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FILIPPOV APPROACH IN NECESSARY CONDITIONS OF OPTIMALITY FOR SINGULAR CONTROL PROBLEM

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Abstract: This paper deals with optimal singular stochastic control problem for systems governed by nonlinear stochastic differential equations (SDEs for short) with Lipschitz coefficients, where the control variable has two components, the first being absolutely continuous and the second singular. We apply Ekeland's variational principle to establish necessary conditions for near-optimality. Filippov approach and stable convergence of probability measure are applied to prove our maximum principle.

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1. Introduction

We consider an optimal singular stochastic control problem of nonlinear dynamical systems governed by SDEs with non differentiable coefficients of the

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form

$$\begin{cases} dx_t = f(t, x_t, u_t)\, dt + \sigma(t, x_t)\, dW_t + G_t d\eta_t, \\ x_0 = \xi, \end{cases} \tag{1.1}$$

where $(W_t)_{t\in[0,T]}$ is a standard $d-$dimentional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The expected cost to be minimized over the set of admissible controls has the form

$$J(u, \eta) = \mathbb{E}\left[h(x_T) + \int_0^T \ell(t, x_t, u_t)\, dt + \int_0^T k_t d\eta_t\right], \tag{1.2}$$

where x_T is a solution of (1.1) at the terminal time T which is called diffusion process. An optimal control (u^*, η^*) is an admissible control satisfying

$$J(u^*, \eta^*) = \inf_{(u,\eta)\in\mathbb{U}} \{J(u, \eta)\}. \tag{1.3}$$

The stochastic control problems have been studied by many authors including, [1, 8, 11, 12, 14, 15]. Pontryagin et al., [12] introduced the maximum principle and gave some necessary conditions for the optimal pairs. Hafayed [8] derived the necessary optimality conditions without differentiability assumptions on the drift and diffusion coefficients by using the Filippov approach.

Stochastic control is an important subfield of control theory which deals with the existence of uncertainty in the data. Singular stochastic control problem is an important and challenging class of problems in control theory, it appear in various fields like mathematical finance, problem of optimal consumption etc. Maximum principle for singular controls was considered by Cadenillas et al., [4] and more recently by [2, 3]. The maximum principle for singular control problems with nonsmooth coefficients and uncontrolled diffusion has been considered in [3], where the authors study two cases: the first when the diffusion coefficient is non-degenerate, in which the authors used Krylov's approach, the second case concerned with the degenerate diffusion, where the result is proved by applying Bouleau-Hirsch flow propriety.

The purpose of this paper is to extend the maximum principle introduced by Hafayed [8] to the singular control problems in which the coefficients f and σ are non-differentiable in x. The main contribution of this paper is to derive a necessary conditions of optimality for singular control in the case when the coefficients f and σ are only Lipschitz continuous. We use Filippov approach for Lipschitz coefficients to overcome the nonsmoothness of the coefficients. This approach will allows us to extend the maximum principle for singular control to nonsmooth case. Our approach is based on the approximation of the nonsmooth coefficients by regular ones. Ekeland's variational principle [5] is applied in order to obtain a sequence of near optimality. By using stable convergence of probability measure and Masur's Lemma, we obtain a first-order adjoint process on enlarged probability space.

This paper is organized as follows. In Section 2, we formulate the stochastic singular control problem. The standard maximum principle for optimal singular control is given in Section 3. The Section 4 is devoted to prove the necessary conditions of near-optimal singular controls. Finally, we prove our result in the last section.

2. Problem Formulation and Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a fixed filtered probability space satisfying the usual conditions in which a $d-$dimentional Brownian motion $W_t = (W_t)_{t\in[0,T]}$ is defined. The columns of W_t are denoted by W_t^j for $j = 1, 2, ..., d$. Let T be a fixed strictly positive real number, $\mathbb{A}_1$ is a closed convex subset of $\mathbb{R}^k$ and $\mathbb{A}_2 := ([0,\infty))^m$. $\mathbb{U}_1$ is the class of measurable adapted processes $u : [0,T] \times \Omega \to \mathbb{A}_1$. $\mathbb{U}_2$ is the class of measurable adapted processes $\eta : [0,T] \times \Omega \to \mathbb{A}_2$.

Definition 2.1. An admissible control is a pair (u, η) of measurable $\mathbb{A}_1\times\mathbb{A}_2$-valued, $\mathcal{F}_t-$adapted processes, such that

(1) η is of bounded variation, nondecreasing continuous on the left with right limits and $\eta_0 = 0$.

