A COMMON FIXED POINT THEOREM FOR
GENERALIZED $\varphi$–TRIPLE ON CONE METRIC SPACES

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Abstract: In the present work, existence of coincidence points and common fixed point theorems for generalized $\varphi$-triple on cone metric spaces are proved. These results extends and generalize the results of F. Sabetghadam and H.P. Masiha [7] and others.

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1. Introduction and Preliminaries

Recently Huang and Zhang [5] have introduced the concept of cone metric spaces and established some fixed point theorems for contractive mappings in these spaces. Subsequently Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [4], [6] and the references mentioned therein). In [4] Di Bari and Vetro have introduced the concept of $\varphi$-map and proved some fixed point theorems generalizing some known results. We define the concept of generalized $\varphi$-triple and prove some results about common fixed points for such mappings. Our results generalize some results of Huang and Zhang [5], Di Bari and Vetro[4], Abbas and Jungck [1] and F. Sabetghadam and H.P. Masiha [7].

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In all that follows, $E$ is a real Banach space and $\theta$ is the zero element of $E$. For the mappings $f, g : X \rightarrow X$, let $C(f, g)$ denotes set of coincidence points of $f, g$ that is $C(f, g) := \{ z \in X : fz = gz \}$.

We recall some definitions of cone metric spaces and some of their properties [5].

**Definition 1.1.** Let $E$ be a real Banach space and $P$ be a subset of $E$. The set $P$ is called a cone if and only if

(a) $P$ is closed, nonempty and $P \neq \{0\}$;
(b) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \implies ax + by \in P$;
(c) $x \in P$ and $-x \in P \implies x = 0$.

**Definition 1.2.** Let $P$ be a cone in a Banach space $E$ define partial ordering $\leq$ with respect to $p$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$ while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of the set $P$. This Cone $P$ is called an order cone.

**Definition 1.3.** Let $E$ be a Banach Space and $P \subset E$ be an order cone. The order cone $P$ is called normal if there exists $K > 0$ such that for all $x, y \in E$,$0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$. The least positive number $K$ satisfying the above inequality is called the normal constant of $P$.

**Definition 1.4.** Let $X$ be a nonempty set of $E$. Suppose that the map $d : X \times X \rightarrow E$ satisfies :

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space. It is obvious that the cone metric spaces generalize metric spaces.

**Definition 1.5.** Let $(X, d)$ be a cone metric space. We say that $\{ x_n \}$ is

(i) a Cauchy sequence if for every $c$ in $E$ with $0 \ll c$, there is $N$ such that for all $n, m > N$, $d(x_n, x_m) \ll c$;
(ii) a convergent sequence if for any $0 \ll c$, there is an $N$ such that for all $n > N$, $d(x_n, x) \ll c$, for some fixed $x$ in $X$. We denote this $x_n \longrightarrow x$ ($n \longrightarrow \infty$). A cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

**Definition 1.6.** Let $f, g : X \rightarrow X$. Then the pair $(f, g)$is said to be(IT)-Commuting at $z \in X$ if $f(g(z)) = g(f(z))$ with $f(z) = g(z)$.

**Definition 1.7.** Let $P$ be an order cone. A non-decreasing function
\( \varphi : P \rightarrow P \) is called a \( \varphi \)-map if:

(i) \( \varphi(\theta) = \theta \) and \( \theta < \varphi(\omega) \) for, \( \omega \in P \setminus \theta \),

(ii) \( \omega \in \text{Int}\,P \) implies \( \omega - \varphi(\omega) \in \text{Int}\,P \),

(iii) \( \lim_{n \to \infty} \varphi^n(\omega) = \theta \) for every \( \omega \in P \setminus \theta \).

**Definition 1.8.** Let \( P \) be a cone and let \( \{\omega_n\} \) be a sequence in \( P \). One say that \( \omega_n \lessdot \theta \) if for every \( \epsilon \in P \) with \( \theta \lessdot \epsilon \) there exists a natural number \( N \) such that \( \omega_n \lessdot \epsilon \) for all \( n \geq N \).

We define the following definitions.

**Definition 1.9.** Let \( P \) be a cone. Let \( F : P \rightarrow P \) be a non-decreasing mapping satisfies the following conditions:

(F1). \( F(\omega) \geq \theta \) for all \( \omega \in p \) and \( F(\omega) = \theta \) if and only if \( \omega = \theta \),

(F2). for every \( \omega_n \in P \), \( \omega_n \lessdot \theta \), if and only if \( F(\omega_n) \lessdot \theta \),

(F3). for every \( \omega_1, \omega_2 \in P \) for every, \( F(\omega_1 + \omega_2) \leq F(\omega_1) + F(\omega_2) \).

