A SPECIAL LINEAR MULTI-STEP METHOD FOR
SPECIAL SECOND-ORDER DIFFERENTIAL EQUATIONS

Z. Eskandari¹, M.Sh. Dahaghin² §
Department of Mathematics
University of Shahrekord
Shahrekord, IRAN

Abstract: In this paper we produce a special general linear multi-step method for special second-order differential equations. We use the one super-future point technique in order to obtain our method with high order and wide region of absolute stability. Finally the numerical results are compared with the method introduced in [3].

AMS Subject Classification: 65L05, 65L06
Key Words: initial value problem, Absolute stability, second-order ordinary differential equations, special multi-step methods, super-future point technique

1. Preliminaries

A general linear $k$-step method for numerical solution of ordinary differential equation

$$y''(x) = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0$$

in the interval $[a, b]$ is given by

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§Correspondence author
\[ \sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \] (1.2)

where \( h \) is step size, \( \alpha_k \neq 0, \alpha_0^2 + \beta_0^2 \neq 0 \) and \( \alpha_j, \beta_j \) are constant. Let introduce the fist and second polynomials, respectively by

\[ \rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \quad \sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j. \] (1.3)

The stability polynomial of the linear multi-step method (1.2) is defined by

\[ \pi(\xi, \bar{h}) = \rho(\xi) - \bar{h}\sigma(\xi) \] (1.4)

where \( \bar{h} = \lambda h^2 \). The roots \( \xi_i \) of the characteristic polynomial (1.4) and \( \bar{h} \) are in general, complex and the region of absolute stability is defined to be the region of the complex \( \bar{h} \)-plane such that the roots of the characteristic equation (1.4) lie within the unit circle whenever \( \bar{h} \) lies in the interior of the region. If we denote the region of absolute stability by \( R \) and boundary by \( \partial R \), then the locus of \( \partial R \) is given by

\[ \bar{h}(\theta) = \rho(e^{i\theta})/\sigma(e^{i\theta}), \quad 0 \leq \theta \leq 2\pi. \] (1.5)

With the linear multi-step method (1.2), for differential equation (1.1), we associate the linear difference operator \( L \) defined by

\[ L[y(x); h] = \sum_{j=0}^{k} [\alpha_j y(x + jh) - h^2 \beta_j y''(x + jh)] \] (1.6)

where \( y(x) \) is an arbitrary continuously differentiable function on any subinterval \([x_0, x]\) of \([a, b]\). If we assume \( y(x) \) has higher derivatives as we require, then by Taylor expansion about the point \( x \), we have

\[ L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_q h^q y^{(q)}(x) + \cdots \] (1.7)

where

\[ C_0 = \sum_{j=0}^{k} \alpha_j, \quad C_1 = \sum_{j=0}^{k} j \alpha_j, \quad C_q = \sum_{j=0}^{k} \frac{j^q}{q!} \alpha_j - \sum_{j=0}^{k} \frac{j^{q-2}}{(q-2)!} \beta_j, \quad q \geq 2 \] (1.8)

**Definition 1.1.** The difference operator (1.7) and the associated linear multi-step method (1.2) are said to be of order \( p \) if in (1.8) \( C_0 = C_1 = C_2 = \cdots = C_{p+1} = 0, \ C_{p+2} \neq 0 \).
2. General Linear Multi-Step Method for Special Second-Order Differential Equations

Implicit multi-step method of order \( k - 1 \) for special second order differential equation (1.1) was introduced by Rama Chandra Rao [3] with the form

\[
\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \beta_k f_{n+k}, \quad \alpha_k = 1. \tag{2.1}
\]

We now introduce a new method for special second-order differential equations (1.1) that has the general form

\[
\sum_{j=0}^{k} \hat{\alpha}_j y_{n+j} = h^2 (\hat{\beta}_k f_{n+k} + \hat{\beta}_{k+1} f_{n+k+1}) \tag{2.2}
\]

where \( \hat{\alpha}_k = 1 \) and the other coefficients are chosen so that (2.2) be of order \( k \). By using the definition 1.1 the coefficients \( \hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_{k-1}, \hat{\beta}_k, \hat{\beta}_{k+1} \) are obtained from

\[
\sum_{j=0}^{k-1} \hat{\alpha}_j = -1, \quad \sum_{j=0}^{k-1} j \hat{\alpha}_j = -k, \\
\sum_{j=0}^{k-1} j^q \hat{\alpha}_j - q(q-1)k^{q-2}\hat{\beta}_k - q(q-1)(k+1)^{q-2}\hat{\beta}_{k+1} = -k^q, \quad q = 2, 3, \ldots, k+1.
\]

The coefficients of \( k \)-step method (2.2) for \( k = 2, 3, \ldots, 9 \) are given in table 2.1. In this method we use the one super-future point technique.

