International Journal of Pure and Applied Mathematics
Volume 77 No. 5 2012, 709-718
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

VARIATIONAL PRINCIPLES ON CONE METRIC SPACES

Seong-Hoon Cho[1 §], Jong-Sook Bae[2]

[1]Department of Mathematics
Hanseo University
Chungnam, 356-706, SOUTH KOREA

[2]Department of Mathematics
Moyngji University
Yongin, 449-728, SOUTH KOREA

Abstract: The aim of this paper is to give an extension of Ekeland's variational principle to cone metric spaces. We give some quivalences of this variational principle.

AMS Subject Classification: 47H10, 54H25
Key Words: variational principle, fixed point, cone metric space

1. Introduction and Preliminaries

In [8, 9], Ekeland proved a result which is one of the most important results obtained in nonlinear analysis. This result is useful tools to solve problems in optiminization, optimal control theory, game theory nonlinear equations and dynamical systems [4, 5, 7, 9, 10, 19]. It was known that the Petal theorem, Daneš drop theorem, Krasnosel"skii-Zabrjeko and Caristi fixed point theorem are equivalent to Ekeland variational principle (see [17] and references therein). These rwsults proved around the same time but independatly of each other. Since the discovery of Ekeland variational principle, there have appeared many extensions and equivalence formulations of Ekeland variational principle (see [1, 15, 16, 20]).

Received: December 12, 2011

© 2012 Academic Publications, Ltd.
url: www.acadpubl.eu

[§]Correspondence author

Recently, in [11], Huang and Zhang introduced the notion of cone metric spaces as a generalization of metric spaces. They introduced the concept of convergence in cone metric spaces and obtained some fixed point theorems for contractive mappings idefined on cone metric spaces.

In this paper, we obtain an extension of Ekeland's variational principle to the case cone metric spaces. We present Caristi's fixed point theorem, Takahashi's minimization theorem and Equilibrium version of Ekeland's variational principle in the setting of complete cone metric spaces. And then we prove that these results and Ekeland variational principle are equivalent.

Consistent with Huang and Zhang [11], the following definitions will be needed in the sequel.

Let E be a topological vector space. A subset P of E is a *cone* if the following conditions are satisfied:
(i) P is nonempty closed and $P \neq \{0\}$,
(ii) $ax + by \in P$, whenever $x, y \in P$ and $a, b \in \mathbb{R}(a, b \geq 0)$,
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$.

For $x, y \in E$, $x \ll y$ stand for $y - x \in int(P)$, where $int(P)$ is the interior of P. A cone P is called *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{u_n\}$ is a sequence such that for some $z \in E$

$$u_1 \leq u_2 \leq \cdots \leq z,$$

then there exists $u \in E$ such that

$$\lim_{n\to\infty} u_n = u.$$

Equivalently, a cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

A cone P is *complete* if every upper bounded non-empty subset A of E, $\sup A$ exists in E. Equivalently, A cone P is complete if every lower bounded nonempty subset A of E, $\inf A$ exists in E.

A complete cone P is *continuous* if, for any bounded chain $\{x_\alpha : \alpha \in \Gamma\}$, $\inf\{\|x_\alpha - \inf\{x_\alpha : \alpha \in \Gamma\}\| : \alpha \in \Gamma\} = 0$ and $\sup\{\|x_\alpha - \sup\{x_\alpha : \alpha \in \Gamma\}\| : \alpha \in \Gamma\} = 0$.

Note that if P is regular, complete and continuous, then for every non-increasing(resp. non-decreasing) sequence $\{x_n\} \subset P$ bounded from below(resp. above), we have $\lim_{n\to\infty} x_n = \inf_n x_n$(resp. $\lim_{n\to\infty} x_n = \sup_n x_n$).

If E is a normed space, a cone P is called *normal* whenever there exists a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

It is well known that every regular cone in a normed space is normal(see [11]).