(2) $\mathbb{E}\left[\sup_{t\in[0,T]} |u_t|^2 + |\eta_T|^2\right] < \infty$.

We denote $\mathbb{U} = \mathbb{U}_1 \times \mathbb{U}_2$, the set of all admissible controls. Since $d\eta_t$ may be singular with respect to Lebesgue measure dt, we call η the singular part of the control and the process u its absolutely continuous part. Throughout this paper, we assume the following

(H1) Let $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{M}_{d\times d}(\mathbb{R})$, $f : [0,T] \times \mathbb{R}^d \times \mathbb{A}_1 \to \mathbb{R}^d$

$$\begin{aligned} |\sigma(t,x) - \sigma(t,y)| + |f(t,x,u) - f(t,y,u)| &\leq c_1 |x-y|, \\ |f(t,x,u)| + |\sigma(t,x)| &\leq c_2 (1+|x|). \end{aligned}$$

(H2) Let $\ell : [0,T] \times \mathbb{R}^d \times \mathbb{A}_1 \to \mathbb{R}$, and $h : \mathbb{R}^d \to \mathbb{R}$ such that ℓ, h are continuously differentiable with respect to x,

(H3) $f(t,x,.) : \mathbb{A}_1 \to \mathbb{R}^d$, $\ell_x(t,x,.) : \mathbb{A}_1 \to \mathbb{R}$ are continuous,

(H4) Let $G : [0,T] \to \mathcal{M}_{d\times m}(\mathbb{R})$ and $k : [0,T] \to \mathbb{R}^d$ such that G is continuous and bounded, and k is continuous, where c_1, c_2 are positive constants.

Assumptions (H1) guarantee the existence and uniqueness of the strong solution of the equation (1.1) such that for any $q \geq 1 : \mathbb{E}\Big(\sup_{t\in[0,T]} |x_t|^q \Big) < \infty$.

We assume that an optimal control (u^*, η^*) exists. The Hamiltonian H associated with the singular stochastic control problem (1.1)-(1.2) is given by

$$H(t, x, u, p_t) := p_t f(t, x, u) + \ell(t, x, u). \tag{2.1}$$

Let us recall Fillipov's set valued map, which will be used in the sequel.

Filippov's Set-Valued Map. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable and bounded function. We associate the following set-valued map called Filippov's regularization of f

$$\Gamma_f(x) := \underset{\lambda(N)=0}{\cap} \ \underset{\delta>0}{\cap} \overline{co} f(B(x, \delta) - N), \tag{2.2}$$

where $\overline{co}(A)$ denotes the closure of the convex hull of A, $B(x, \delta)$ open ball of radius δ centered at x, and $\underset{\lambda(N)=0}{\cap}$ denotes the intersection over all sets N of Lebesgue measure zero. We associate the following ordinary differential equation to f

$$dx(t) = f(x(t))dt, \ t \geq 0, \ x(0) = \xi. \tag{2.3}$$

Definition 2.2. (see [6]) An absolutely continuous solution $t \in [0, +\infty) \mapsto x(t) \in \mathbb{R}^n$ is a Filippov's solution of the ordinary differential equation (2.4) if and only if it is a solution of the following differential inclusion

$$x'(t) \in \Gamma_f(x), t \geq 0, x(0) = \xi. \tag{2.4}$$

Some properties of Filippov approach are obtained in [8] and the references therein.

3. Maximum Principle of Optimality for Singular Control in Standard Form

The purpose of this section is to introduce the maximum principle of optimality for the singular control problem (1.1)-(1.2) in the regular case. In this case we assume:

$$f, \ \sigma \text{ are continuously differentiable with respect to } x. \tag{3.1}$$

To achieve these necessary conditions for optimal singular control, we use the spike variation for (u_t^*, η_t^*) , where the first is the strong perturbation for the continuous component and the second is convex perturbation for the singular part. This method is defined as follows. For $t_0 \in [0,T]$, $v \in \mathbb{A}_1$, $\xi \in \mathbb{A}_2$, let

$$(u_t^\theta, \eta_t^*) = \begin{cases} (v, \eta_t^*) \text{ if } t \in [t_0, t_0+\theta], \\ (u_t^*, \eta_t^*) \text{ otherwise}, \end{cases} \tag{3.2}$$

and

$$(u_t^*, \eta_t^\theta) = (u_t^*, \eta_t^* + \theta(\xi - \eta_t^*)), \tag{3.3}$$

where $\theta > 0$ is sufficiently small. From the optimality we have

$$J(u^*, \eta^*) \le J(u^\theta, \eta^*) \text{ and } J(u^*, \eta^*) \le J(u^*, \eta^\theta). \tag{3.4}$$