**Definition 1.10.** The self-mappings \( f, g, h : X \rightarrow X \) are called generalized \( \varphi - \)triple, if there exists a \( \varphi \)-mapping and a mapping \( F \) satisfying the conditions (F1), (F2) and (F3) such that

\[
F(d(fx, gy)) \leq \varphi(F(d(hx, hy))) \quad \text{for every } x, y \in X.
\]

### 2. Common Fixed Point Theorem

The following theorem extends and generalization of Theorem 2.3 of [7] for \( \varphi \)-triple.

**Theorem 2.1.** The self-mappings \( f, g, h : X \rightarrow X \) are called generalized \( \varphi - \)triple, if there exists a \( \varphi \)-mapping and a mapping \( F \) satisfying the conditions (F1), (F2) and (F3) such that

\[
F(d(fx, gy)) \leq \varphi(F(d(hx, hy))), \quad \text{for all } x, y \in X.
\]  

If \( f(X) \cup g(X) \subset h(X) \) and \( h(X) \) is a complete subspace of \( X \). Then the maps \( f, g \) and \( h \) have a coincidence point \( p \) in \( X \). Moreover if \((f, h)\) and \((g, h)\) are (IT)–Commuting at \( p \), then \( f, g \) and \( h \) have a unique common fixed point.

**Proof.** Suppose \( x_0 \) is an arbitrary point of \( X \). Since \( f(X) \cup g(X) \subset h(X) \), then we can define the sequence \( \{x_n\} \) by

\[
fx_{2n} = hx_{2n+1} \quad \text{and} \quad gx_{2n+1} = hx_{2n+2}.
\]
for all \( n=0,1,2, \ldots \). Applying the contractive conditions and using the functions that \( f, g, h \) and \( \varphi \)-triple, we get

\[
F(d(h_{2n+1}, h_{2n+2})) = F(d(f x_{2n}, g x_{2n+1})) \\
\leq \varphi(F(d(h_{2n}, h_{2n+1})))
\]

(2)

Similarly, \( F(d(h_{2n+2}, h_{2n+3})) = F(d(g x_{2n+1}, f x_{2n+2})) \)

\[
\leq \varphi(F(d(h_{2n+1}, h_{2n+2}))).
\]

(3)

That is \( F(d(h_{2n+2}, h_{2n+3})) \leq \varphi(F(d(h_{2n}, h_{2n+1}))). \) (by (2))

(4)

From (2) and (3) and by induction, we obtain

\[
F(d(h_{2n+1}, h_{2n+2})) \leq \varphi(F(d(h_{2n}, h_{2n+1}))) \leq \varphi^2(F(d(h_{2n-1}, h_{2n})))
\]

\[
\leq \ldots \ldots \leq \varphi^{2n+1}(F(d(h_{x0}, h_{x1}))).
\]

(5)

And \( F(d(h_{2n+1}, h_{2n+2})) \leq \varphi^{2n+2}(F(d(h_{x0}, h_{x1}))). \)

(6)

Fix \( \theta \ll \epsilon \) and we choose a positive real number \( \delta \) such that \( \epsilon - \varphi(\epsilon) + I(\theta, \delta) \subset IntP \), where \( I(\theta, \delta) = \{y \in E : \|y\| < \delta\} \).

Since \( \varphi(\omega) < \omega \) for all \( \omega \in P - \{\theta\} \). Let we choose a natural number \( N \) such that \( \varphi^m(F(d(h_{x0}, h_{x1}))) \in I(\theta, \delta) \) for all \( m \geq N \), then

\[
\varphi^m(F(d(h_{x0}, h_{x1}))) \ll \epsilon - \varphi(\epsilon) \text{ for all } m \geq N.
\]

Consequently,

\[
F(d(h_{x_m}, h_{x_{m+1}})) \ll \epsilon - \varphi(\epsilon) \text{ for all } m \geq N.
\]

(7)

Fix \( m \geq N \) and we prove

\[
F(d(h_{x_m}, h_{x_{m+1}})) \ll \epsilon \text{ for all } n \geq m.
\]

(8)

We note that (8) holds when \( n = m \). We assume that (8) holds for \( n \geq m \). Now we prove for \( n + 1 \), then we have by the triangle inequality

\[
F(d(h_{x_m}, h_{x_{n+2}}) \leq F(d(h_{x_m}, h_{x_{m+1}}) + F(d(h_{x_{m+1}}, h_{x_{n+2}}))
\]
\begin{align*}
\epsilon - \varphi(\epsilon) + F(d(fx_m, gx_{n+1}) & \leq \epsilon - \varphi(\epsilon) + \varphi(F(d(fx_m, gx_{n+1}))) \\
& \leq \epsilon - \varphi(\epsilon) + \varphi(\epsilon) = \epsilon \text{ (by induction)}
\end{align*}

Therefore \((8)\) \(n = n + 1\). By induction and \((F_2)\), we deduce \((8)\) holds for all \(n \geq m\). Hence \(\{hx_n\}\) is a Cauchy sequence. Since \(h(X)\) is complete, there exists \(q\) in \(h(X)\) such that \(\{hx_n\} \longrightarrow q\) as \(n \longrightarrow \infty\). Consequently, we can find \(p\) in \(X\) such that \(h(p) = q\). We shall show that \(hp = fp = gp\).