The authors [2, 3, 11, 12, 13, 18, 21, 23] obtained fixed point theorems on cone metric spaces under assumpution that the cone is normal. Also, the auothors [6, 14, 22] proved fixed point results under assumpution that the cone is regular.

In this paper, we use the concept of regularity to obtain our results.

Without special mention, we assume that E is a normed space, P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

For a non-empty set X, a mapping $d : X \times X \to E$ is called *cone metric* on X if the following conditions are satisfied:
(i) $0 \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A sequence $\{x_n\}$ of points in a cone metric space (X, d) *converges* [11] to a point $x \in X$(denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$) if for any $c \in int(P)$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \ll c$. A sequence $\{x_n\}$ of points in a cone metric space (X, d) is *Cauchy* [11] if for any $c \in int(P)$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$, $d(x_n, x_m) \ll c$. A cone metric space (X, d) is called *complete* if every Cauchy sequence is convergent.

Note that if $\lim_{n\to\infty} d(x_n, x) = 0$, then $\lim_{n\to\infty} x_n = x$. The converse is true if E is a normed space and P is a normal cone. Also, note that if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, then $\{x_n\}$ is a Cauchy sequence in X. If E is a normed space and P is a normal cone, then $\{x_n\}$ is a Cauchy sequence in X if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

Let (X, d) be a cone metric space with a partial ordering $\lhd$.

A sequence $\{x_n\}$ of points in X is called *decreasing* if $x_{n+1} \lhd x_n$ for all $n \geq 0$. The set $S(x) = \{y \in X : y \lhd x\}$ is $\lhd$-*complete* if every decreasing Cauchy sequence in $S(x)$ converges in it.

Let (X, d) be a cone metric space, and let $\phi : X \to E$ be a mapping. We define an ordering $\preceq$ on X as

$$y \preceq x \text{ if and only if } d(x,y) \le \phi(x) - \phi(y). \tag{1.1}$$

Note that $\preceq$ is a partial ordering, and $y \preceq x$ if and only if $\phi(x) - \phi(y) - d(x,y) \in P$.

A function $f : X \to E$ is called *lower semi-continuous from above* if, for every sequence $\{x_n\} \subset X$ conversing to some point $x \in X$ and satisfying $fx_{n+1} \le fx_n$ for all $n \in \mathbb{N}$, we have $fx \le \lim_{n\to\infty} fx_n$.

2. Main Results

We start with the following theorem which is an extesion of Brezis-Browder theorem to a normed space.

Theorem 2.1. *Let $P \subset E$ be a complete and continuous cone, and let $(X, \preceq)$ be a preodered set. Suppose that a mapping $\psi : X \to E$ is satisfying the followig conditions:*
(1) $x \preceq y$ and $x \neq y$ imply $\psi(x) < \psi(y)$;
(2) for every $\preceq$-decreasing sequence $\{x_n\} \subset X$, there exists $y \in X$ such that $y \preceq x_n$ for all $n \in \mathbb{N}$;
(3) ψ is bounded from below.

Then, for each $x \in X$, $S(x)$ has a minimal element in $S(x)$, where $S(x) = \{y \in X : y \preceq x\}$.

Proof. Suppose that, for some $x_0 \in X$, the result is false.

Then $S(x_0)$ has no minimal element in $S(x_0)$.

Let $\{x_\alpha : \alpha \in \Gamma\}$ be a chain in $S(x_0)$. Then $\{\psi(x_\alpha) : \alpha \in \Gamma\}$ is a chain in E. By (3), $\{\psi(x_\alpha) : \alpha \in \Gamma\}$ is bounded from below, and hence it is bounded. Let $\alpha_1 \in \Gamma$ be fixed. Since P is complete, $r = \inf\{\psi(x_\alpha) : x_\alpha \preceq x_{\alpha_1}\}$ exists in E. Also, we obtain $\inf\{\|\psi(x_\alpha) - r\| : x_\alpha \preceq x_{\alpha_1}\} = 0$, because P is continuous.