Lemma 3.1. *If* $x_t^{(u^\theta,\eta^*)}$, $x_t^{(u^*,\eta^\theta)}$ *and* x_t^* *are the trajectories corresponding to* (u^θ, η^*), (u^*, η^θ) *and* (u^*, η^*) respectively, *then we have the following estimates:*

$$\begin{aligned} \lim_{\theta\to 0} \mathbb{E}\Big(\sup_{t\in[0,T]} \left|x_t^{(u^\theta,\eta^*)} - x_t^*\right|^2\Big) &= 0, \\ \lim_{\theta\to 0} \mathbb{E}\Big(\sup_{t\in[0,T]} \left|x_t^{(u^*,\eta^\theta)} - x_t^*\right|^2\Big) &= 0, \\ \lim_{\theta\to 0} \mathbb{E}\Big(\left|\frac{x_t^{(u^*,\eta^\theta)} - x_t^*}{\theta} - z_t\right|^2\Big) &= 0, \end{aligned}$$

where z_t is given by

$$z_t = \int_0^t f_x(t, x_s^*, u_s^*)\, z_s ds + \int_0^t \sigma_x(s, x_s^*)\, z_s dW_s + \int_0^t G_s d(\eta - \eta^*)_s .$$

The proof of this lemma can be found in ([2]).

Let $\varphi(t,s)$, $t \ge s$ the fundamental solution of the linearized equation

$$\begin{cases} d\varphi(t) = f_x(t, x_t^*, u_t^*)\, \varphi(t)\, dt + \sum_{j=1}^{d} \sigma_x^j(t, x_t^*)\, \varphi(t)\, dW_t^j, \\ \varphi(s,s) = I_d. \end{cases} \tag{3.5}$$

Under the conditions *(3.1)* on the initial data, the standard version of maximum principle for optimal singular control can be formulated as follows.

Lemma 3.2. *Let (u_t^*, η_t^*) be an optimal control. Then there exists an $\mathcal{F}_t$ adapted process $p_t : [0,T] \to \mathbb{R}^d$ satisfying*

$$p_t = -\mathbb{E}\left\{ \int_t^T \varphi^\tau(s,t)\,\ell_x(s, x_s^*, u_s^*)ds + \varphi^\tau(T,t)\,h_x(x_T^*) \mid_{\mathcal{F}_t} \right\}, \tag{3.6}$$

such that, for any $v \in \mathbb{A}_1$ and any $\eta \in \mathbb{A}_2$

$$\mathbb{E}\left[H(t, x_t^*, u_t^*, p_t)\right] = \max_{v\in\mathbb{A}_1} \mathbb{E}\left[H(t, x_t^*, v, p_t)\right] \; dt - a.e., \tag{3.7}$$

and

$$\mathbb{E}\left[\int_0^T (k_t + G_t^\tau p_t) d(\eta - \eta^*)_t\right] \geq 0, \tag{3.8}$$

where $\varphi^\tau(T,t)$ is the transpose of $\varphi(T,t)$ which is the solution to problem (3.4). We call p_t the adjoint process and (3.7)-(3.8) the maximum conditions.

The proof of the above lemma can be found in ([2], Theorem 3.7).

4. Necessary Conditions of Near-Optimality for Singular Control

In this section, we use Ekeland's variational principle [5] to establish necessary conditions of near optimality for singular control, called $\varepsilon-$optimality, which are satisfied by a sequence of nearly optimal controls. A control $(u^\varepsilon, \eta^\varepsilon)$ is called near-optimal if for every $\varepsilon > 0$ there exists $(u^\varepsilon, \eta^\varepsilon)$ such that

$$J(u^\varepsilon, \eta^\varepsilon) \leq \inf_{(u,\eta)\in\mathbb{U}} \{J(u,\eta)\} + \varepsilon, \tag{4.1}$$

for a sufficiently small $\varepsilon > 0$. Now we introduce Ekeland's variational principle [5].