Note that \(F(d(hp, fp) = F(d(q, fp))\). Let us estimate \(F(d(hp, fp))\). We have by the triangle inequality and \((F_3)\)

\[F(d(hp, fp) \leq F(d(hp, hx_{2n+1})) + F(d(hx_{2n+1}, hx_{2n+2})) + F(d(hx_{2n+1}, fp)) \]

\[= F(d(q, hx_{2n+1})) + F(d(fp, gx_{2n+1})),\]

for large \(n\).

By the contractive condition, we get

\[F(d(fp, gx_{2n+1}) \leq \varphi(F(d(hp, hx_{2n+1})))) \]

\[= \varphi(F(d(hp, hx_{2n+1}))) \longrightarrow \theta \text{ as } n \longrightarrow \infty.\]

Therefore, for large \(n\), we have \(F(d(hp, fp)) = \theta\) which leads to \(F(d(hp, fp)) = \theta\) and hence

\[hp = q = fp. \quad (9)\]

Similarly we can show that \(hp = q = gp. \quad (10)\)

From \((9)\) and \((10)\) it follows

\[q = hp = fp = gp. \quad (11)\]

Therefore, \(p\) is a coincidence point of \(f, g, h\). Since, \((f, h), (g, h)\) are \(IT\)-commuting at \(p\) and since \(\varphi\)-triple we get

\[F(d(ffp, fp)) = F(d(ffp, gp)) \leq \varphi(F(d(hfp, hp))) = \varphi(F(d(fp, hp))) = \varphi(F(d(ffp, fp))) < F(d(ffp, fp))(\text{since } \varphi - \text{map}),\]
which is a contradiction. Therefore \( ffp = fp, \) \( fp = ffp = fhp = hfp, \)
which implies \( ffp = hfp = fp = q. \) Therefore,
\[
(fp(=q) \text{ is a common fixed point of } f \text{ and } h. \tag{12}
\]
similarly, we get,
\[
gp = ggp = ghp = hgp, \Rightarrow ggp = hgp = gp = q. \tag{13}
\]
Therefore, \( gp = fp(=q) \) is a common fixed point of \( g \) and \( h \) \( \tag{14} \)

In view of (12) and (14) it follows that \( f, g \) and \( h \) have a common fixed point namely \( q. \) The uniqueness of the common fixed point of \( q \) follows equation (1) and \( \phi \)-triple. Indeed, let \( q_1 \) be another common fixed point of \( f, g \) and \( h. \) Consider,
\[
0 \leq d(q, q_1) = F(d(fq, gq_1)) \leq \phi(F((dhq, hq_1))).
\]
\[
= \phi(F((d(q, q_1))) = \theta.
\]
Thus \( q = q_1. \) Therefore \( f, g \) and \( h \) have a unique common fixed point. □

If we let the mapping \( F \) be the identity mapping in Theorem 2.1, then we obtain the following corollary.

**Corollary 2.2.** Let \((X, d)\) be a cone metric space, and let \( f, g, h : X \rightarrow X \) be generalized \( \phi \)-triple. That is there exists a \( \phi \)-map such that
\[
d(fx, gy) \leq \varphi(d(hx, hy)) \text{ for every } x, y \in X. \tag{15}
\]
If \( f(X) \cup g(X) \subset h(X) \) and \( h(X) \) is a complete subspace of \( X. \) Then the maps \( f, g \) and \( h \) have a coincidence point \( p \) in \( X. \) Moreover if \((f, h)\) and \((g, h)\) are \((IT) \)–Commuting at \( p \), then \( f, g \) and \( h \) have a unique common fixed point.

**Remark 2.3.** If we choose the \( \phi \)-mapping defined by \( \varphi(\omega) = k\omega, \) where \( k \in [0, 1) \) is a constant and we take \( g = f \) and \( h = g \) in Theorem 2.1, then generalizes the Theorem 2.1 of [1]. Further if we let \( g = f, h = g, \) and \( g \) is identity map on \( X \) in Theorem 2.1, then we obtain Theorem 1 of [5], that is, the extension of Banach Fixed Point Theorem for metric spaces.
References