Hence we can choose $x_{\alpha_2} \in S(x_{\alpha_1})$ such that

$$\|\psi(x_{\alpha_2}) - \inf\{\psi(x_\alpha) : x_\alpha \preceq x_{\alpha_1}\}\| < \frac{1}{2}.$$

Again, we can choose $x_{\alpha_3} \in S(x_{\alpha_2})$ such that

$$\|\psi(x_{\alpha_3}) - \inf\{\psi(x_\alpha) : x_\alpha \preceq x_{\alpha_2}\}\| < \frac{1}{3}.$$

Inductively, we obtain a sequence $\{x_{\alpha_n}\}$ in X such that

$$x_{\alpha_{n+1}} \in S(x_{\alpha_n})$$

and

$$\|\psi(x_{\alpha_{n+1}}) - \inf\{\psi(x_\alpha) : x_\alpha \preceq x_{\alpha_n}\}\| < \frac{1}{n+1} \tag{2.1}$$

for all $n \in \mathbb{N}$.

Then $x_{\alpha_{n+1}} \preceq x_{\alpha_n}$ for all $n \in \mathbb{N}$. By (2), there exists $y \in X$ such that $y \preceq x_{\alpha_n}$ for all $n \in \mathbb{N}$. Hence

$$\psi(y) \leq \psi(x_{\alpha_n}) \tag{2.2}$$

for all $n \geq 0$.

By assumption, y is not a minimal element in $S(x_0)$. So there exists $u \in X$ such that $u \preceq y$ and $u \neq y$. By (1), we have

$$\psi(u) < \psi(y).$$

Since $u \preceq x_{\alpha_n}$ for all $n \in \mathbb{N}$,

$$r = \inf\{\psi(x_{\alpha_n}) : x_\alpha \preceq x_{\alpha_n}\} \leq \psi(u). \tag{2.3}$$

From (2.1) we have $\lim_{n\to\infty} \psi(x_{\alpha_n}) = r$. From (2.2) and (2.3) we obtain $\psi(y) \leq r \leq \psi(u)$, which is a contradiction. □

Theorem 2.2. *Let (X, d) be a cone metric space such that P is complete and continuous, and let $\phi : X \to E$ be a mapping bounded from below. Suppose that, for each $x \in X$, $S(x) = \{y \in X : y \preceq x\}$ is $\preceq$-complete, where $\preceq$ is a partial ordering on X defined as* (1.1)*.*

Then, for each $x_0 \in X$, there exists $\overline{x} \in X$ such that
(a) $\phi(x_0) - \phi(\overline{x}) - d(x_0, \overline{x}) \in P$,
(b) $\phi(\overline{x}) - \phi(x) - d(\overline{x}, x) \notin P$ for all $x \in X$ with $x \neq \overline{x}$.

Proof. We define a partial ordering $\preceq$ on X as (1.1).

If $x \preceq y$ and $x \neq y$, then $0 < d(y, x) \leq \phi(y) - \phi(x)$, and so $\phi(x) < \phi(y)$.

Let $\{x_n\}$ be a decreasing sequence in X. Then $x_{n+1} \in S(x_n)$ for all $n \geq 0$, and $\{\phi(x_n)\}$ is bounded from below, because ϕ is bounded from below. Hence $\{\phi(x_n)\}$ is bounded. Since P is complete, $u = \inf \phi(x_n)$ exists in E. Also, since P is continuous, $\inf\{\|\phi(x_n) - u\| : n \in \mathbb{N}\} = 0$. Hence $\lim_{n\to\infty} \phi(x_n) = u$ and $u \leq \phi(x_n)$ for all $n \geq 0$.

For $m > n$, since $x_m \preceq x_n$, $d(x_n, x_m) \leq \phi(x_n) - \phi(x_m) \leq \phi(x_n) - u$. Hence $\{x_n\}$ is a decreasing Cauchy sequence in $S(x_0)$. Since $S(x_n)$ is $\preceq$-complete and

$x_{n+1} \in S(x_n)$ for all $n \geq 0$, there exists $x \in S(x_n)$ such that $\lim_{n\to\infty} x_n = x$. Thus, $x \preceq x_n$ for all $n \geq 0$.