Lemma 4.1. (Ekeland's Lemma) *Let (F, ρ) be a complete metric space and $f : F \to \overline{\mathbb{R}}$ be a lower semi-continuous function which is bounded below. For a given $\varepsilon > 0$, suppose that $u^\varepsilon \in F$ satisfying $f(u^\varepsilon) \leq \inf(f) + \varepsilon$, then for any $\lambda > 0$, there exists $v \in F$ such that*

(1) $f(v) \leq f(u^\varepsilon)$. (2) $\rho(u^\varepsilon, v) \leq \lambda$. (3) $f(v) < f(w) + \dfrac{\varepsilon}{\lambda}\rho(v,w)$ for all $w \neq v$.

To apply Ekeland's variational principle to our problem, we define a distance function ρ on the space of admissible controls such that $(\mathbb{U}, \rho)$ becomes a complete metric space. To achieve this goal, we define for any (u,η) and $(v,\zeta) \in \mathbb{U}$:

$$\rho((u,\eta),(v,\zeta)) = d_1(u,v) + d_2(\eta,\zeta), \tag{4.2}$$

where

$$d_1(u,v)=\mathbb{P}\otimes dt\left\{(w,t)\in\Omega\times[0,T]:u(w,t)\neq v(w,t)\right\},$$

and

$$d_2(\eta,\zeta)=\left[\mathbb{E}(\sup_{t\in[0,T]}|\eta_t-\zeta_t|^2)\right]^{\frac{1}{2}},$$

here $\mathbb{P}\otimes dt$ is the product measure of $\mathbb{P}$ with the Lebesgue measure dt on $[0,T]$. It is easy to see that $(\mathbb{U}_2,d_2)$ is a complete metric space. Moreover, it has been shown in Yong *et al.*, [14] that $(\mathbb{U}_1,d_1)$ is a complete metric space. Hence $(\mathbb{U},\rho)$ as a product of two complete metric spaces is a complete metric space.

By using Ekeland's variational principle [5] with $\lambda=\frac{1}{2}$ there exists $(u^\varepsilon,\eta^\varepsilon)$ such that $J(u^\varepsilon,\eta^\varepsilon)\leq J(u,\eta)+\varepsilon^{\frac{1}{2}}\rho((u,\eta),(u^\varepsilon,\eta^\varepsilon))$ for any $(u,\eta)\in\mathbb{U}$. Notice that $(u^\varepsilon,\eta^\varepsilon)$ which is near-optimal for the initial cost J defined in (1.2) is optimal for the new cost J^ε given by

$$J^\varepsilon(u,\eta)=J(u,\eta)+\varepsilon^{\frac{1}{2}}\rho((u^\varepsilon,\eta^\varepsilon),(u,\eta)). \tag{4.3}$$

Let x_t^ε is the unique solution of the equation (1.1) associated with $(u^\varepsilon,\eta^\varepsilon)$.

Let $\varphi^\varepsilon(t,s)\}$, $t\geq s$ be the fundamental solution of the following linear stochastic differential equation

$$\begin{cases} d\varphi^\varepsilon(t,s)=f_x(t,x_t^\varepsilon,u_t^\varepsilon)\varphi^\varepsilon(t,s)\,dt+\sum\limits_{j=1}^{d}\sigma_x^j(t,x_t^\varepsilon)\varphi^\varepsilon(t,s)\,dW_t^j, \\ \varphi^\varepsilon(s,s)=I_d. \end{cases} \tag{4.4}$$

The adjoint process associated with $(u^\varepsilon,\eta^\varepsilon)$ is given by

$$p_t^\varepsilon=-\mathbb{E}\left[\int_t^T\varphi^{\varepsilon,\tau}(s,t)\,\ell_x(s,x_s^\varepsilon,u_s^\varepsilon)\,ds+\varphi^{\varepsilon,\tau}(T,t)\,h_x(x_T^\varepsilon)\,|_{\mathcal{F}_t}\right]. \tag{4.5}$$