Assume that the solution values \( y_n, y_{n+1}, \ldots, y_{n+k-1} \) are available. The method (2.2) is used to solve differential equation (1.1) by the following stages.

**Stage 1.** Compute \( \bar{y}_{n+k} \) as the solution of

\[
\sum_{j=0}^{k-1} \alpha_j y_{n+j} + \alpha_k \bar{y}_{n+k} = h^2 \beta_k \bar{f}_{n+k} \tag{2.3}
\]

where \( \bar{f}_{n+k} = f(x_{n+k}, \bar{y}_{n+k}) \) and \( \alpha_k = 1 \) and the other coefficients are chosen so that (2.3) be of order \( k - 1 \). The coefficients of \( k \)-step method of class (2.3) are given in table 2.2 for \( k = 2, 3, \ldots, 9 \).

**Stage 2.** Compute \( \bar{y}_{n+k+1} \) as the solution of

\[
\sum_{j=0}^{k-2} \alpha_j y_{n+j+1} + \alpha_{k-1} \bar{y}_{n+k} + \alpha_k \bar{y}_{n+k+1} = h^2 \beta_k \bar{f}_{n+k+1} \tag{2.4}
\]
Table 2.1: Coefficients of (2.2) for \( k = 2, 3, \cdots, 9 \)

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<thead>
<tr>
<th>( k )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>12060</td>
<td>5315580</td>
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Table 2.2: Coefficients of (2.3) for \( k = 2, 3, \cdots, 9 \)

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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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<td>45</td>
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<td>-126</td>
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<td>( d \times \alpha_3 )</td>
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</table>

where \( \bar{f}_{n+k+1} = f(x_{n+k+1}, \bar{y}_{n+k+1}) \).

**Stage 3.** Compute \( y_{n+k} \) by

\[
\sum_{j=0}^{k} \hat{\alpha}_j y_{n+j} = h^2(\hat{\beta}_k f_{n+k} + \hat{\beta}_{k+1} \bar{f}_{n+k+1}).
\]  

(2.5)

**Theorem 2.1.** In order to method introduced in (2.2) we have:
(i) Relation (2.3) is of order $k - 1$.

(ii) Relation (2.2) is of order $k$.

(iii) If the implicit algebra equations defining $\bar{y}_{n+k}$, $\bar{y}_{n+k+1}$ are solved exactly, then the scheme (2.5) is of order $k$.

Proof. Suppose the values $y_n, y_{n+1}, \ldots, y_{n+k-1}$ be exact. From (2.3) we have

$$y(x_{n+k}) - \bar{y}_{n+k} = C_1h^{k+1}y^{(k+1)}(x_{n+k}) + O(h^{k+2})$$  \hspace{1cm} (2.6)

where $C_1$ depends on $\alpha_0, \alpha_1, \cdots, \alpha_{k-1}, \beta_k$. From (2.6) with one super-future point and $y(x_{n+k}) = y_{n+k}$ we have

$$y(x_{n+k+1}) - \bar{y}_{n+k+1} = C_1h^{k+1}y^{(k+1)}(x_{n+k+1}) + O(h^{k+2})$$

$$= C_1h^{k+1}y^{(k+1)}(x_{n+k}) + C_1h^{k+2}y^{(k+2)}(x_{n+k}) + O(h^{k+2}).$$

But since in (2.4) we apply $\bar{y}_{n+k}$ therefore we must add the error of $(y(x_{n+k}) - \bar{y}_{n+k})$ to the above expression. Hence

$$y(x_{n+k+1}) - \bar{y}_{n+k+1} = C_1(1 - \alpha_{k-1})h^{k+1}y^{(k+1)}(x_{n+k}) + O(h^{k+2}).$$  \hspace{1cm} (2.7)

If $C_2h^{k+2}y^{(k+2)}(x_{n+k})+O(h^{k+3})$, where $C_2$ depends on $\hat{\alpha}_0, \hat{\alpha}_1, \cdots, \hat{\alpha}_{k-1}, \hat{\beta}_k, \hat{\beta}_{k+1}$, be the defect of formula (2.2) then by replacing $f(x_{n+k+1}, y(x_{n+k+1}))$ with $f(x_{n+k+1}, \bar{y}_{n+k+1})$ and adding the expression obtained in (2.7) to this error we get

$$y(x_{n+k}) - y_{n+k} = C_2h^{k+2}y^{(k+2)}(x_{n+k})$$

$$- h^2\hat{\beta}_{k+1}[f(x_{n+k+1}, y(x_{n+k+1})) - f(x_{n+k+1}, \bar{y}_{n+k+1})].$$