By Theorem 2.1, $S(x_0)$ has a minimal element $\overline{x}$ in $S(x_0)$. Hence $\overline{x} \preceq x_0$, and hece $d(x_0, \overline{x}) \leq \phi(x_0) - \phi(\overline{x})$. Thus, $\phi(x_0) - \phi(\overline{x}) - d(x_0, \overline{x}) \in P$, and (a) is proved.

We prove (b).

Assume that (b) is false.

Then there exists $x \neq \overline{x}$ such that $\phi(\overline{x}) - \phi(x) - d(\overline{x}, x) \in P$. Then, $d(\overline{x}, x) \leq \phi(\overline{x}) - \phi(x)$. Thus $x \preceq \overline{x}$, which is a contradiction. □

Theorem 2.3. *Let (X, d) be a complete cone metric space such that P is complete and continuous. Suppose that $\phi : X \to E$ is lower semi-continuous from above and bounded from below.*

Then, for each $x_0 \in X$, there exists $\overline{x} \in X$ such that
(a) $\phi(x_0) - \phi(\overline{x}) - d(x_0, \overline{x}) \in P$,
(b) $\phi(\overline{x}) - \phi(x) - d(\overline{x}, x) \notin P$ for all $x \in X$ with $x \neq \overline{x}$.

Proof. We define a partial ordering $\preceq$ on X as (1.1). It suffice to show that, for each $x_0 \in X$, $S(x_0)$ is $\preceq$-complete.

Let $x_0 \in X$ be a fixed, and let $\{x_n\}$ be a decreasing Cauchy sequence in $S(x_0)$. Then it is a decreasing Cauchy sequence in X. Hence, $\phi(x_{n+1}) \leq \phi(x_n)$ for all $n \in \mathbb{N}$. Since X is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since ϕ is lower semi-continuous from above, $\phi(x) \leq \lim_{n\to\infty} \phi(x_n)$. Thus $\phi(x) \leq \phi(x_n)$ for all $n \in \mathbb{N}$. Since $x_m \preceq x_n$ for $m > n$, we obtain

$$d(x_n, x_m) \leq \phi(x_n) - \phi(x_m) \leq \phi(x_n) - \phi(x).$$

Letting $m \to \infty$ in above inequality, we have $d(x_n, x) \leq \phi(x_n) - \phi(x)$ because d is continuous. Thus we have $x \preceq x_n$, and so $x \preceq x_n \preceq x_0$. Hence $x \in S(x_0)$, and hence $S(x_0)$ is $\preceq$-complete. From Theorem 2.1 conclusions are satisfied. □

Theorem 2.4. *Let (X, d) be a complete cone metric space such that P is complete and continuous. Suppose that $\phi : X \to E$ is lower semi-continuous from above and bounded from below.*

Then the following are equialent.

(1) Ekeland variational principle:
For each $x_0 \in X$, there exists $\overline{x} \in X$ such that
$(1a)$ $\phi(x_0) - \phi(\overline{x}) - d(x_0, \overline{x}) \in P$,
$(1b)$ $\phi(\overline{x}) - \phi(x) - d(\overline{x}, x) \notin P$ for all $x \neq \overline{x}$.

(2) Caristi fixed point theorem:

If a mapping $f : X \to X$ is satisfying

$$\phi(x) - \phi(fx) - d(x, fx) \in P \tag{2.4}$$

for all $x \in X$, then f has a fixed point in X.

(3) Takahashi minization theorem:
If, for each $x_* \in X$ with $\inf_{z\in X} \phi(z) \neq \phi(x_*)$ there exists $x \in X$ such that $x \neq x_*$ and $d(x_*, x) \leq \phi(x_*) - \phi(x)$, then there exists $\overline{x} \in X$ such that $\phi(\overline{x}) = \inf_{z\in X} \phi(z)$.