Lemma 4.2. (Necessary Conditions for Near-Optimality of Singular Control) *For each* $\varepsilon>0$ *there exists* $(u^\varepsilon,\eta^\varepsilon)$ *and an adapted process* p_t^ε *given by (4.6) such that for all* $v\in\mathbb{A}_1$, $\eta\in\mathbb{A}_2$, *the following variational inequalities hold*

$$\mathbb{E}[H(t,x_t^\varepsilon,u_t^\varepsilon,p_t^\varepsilon)]\geq\mathbb{E}[H(t,x_t^\varepsilon,v,p_t^\varepsilon)]-c_1\varepsilon^{\frac{1}{2}},\ dt-a.e., \tag{4.6}$$

and

$$\mathbb{E}\left[\int_0^T(k_t+G_t^\tau p_t^\varepsilon)d(\eta-\eta^\varepsilon)_t\right]\geq-c_2\varepsilon^{\frac{1}{2}}. \tag{4.7}$$

Proof. Since the control $(u^\varepsilon, \eta^\varepsilon)$ is optimal for the cost $J^\varepsilon(u, \eta)$, we apply the results of the last section to derive the adjoint process p_t^ε and the inequality between Hamiltonians. By using the spike variation for $(u_t^\varepsilon, \eta_t^\varepsilon)$ given by

$$(u_t^{\varepsilon,\theta}, \eta_t^\varepsilon) = \begin{cases} (v, \eta_t^\varepsilon) & \text{if } t \in [t_0, t_0 + \theta], \\ (u_t^\varepsilon, \eta_t^\varepsilon) & \text{otherwise,} \end{cases}$$

and

$$(u_t^\varepsilon, \eta_t^{\varepsilon,\theta}) = (u_t^\varepsilon, \eta_t^\varepsilon + \theta(\xi - \eta_t^\varepsilon)),$$

also using the fact that $(u^\varepsilon, \eta^\varepsilon)$ is optimal for the cost J^ε, we get

$$J^\varepsilon(u^\varepsilon, \eta^\varepsilon) \le J^\varepsilon(u^{\varepsilon,\theta}, \eta^\varepsilon) \text{ and } J^\varepsilon(u^\varepsilon, \eta^\varepsilon) \le J^\varepsilon(u^\varepsilon, \eta^{\varepsilon,\theta}).$$

Then we have

$$\begin{aligned} J(u^\varepsilon, \eta^\varepsilon) &\le J\left(u^{\varepsilon,\theta}, \eta^\varepsilon\right) + \varepsilon^{\frac{1}{2}} d_1\left(u^\varepsilon, u^{\varepsilon,\theta}\right), \\ J(u^\varepsilon, \eta^\varepsilon) &\le J\left(u^\varepsilon, \eta^{\varepsilon,\theta}\right) + \varepsilon^{\frac{1}{2}} d_2\left(\eta^\varepsilon, \eta^{\varepsilon,\theta}\right). \end{aligned} \tag{4.8}$$

By using similar argument as used by [2] for inequalities (**??**), we obtain the maximum conditions (see 4.6 and 4.7). □

5. Main Result

We employ Filippov approach to prove the necessary conditions of optimality for singular control in non-convex control domain for diffusions without assuming any differentiability restriction on the coefficients. In order to achieve this goal, we introduce some new terms and notations generalizing the usual derivatives.

5.1. Sequence of Approximate Singular Control Problem

We assumed that the coefficients f and σ^j satisfy the assumptions (H1) and but they are not differentiable with respect to x. So we will weaken the differentiability assumptions on these coefficients. This method is described briefly as follows.

Let Φ be a non-negative smooth function defined on $\mathbb{R}^d$ with support in the unit ball such that $\int_{\mathbb{R}^d} \Phi(y)\, dy = 1$. We define the following smooth functions by convolution as follows

$$\begin{aligned} f^n(t, x, u) &= n^d \int_{\mathbb{R}^d} f(t, x - y, u)\, \Phi(ny)\, dy. \\ \sigma^{j,n}(t, x) &= n^d \int_{\mathbb{R}^d} \sigma^j(t, x - y)\, \Phi(ny)\, dy. \end{aligned} \tag{5.1}$$

It is well known that f^n and $\sigma^{j,n}$ are Borel measurable, bounded functions which are Lipschitz continuous in x and satisfy

$$\left|f^n(t,x,u) - f(t,x,u)\right| + \left|\sigma^{j,n}(t,x) - \sigma^j(t,x)\right| \leq \frac{C}{n} = \varepsilon_n, \tag{5.2}$$

where $C > 0$ is a constant (independent of t, x and n). (for more details, see [8], or [7]).

