) \le \varepsilon$. By the definition of d_{M_1} it follows that

$$N(\psi_1(x) - \psi_2(x), \varepsilon t) \ge M_1(x,t) \quad \text{for all } x \in A \text{ and } t > 0 \tag{3.24}$$

Therefore,

$$\begin{aligned} N\left(J\psi_1(x) - J\psi_2(x), \frac{\alpha}{3}\varepsilon t\right) &= N\left(\frac{1}{3}\psi_1(3x) - \frac{1}{3}\psi_2(3x), \frac{\alpha}{3}\varepsilon t\right) \\ &= N(\psi_1(3x) - \psi_2(3x), \alpha\varepsilon t) \\ &\ge M_1(3x, \alpha t) = M_1(x,t), \\ &\qquad \text{for all } x \in A \text{ and } t > 0. \end{aligned} \tag{3.25}$$

Hence it holds that $d_{M_1}(J\psi_1, J\psi_2) \le (\alpha/3)\varepsilon$ that is,

$$d_{M_1}(J\psi_1, J\psi_2) \le (\alpha/3) d_{M_1}(\psi_1, \psi_2)$$

for all. It follows that $d_{M_1}(f, Jf) \le 1$. From the theorem (3.6) of fixed point alternative, we deduce the existence of a fixed point of J, the existence of mapping $T : A \to B$ such that $T(3x) = 3T(x)$ for each $x \in A$. We have $d_{M_1}(J^n f, T) \to 0$, which implies that

$$T(x) = N - \lim_{n\to\infty} \frac{f(3^n x)}{3^n} \quad \text{for all } x \in A \tag{3.26}$$

The condition $d_{M_1}(f, T) \le (1/(1-L)) d_{M_1}(f, Jf)$ implies that

$$d_{M_1}(f, T) \le \frac{1}{1 - (\alpha/3)} = \frac{3}{3 - \alpha} \tag{3.27}$$

If ε_n is a decreasing sequence converging to $3/(3-\alpha)$, then we have

$$N(T(x) - f(x), \varepsilon_n t) \ge M_1(x,t)$$

$$\Rightarrow \quad N(T(x) - f(x), t) \geq M_1\left(x, \frac{t}{\varepsilon_n}\right)$$
$$\Rightarrow \quad N(T(x) - f(x), t) \geq M_1\left(x, \frac{3-\alpha}{3}t\right) \text{ for all } x \in A \text{ and } t > 0 \tag{3.28}$$

Now to prove that inequality (3.19) holds.

$$\begin{aligned} N&\left(T(x) - g(x), \frac{12-\alpha}{9}t\right) \\ &\geq \min\left\{N(T(x) - f(x), t), N\left(f(x) - g(x), \frac{3-\alpha}{9}t\right)\right\} \\ &\geq \min\left\{M_1\left(x, \frac{3-\alpha}{3}t\right), N\left(f(x) - g(x), \frac{3-\alpha}{9}t\right)\right\} \\ &\geq M_1\left(x, \frac{3-\alpha}{3}t\right) \end{aligned}$$

$$N(T(x) - g(x), t) \geq M_1\left(x, \frac{9-3|\alpha|}{12-|\alpha|}t\right)$$

Thus, similarly inequality (3.20) and (3.21) for h and j also holds. □

Corollary 3.9. *Let X be a Banach space and let $\epsilon > 0$ be a real number. Consider $f, g, h, j : X \to X$ be mappings satisfying*

$$\|f(x+y+z) - g(x) - h(y) - j(z)\| \leq \varepsilon, \tag{3.29}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $T : X \to X$ which satisfies (0.2) such that

$$\|T(x) - f(x)\| \leq 3\varepsilon, \tag{3.30}$$
$$\|T(x) - g(x)\| \leq 4\varepsilon, \tag{3.31}$$
$$\|T(x) - h(x)\| \leq 4\varepsilon, \tag{3.32}$$
$$\|T(x) - j(x)\| \leq 4\varepsilon, \tag{3.33}$$

for all $x \in X$. where $T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n}$

Proof. Let X be a linear space and (X, N) be the fuzzy normed space where norm $N : X \times R \to [0,1]$ is defined by

$$N(x,t) = \begin{cases} 1 - \dfrac{1}{\exp(t/\|x\|)}, & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

for all $x \in X, t \in R$. Let $\phi : X \times X \times X \to Z$ the mapping defined by $(x, y, z) \to \epsilon$. It follows that the inequality (3.2) implies

$$N(f(x+y+z) - g(x) - h(y) - j(z)) \geq N^1(\varepsilon, t),$$

The fuzzy norm on R, is given by $N^1 : R \times R \to [0,1]$ such that

$$N^1(x,t) = \begin{cases} 1 - \dfrac{1}{\exp(t/\|x\|)}, & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Therefore, by Theorem 3.8, there exist a unique mapping $T : X \to X$ which satisfies the inequality (1.2) and the inequality

$$\begin{aligned} 1 - \frac{1}{\exp(t/\|T(x) - f(x)\|)} &= N(T(x) - f(x), t) \\ &\geq N^1\left(\varepsilon, \frac{t}{3}\right) = 1 - \frac{1}{\exp(t/3\varepsilon)} \\ &\quad \text{for all } x \in X,\ \epsilon > 0 \text{ and } t > 0 \end{aligned}$$

This implies that

$$\|T(x) - f(x)\| \leq 3\varepsilon$$

Thus, similarly inequality (3.31), (3.32) and (3.33) for g, h and j also holds. Hence proved the desired result. □

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