(4) Equilibrium version of Ekeland variational principle:
Suppose that a mapping $F : X \times X \to E$ is satisfying
$(E1)$ *$F(x, z) \leq F(x, y) + F(y, z)$ for all $x, y, z \in X$,*
$(E2)$ *for each $x \in X$, the mapping $F(x, \cdot) : X \to E$ is lower semi-continuous from above and bounded from below.*
Then, for each $x_0 \in X$, there exists $\overline{x} \in X$ such that
$(4a)$ *$F(x_0, \overline{x}) + d(x_0, \overline{x}) \in (-P)$,*
$(4b)$ *$F(\overline{x}, x) + d(\overline{x}, x) \notin (-P)$ for all $x \in X$ with $x \neq \overline{x}$.*

Proof. (1) $\Rightarrow$ (2): Assume that f has no fixed point in X. Then $\overline{x} \neq f\overline{x}$. Thus from $(1b)$ $\phi(\overline{x}) - \phi(f\overline{x}) - d(\overline{x}, f\overline{x}) \notin P$, and from (3.1) we have $\phi(\overline{x}) - \phi(f\overline{x}) - d(\overline{x}, f\overline{x}) \in P$, which is a contradiction. Thus the implication (1) $\Rightarrow$ (2) is proved.

(2) $\Rightarrow$ (3): Define a mapping $f : X \to X$ by $fx = y$ such that $d(x, y) \leq \phi(x) - \phi(y)$. Then (2.4) is satisfied, and hence there exists $\overline{x} \in X$ such that $\overline{x} = f\overline{x}$. By assumpution, for each $x_* \in X$ with $\inf_{z\in X} \phi(z) \neq \phi(x_*)$ there exists $x \in X$ such that $x \neq x_*$ and $d(x_*, x) \leq \phi(x_*) - \phi(x)$. By definition of f, $fx_* = x$. Thus, $x_* \neq fx_*$ whenever $\inf_{z\in X} \phi(z) \neq \phi(x_*)$. Hence $\overline{x} = f\overline{x}$ implies $\phi(\overline{x}) = \inf_{z\in X} \phi(z)$.

(3) $\Rightarrow$ (4): Let $x_0 \in X$ be fixed, and let $\phi(x) = F(x_0, x)$ for all $x \in X$. Since $F(x_0, \cdot)$ is bounded from below and P is complete, $\inf_{z\in X} \phi(z)$ exists in E.
Suppose that $(4b)$ is not satisfied.
Then, for all $x \in X$, there exists $y \in X$ with $x \neq y$ such that $F(x, y) + d(x, y) \in (-P)$. So $x \neq y$ and $F(x, y) + d(x, y) \leq 0$.
From $(E1)$ we have $F(x_0, y) - F(x_0, x) \leq F(x, y)$.
Thus we obtain

$$\phi(y) - \phi(x) + d(x, y)$$

$$=F(x_0, y) - F(x_0, x) + d(x, y)$$
$$\leq F(x, y) + d(x, y)$$
$$\leq 0.$$

Hence
$$x \neq y \text{ and } d(x, y) \leq \phi(x) - \phi(y). \tag{2.5}$$

By (3), there exists $\overline{x} \in X$ such that $\phi(\overline{x}) \leq \phi(z)$ for all $z \in X$.

Substitute x by $\overline{x}$ in (2.5), we obtain that for some $y \in X$, $y \neq \overline{x}$ and $d(\overline{x}, y) \leq \phi(\overline{x}) - \phi(y)$. That is, $0 < d(\overline{x}, y) \leq \phi(\overline{x}) - \phi(y)$. Thus we have that, for some $y \in X$
$$\phi(y) < \phi(\overline{x}),$$
which is a contradiction.