Let y_t^n be the solution of the stochastic differential equation

$$\begin{cases} dy_t^n = f^n(t, y_t^n, u_t)\,dt + \sigma^n(t, y_t^n)\,dW_t + G_t d\eta_t, \\ y_0^n = \xi. \end{cases} \tag{5.3}$$

The corresponding cost is given by

$$J^n(u,\eta) = \mathbb{E}\left[h(y_T^n) + \int_0^T \ell(t, y_t^n, u_t)\,dt + \int_0^T k_t d\eta_t\right]. \tag{5.4}$$

We have the following estimates.

Lemma 5.1. *Let x_t and y_t^n are the solutions of (1.1) and (5.3) respectively corresponding to (u,η). Then there exists positive constants K_1 and K_2 such that*

(i) $\mathbb{E}(\sup_{0\leq t\leq T}|x_t - y_t^n|^2) \leq K_1\varepsilon_n^2$. *(ii)* $|J^n(u,\eta) - J(u,\eta)| \leq K_2\varepsilon_n$.

Proof. Since $(x_t - y_t^n)$ and $J^n(u,\eta) - J(u,\eta)$ does not depend on the singular part, this lemma follows easily by using a classical argument of stochastic calculus and the inequality (5.2). □

By Lemma 5.2, there exists a positive constant $\delta_n = K_2\varepsilon_n$ such that

$$J^n(u^*,\eta^*) \leq \inf_{(u,\eta)\in\mathbb{U}} \{J^n(u,\eta)\} + \delta_n.$$

By applying Ekeland's variational principle [5] to J_δ^n and (u^*,η^*) with $\lambda_n = \delta_n^{\frac{1}{2}}$, we get that there exists an admissible control (u^n,η^n) such that

(1) $\rho((u^n,\eta^n),(u^*,\eta^*)) \leq \delta_n^{\frac{1}{2}}$,

(2) $J_\delta^n(u^n,\eta^n) \leq J_\delta^n(u,\eta)$, for any $(u,\eta)\in\mathbb{U}$,

where J_δ^n is given by

$$J_\delta^n(u,\eta) = J^n(u,\eta) + \delta_n^{\frac{1}{2}}\rho((u,\eta),(u^n,\eta^n)). \tag{5.5}$$

The above analysis implies that that (u^n,η^n) is optimal for the problem *(5.3)-(5.5)*.

Let x_t^n be the solution of the SDEs *(1.1) associated with* (u^n, η^n).

Lemma 5.2. *Let x_t^* the solution of (1.1) corresponding to (u^*, η^*). Then*

$$\lim_{n\to+\infty} \mathbb{E}(\sup_{0\leq t\leq T} |x_t^n - x_t^*|^2) = 0. \tag{5.6}$$

Proof. The proof of the above lemma can be found in ([3], Lemma 3.3). □

Let $\varphi^n(t,s), t \geq s$ be the fundamental solution of the following linear SDE:

$$\begin{cases} d\varphi^n(t,s) = f_x^n(t, x_t^n, u_t^n)\,\varphi^n(t,s)\,dt + \sum_{j=1}^{d} \sigma_x^{j,n}(t, x_t^n)\,\varphi^n(t,s)\,dW_t^j, \\ \varphi_0^n = I_d. \end{cases} \tag{5.7}$$

Since the equation (5.7) is linear with bounded coefficients, it admits a unique strong solution.

The Hamiltonian H_n associated with f^n, σ^n is defined by

$$H_n(t, x, u, p_t) := p_t f^n(t, x_t, u_t) + \ell(t, x_t, u_t). \tag{5.8}$$

Let us recall the following proposition which will be very useful for the rest of the paper.

Corollary 5.1. *There exists an admissible control (u^n, η^n) and a real number $\delta_n = K_2\varepsilon_n$ such that for every $v \in \mathbb{A}_1$ and $\eta \in \mathbb{U}_2$*

$$\mathbb{E}\left[H_n(t, x^n, u^n, p^n)\right] \geq \mathbb{E}\left[H_n(t, x^n, v, p^n)\right] - c_1\delta_n^{\frac{1}{2}},\ dt - a.e. \tag{5.9}$$

and

$$\mathbb{E}\left[\int_0^T (k_t + G_t^\tau p_t^n)\,d(\eta - \eta^n)_t\right] \geq -c_2\delta_n^{\frac{1}{2}}, \tag{5.10}$$

where the associated adjoint process is given by

$$p_t^n = -\mathbb{E}\left\{\int_t^T \varphi^{n,\tau}(s,t)\,\ell_x(s, x_s^n, u_s^n)\,ds + \varphi^{n,\tau}(T,t)\,h_x(x_T^n)\,|_{\mathcal{F}_t}\right\}. \tag{5.11}$$

Proof. We proceed by similar way of Lemma 4.2 to derive the necessary conditions for the control (u^n, η^n).