Also from (2.7) we have

$$f(x_{n+k+1}, y(x_{n+k+1})) - f(x_{n+k+1}, \bar{y}_{n+k+1})$$

$$= \frac{\partial f}{\partial y}(x_{n+k+1}, \eta)(y(x_{n+k+1}) - \bar{y}_{n+k+1})$$

$$= \frac{\partial f}{\partial y}(x_{n+k+1}, \eta)C_1(1 - \alpha_{k-1})h^{k+1}y^{(k+1)}(x_{n+k}) + O(h^{k+2}).$$

where $\eta$ is between $y(x_{n+k+1})$ and $\bar{y}_{n+k+1}$. This yields

$$y(x_{n+k}) - y_{n+k} = h^{k+2}[C_2y^{(k+2)}(x_{n+k})$$

$$- \frac{\partial f}{\partial y}(x_{n+k+1}, \eta)C_1(1 - \alpha_{k-1})h^1\hat{\beta}_{k+1}y^{(k+1)}(x_{n+k}) + O(h^{k+3})$$

which shows that the order of scheme (2.5) is $k$. □
3. Stability Analysis of the New Method

We now examine the stability of our method. If we apply (2.2) to the test problem $y'' = \lambda y$ we get

$$
\sum_{j=0}^{k} C_j(\bar{h}) y_{n+j} = 0 \tag{3.1}
$$

where $\bar{h} = \lambda h^2$, $A = 1 - \bar{h} \beta_k$ and $d_0 = \frac{\alpha \alpha_{k-1}}{A^2}$, $d_j = \frac{\alpha_j \alpha_{k-1}}{A^2} - \frac{\alpha_{j-1}}{A}$, $j = 1, 2, \cdots, k - 1$, $C_k(\bar{h}) = 1 - \bar{h} \hat{\beta}_{k+1}$, $C_j(\bar{h}) = \hat{\alpha}_j - \bar{h} \hat{\beta}_{k+1} d_j$, $j = 0, 1, \cdots, k - 1$.

Therefore, the corresponding characteristic equation of the $k$th order difference equation of the method (2.2) is

$$
\pi(\xi, \bar{h}) = \sum_{j=0}^{k} C_j(\bar{h}) \xi^j = 0. \tag{3.2}
$$

To see the zero-stability of this method, one can easily show that by substituting $\bar{h} = 0$ in (3.2) the resulted characteristic polynomial satisfies the root condition and so the method is zero-stable. For more details see [1], [2]. To obtain the region of absolute stability we use the boundary locus method. We see that (3.2) is of the form

$$
\sum_{j=0}^{3} A_j \bar{h}^j = 0 \tag{3.3}
$$

where the coefficients $A_j$, $j = 0, 1, 2, 3$ of different powers of $\bar{h}$ are functions of $\xi$ and $k$. If we set $\xi = e^{i\theta}$ then Eq. (3.3) gives 3 roots $\bar{h}_i(\theta)$, $i = 1, 2, 3$ which describe the stability domain. The region of absolute stability of the methods (2.1) for $k = 2, 3, \ldots, 7$ and (2.2) for $k = 2, 3, 4, 5$ are shown in figures 3.1 and 3.2 respectively. This figures show that the region of absolute stability of the method (2.2) is better than the method (2.1). This region lies outside the boundary.

4. Numerical Result

Example 4.1. In this section, we compare method given in [3] with our method for $k = 4, 5$ to solve the differential equation

$$
y'' = 2e^x + y, \quad y(0) = 0, \quad y'(0) = -1 \tag{4.1}
$$

in the interval $[0,1]$ with $h = 0.01$. The results are given in table 4.1.
Figure 3.1: The region of absolute stability of method (2.1) for $k = 2, 3, \cdots, 7$.

Figure 3.2: The region of absolute stability of method (2.2) for $k = 2, 3, 4, 5$. 
Table 4.3: The results of example 4.1 for the case $k = 4, 5$.

<table>
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<tr>
<th>$k$</th>
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<th>Error in the new method</th>
<th>Error in method given in [3]</th>
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</thead>
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<td>0.5</td>
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References