(4) $\Rightarrow$ (1): Let $F(x, y) = \phi(y) - \phi(x)$ for all $x, y \in X$, and let $x_0 \in X$. By (4), there exists $\overline{x} \in X$ such that $\phi(\overline{x}) - \phi(x_0) + d(x_0, \overline{x}) = F(x_0, \overline{x}) + d(x_0, \overline{x}) \in (-P)$. Hence $\phi(x_0) - \phi(\overline{x}) - d(x_0, \overline{x}) \in P$.

Also, for all $x \neq \overline{x}, \phi(x) - \phi(\overline{x}) + d(x, \overline{x}) = F(\overline{x}, x) + d(\overline{x}, x) \notin (-P)$ by $(4b)$. Thus we have $\phi(\overline{x}) - \phi(x) - d(\overline{x}, x) \notin P$ for all $x \neq \overline{x}$. □

Acknowledgments

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(No. 2011-0012118).

References

[1] S. Al-Homidan, Q.H. Ansaria, J.C. Yao, Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory, *Nonlinear Analysis* **69** (2008), 126-139.

[2] M. Abbas, G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl.* **341**(2008), 416-420.

[3] M. Abbas, B. E. Rhoades, Fixed and periodic point results in cone metric spaces, *Applied Mathematics Letters* **22**(4), 511-515.

[4] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhauser, Boston, Basel, Berlin, 1990.

[5] M. Bianchi, G. Kassay, R. Pini, Existence of equilibria via Ekelands principle, *J. Math. Anal. Apl.* **305** (2005), 502-512.

[6] S.H. Cho, J.S. Bae, Common fixed point theorems for mappings satisfying property $(E.A)$ on cone metric spaces, *Mathematical and Computer Modelling* **53** (2011), 945-951.

[7] D.G. De Figueiredo, The Ekeland Variational Principle with Applications and Detours, Tata Institute of Fundamental Research, Bombay, 1989.

[8] I. Ekeland, Sur les problems variationnels, *C. R. Acad. Sci. Par* **257** (1972), 1057-1059.

[9] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974), 324-353.

[10] I. Ekeland, On convex minimization problems, *Bull. Amer. Math. Soc.* **1** (3) (1979), 445-474.

[11] L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.* **332**(2) (2007), 1468-1476.

[12] D. Ilić, V. Rakočević, Common fixed points for maps on cone metric spaces, *J. Math. Anal. Appl.* **341** (2008), 876-882.

[13] D. Ilić, V. Rakočević, Quasi-contraction on cone metric spaces, *Applied Mathematics Letters* **22**(5), 728-731.

[14] D. Klim, D. Wardowski, Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces, *Nonlinear Analysis* **71**(2009), 5170-5175.

[15] O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, *Math. Japonica* **44** (2)(1996), 381-391.

[16] L.-J. Lin, W.-S. Du, Ekelends variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces, *J. Math. Anal. Appl.* **323** (2006), 360-370.

[17] J.-P. Penot, The drop theorem, the petal theorem and Ekelands variational principle, *Nonlinear Anal.* **10** (1986), 813-822.

[18] S. Radenović, Common fixed points under contractive conditions in cone metric spaces, *Computers and Math. with Appl.* **58**(2009), 1273-1278.

[19] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, Japan, 2000.

[20] T. Suzuki, Generalized distance and existence theorems in complete metric spaces, *J. Math. Anal. Appl.* **253** (2001), 440-458.

[21] K. Wlodarczyk, R. Plebaniak, C. Obczyński, Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces, *Nonlinear Analysis* **71** (2010), 794-805.

[22] S.K. Yang, J.S. Bae, S.H. Cho, Coincidence and common fixed and periodic point theorems in cone metric spaces, *Computers and Mathematics with Applications* **61** (2011), 170-177.

[23] Z. Zhao, X. Chen, Fixed points of decreasing operators in ordered Banach spaces and applications to nonlinear second order elliptic equations, *Computers and Math. with Appl.* **58** (2009), 1223-1229.