5.2. Weak Convergence in Enlarged Space

The weak limit of φ^n is proved by using the stable convergence in an enlarged space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{\mathbb{P}})$ introduced by Jacod [9].

Let us first introduce the notion of stable convergence and good extension of a filtered probability space defined in [9].

Definition 5.1. (Stable Convergence) *The sequence $\overline{\mathbb{P}}_n$ converges with respect to stable convergence to a limit $\overline{\mathbb{P}}$ if and anly if:* $\lim_{n\to\infty} \overline{\mathbb{P}}_n(\Psi(w,\widetilde{w})) = \overline{\mathbb{P}}(\Psi(w,\widetilde{w}))$, *for every function $\Psi : \overline{\Omega} \to \mathbb{R}$ which is measurable, bounded such that $\Psi(w,.)$ is continuous $\forall w \in \Omega$.*

We now introduce the enlarged space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{\mathbb{P}}_n)$ as follows, let $\overline{\Omega} = \Omega \times \Omega_1 \times \Omega_2 \times \Omega_3 \times \Omega_4$, $\overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}^1 \otimes \mathcal{F}^2 \otimes \mathcal{F}^3 \otimes \mathcal{F}^4$ and $\overline{\mathcal{F}}_t = \underset{s\geq t}{\cap} \mathcal{F}_s \otimes \mathcal{F}_s^1 \otimes \mathcal{F}_s^2 \otimes \mathcal{F}_s^3 \otimes \mathcal{F}_s^4$, where $(\Omega_1, \mathcal{F}^1, \mathcal{F}_t^1)$, $(\Omega_2, \mathcal{F}^2, \mathcal{F}_t^2)$, $(\Omega_3, \mathcal{F}^3, \mathcal{F}_t^3)$, $(\Omega_4, \mathcal{F}^4, \mathcal{F}_t^4)$ are the canonical spaces associated with the processes $f_x^n(t, x^n, u^n)$, $\sigma_x^{j,n}(t, x^n)$, $\sigma_x^{j,n}(t,x)^\tau \sigma_x^{j,n}(t,x)$ and φ_n. (For more details, see [8]). The randomized probability measure $\overline{\mathbb{P}}_n$ defined on $(\overline{\Omega}, \overline{\mathcal{F}})$ is given by

$$\overline{\mathbb{P}}_n(w, w_1, w_2, w_3, w_4) = \mathbb{P}(w)\, \delta_{f_x^n}(dw_1)\, \delta_{\sigma_x^n}(dw_2)\, \delta_{a_n}(dw_3)\, \delta_{\varphi_n}(dw_4), \quad (5.12)$$

where δ_x is the *Dirac measure* at x and φ_n is the solution of the equation *(5.7)*.

5.3. Maximum Principle for Singular Control without Regular Assumptions

In this section we derive a general stochastic maximum principle for optimal singular stochastic control using the Filippov approach for Lipschitz coefficients. This approach allows us to extend the result of Hafayed [8] to the singular case where the coefficients f and σ are only Lipschitz continuous with a linear growth conditions and the control domain is not necessarily convex.

Lemma 5.3. (Compactness Criteria) *(1) Let φ_n be the solution of (5.7), then there exists a positive constant c such that for $s, t \in [0, T]$:*

$$\mathbb{E}\left(\| \varphi_n(t) - \varphi_n(s) \|^4\right) \leq c \mid t - s \mid^2 . \quad (5.13)$$

(2) The sequence $\overline{\mathbb{P}}_n$ is relatively compact with respect to the topology of stable convergence.

The proof of the above lemma can be found in ([8], Lemma 5.5).

Now we show the connection between the support of $\overline{\mathbb{P}}$ and the Filippov approach of f, σ^j at x. Let S_w^1, and $S_w^{j;2}$ are the sets of the limit points of the subsequences of $f_x^n(t, x_t^n(w), u_t^n(w))$ and $\sigma_x^{j,n}(t, x_t^n(w))$ respectively, where $w \in \overline{\Omega}$ is fixed.

Theorem 5.1. *(1) There exists subsequences (we do not reindex)* $f_x^n(t, x_t^n(w), u_t^n(w))$ *and* $\sigma_x^{j,n}(t, x_t^n(w))$ *which converge weakly in* $\mathbb{L}^1(dt)$ *to* $\alpha_1(t)$ *and* $\alpha_2^j(t)$ respectively.

(2) For almost every $t \in [0,T]$, *we have* $\alpha_1(t) \in \Gamma_{\partial f}(t, x_t^*, u_t^*)$ and $\alpha_2^j(t) \in \Gamma_{\partial \sigma^j}(t, x_t^*)$.

Proof. The proof is similar to the proof of Theorem 5.8 of Hafayed [8]. □

Let $\overline{\mathbb{E}}_n$ and $\overline{\mathbb{E}}$ are the expectations with respect to the randomized probability $\overline{\mathbb{P}}_n$ and $\overline{\mathbb{P}}$ respectively. Now we state our next result.

Theorem 5.2. *Let* $\overline{\mathbb{P}}$ *be the limit of* $\overline{\mathbb{P}}_n$ *(in the sense of stable convergence), then* $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{\mathbb{P}})$ *is a good extension of the space* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. *Moreover the canonical process* φ_t *satisfies*

$$\begin{cases} d\varphi(t) = \alpha_1(t)\varphi(t)\,dt + \sum_{j=1}^{d} \alpha_2^j(t)\varphi(t)\,dW_t^j + \sum_{j=1}^{d} \widehat{\alpha}_2^j(t)\varphi(t)\,d\widehat{W}_t^j \\ \varphi_0 = I_d, \end{cases} \tag{5.14}$$

where $\widehat{W}_t$ is a Brownian motion which is independent of W_t.

Proof. We need the same techniques used in [11] or [8] in order to prove this result. Hence it is only sufficient to prove that all $(\mathcal{F}_t - \mathbb{P})$ martingale is $(\widetilde{\mathcal{F}}_t - \widetilde{\mathbb{P}})$ martingale, which is true (See Hafayed [8], Theorem 5.4 & Theorem 5.6). □

Theorem 5.3. (Nonsmooth Maximum Principle for Optimal Singular Control) *Let* (u_t^*, η_t^*) *be an optimal control for the problem (1.1)- (1.2). Then there exists an* $\overline{\mathcal{F}}_t$*−adapted process* $\overline{p}_t$ *over the enlarged space* $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{\mathbb{P}})$ *which is the solution of the following equation*

$$\overline{p}_t = -\overline{\mathbb{E}}\left[\varphi^\tau(T,t)\,h_x(x_T^*) + \int_0^T \varphi^\tau(s,t)\,\ell_x(s, x_s^*, u_s^*)\,ds \mid_{\overline{\mathcal{F}}_t}\right], \tag{5.15}$$

such that, for any $v \in \mathbb{A}_1$ and any $\eta \in \mathbb{U}_2$

$$\overline{\mathbb{E}}\left[H(t, x_t^*, u_t^*, \overline{p}_t)\right] = \max_{v \in \mathbb{A}_1} \overline{\mathbb{E}}\left[H(t, x_t^*, v, \overline{p}_t)\right],\ dt - a.e., \tag{5.16}$$

and

$$\overline{\mathbb{E}}\left[\int_0^T (k_t + G_t^\tau \overline{p}_t) d\left(\eta - \eta^*\right)_t\right] \geq 0. \tag{5.17}$$

Proof. According to Corollary 5.1, there exists a control (u^n, η^n) defined on the enlarged space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{\mathbb{P}}_n)$ such that $\rho\left((u^n, \eta^n), (u^*, \eta^*)\right) \leq \delta_n^{\frac{1}{2}}$, where $\lim_{n\to+\infty} \delta_n = 0$. We obtain our maximum principle by passing the weak limit. Since k_t, G_t and ℓ_x are continuous (Assumptions (H3) and (H4)), hence it is only sufficient to prove that $\mathbb{E}\left[H_n\left(t, x_t^n, u_t^n, p_t^n\right)\right]$ converges weakly to $\mathbb{E}\left[H(t, x_t^*, u_t^*, \overline{p}_t)\right]$ as $n \to +\infty$, which is true.

Remark 5.1. *Our result may be interpreted as a nonsmooth version of the regular one. Moreover, if we take $G_t = k_t = \ell = 0$, then we can recover the stochastic maximum principle introduced in Hafayed* [8].

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