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ONCE MORE ON THE GRACEFULNESS OF THE DIGRAPHS $n - \vec{C}_m$

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Abstract: A digraph $D(V, E)$ is said to be graceful if there exists an injection $f : V(D) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f' : E(D) \to \{1, 2, \cdots, |E|\}$ which is defined by $f'(u, v) = [f(v) - f(u)] \pmod{(|E| + 1)}$ for every directed edge (u, v) is a bijection. Here, f is called a graceful labeling (graceful numbering) of digraph $D(V, E)$, and f' is called the induced edge's graceful labeling of digraph $D(V, E)$. In this paper we discuss the gracefulness of the digraph $n - \vec{C}_m$ and prove the digraph $n - \vec{C}_{19}$ is graceful for even n with more regular labeling than [9].

Key Words: digraph, directed cycles, graceful graph, graceful labeling

1. Introduction

A graph $G(V, E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f' : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f'(u, v) = |f(u) - f(v)|$ for every edge (u, v) is a bijection. Here,

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f is called a graceful labeling (graceful numbering) of G, while f' is called the induced edge's graceful labeling of G. A digraph $D(V,E)$ is said to be graceful if there exists an injection $f : V(D) \to \{0,1,\cdots,|E|\}$such that the induced function $f' : E(D) \to \{1,2,\cdots,|E|\}$ which is defined by $f'(u,v) = [f(v)-f(u)]$ (mod $|E|+1$) for every directed edge (u,v) is a bijection, where $[v]$ (mod n) denotes the least positive residue of v modulo n. And, for any integers $a \leq b$, let $[a,b]$ denote the set of all consecutive integers from a to b. Let $\vec{C}_m$ denote the directed cycle on m vertices, $n-\vec{C}_m$ denote the graph obtained from any n copies of $\vec{C}_m$ which have just one common edge. Digraph $n-\vec{C}_m$ has a potential application in the password and net work. In [3], Ma has showed that $n-\vec{C}_3$ is a graceful graph. Xu has proved that $n-\vec{C}_m$ is a graceful digraph for $m = 4,6,8,10$ and even n in [5]. In [6], Jirimutu put forward a conjecture and a problem as following:

Conjecture 1. For any positive even n and any integer $m \geq 14$, the digraph $n-\vec{C}_m$ is graceful.

Problem 1. For any positive odd n and any odd $m \geq 14$, whether the digraph $n-\vec{C}_m$ is graceful ?

After, Zhao has proved that $n-\vec{C}_m$ is a graceful digraph for $m = 15$ and even n in [7], and for $m = 17$ and even n in [8], and for $m = 19$ and even n in [9], respectively. In this paper we discuss the gracefulness of the digraph $n-\vec{C}_m$ and prove the digraph $n-\vec{C}_{19}$ is graceful for even n with more regular labeling than[9].

Lemma 1. *For any positive integer n, and $m \geq 3$, the necessary condition of the digraph $n-\vec{C}_m$ to be graceful is $nm \equiv 0$ (mod2).*

Lemma 2. *If $nm \equiv 1$(mod2), then the digraph $n-\vec{C}_m$ is not graceful.*

Lemma 3. *If the digraph $n-\vec{C}_m$ is graceful, then$f(v_0) = 0$ and $f(v_{m-1}) = \frac{q+1}{2}$ (v_0 and v_{m-1} are two vertices of common edge).*

2. Main Results

Theorem 1. *For any positive even n, the digraph $n-\vec{C}_{19}$ is graceful.*

Proof. Let $\vec{C}^1_{19}$, $\vec{C}^2_{19}$, $\cdots$, $\vec{C}^n_{19}$ denote the n directed cycles of $n-\vec{C}_{19}$. Two vertices of common edge of $n-\vec{C}_{19}$ are denoted by v_0 and v_{18}, respectively. Other 17 vertices of $n-\vec{C}^i_{19}$ is denoted by v^i_j for $j = [1,17]$ and $i = [1,n]$. For convenience, we put $v^1_0 = v^2_0 = \cdots = v^n_0 = v_0$, $v^1_{18} = v^2_{18} = \cdots = v^n_{18} = v_{18}$, and take subscripts j's modulo 19. Based on such notations, we define the vertex label f of $n-\vec{C}_{19}$ as follows:

$$f(v_0)=0,\ f(v_{18})=9n+1$$

.

$$f(v_j^i)=\begin{cases}(\frac{j-1}{2})n+i & j=1,3,5,i=1,\ 2,\ \cdots,n\\ (2j+\lceil\frac{3}{j}\rceil)n+1-i & j=2,4,i=1,2,\cdots,n\\ 11n+2-i & j=6,i=1,2,\cdots,n\\ 4n+i & j=7,i=1,\ 2,\ \cdots,n\\ 15n+3-i & j=8,i=1,2,\cdots,n\\ (j+4)n+\frac{j-7}{2}+i & j=9,11,i=1,2,\cdots,n\\ 17n+3-i & j=10,i=1,2,\cdots,n\\ 12n+2-i & j=12,i=1,2,\cdots,n\\ 3n+i & j=13,i=1,\ 2,\ \cdots,n\\ 7n+1-i & j=14,i=1,2,\cdots,n\\ 12n+1+i & j=15,i=1,2,\cdots,n\\ 7n+i & j=17,i=1,\ 2,\ \cdots,n\end{cases}$$

$$f(v_j^i)=\begin{cases}10n+1-i & j=16,i=1,\ 2,\ \cdots,\ \frac{n}{2}\\ 18n+3-i & j=16,i=\frac{n}{2}+1,\ \frac{n}{2}+2,\cdots,\ n\end{cases}$$

Firstly, we show that f is an injective mapping from $V(n-\vec{C}_{19})$ into $[0,18n+1]$.

For $j\in[0,18]$, put $S_j=\{\,f(v_j^i)|i\in[1,n]\}$, and set $S_{j,1}=\{\,f(v_j^i)|i\in[1,\frac{n}{2}]\}$, $S_{j,2}=\{f(v_j^i)|,i\in[\frac{n}{2}+1,n]\}$. Then:

$$\begin{aligned}S_0&=\{f(v_0)\}=\{0\},\\ S_1&=\{f(v_1^i)\}=\{1,2,\cdots,n\},\\ S_3&=\{f(v_3^i)\}=\{n+1,n+2,\cdots,2n\},\\ S_5&=\{f(v_5^i)\}=\{2n+1,2n+2,\cdots,3n\},\\ S_{13}&=\{f(v_{13}^i)\}=\{3n+1,3n+2,\cdots,4n\},\\ S_7&=\{f(v_7^i)\}=\{4n+1,4n+2,\cdots,5n\},\\ S_2&=\{f(v_2^i)\}=\{6n,\cdots,5n+2,5n+1\},\\ S_{14}&=\{f(v_{14}^i)\}=\{7n,\cdots,6n+2,6n+1\},\\ S_{17}&=\{f(v_{17}^i)\}=\{7n+1,7n+2,\cdots,8n\},\end{aligned}$$

$$
\begin{aligned}
S_4 &= \{f(v_4^i)\} = \{9n, \cdots, 8n+2, 8n+1\},\\
S_{18} &= \{f(v_{18}^i)\} = \{9n+1\},\\
S_{16,1} &= \{f(v_{16}^i)\} = \{10n, \cdots, 9n+\frac{n}{2}+2, 9n+\frac{n}{2}+1\},\\
S_6 &= \{f(v_6^i)\} = \{11n+1, \cdots, 10n+3, 10n+2\},\\
S_{12} &= \{f(v_{12}^i)\} = \{12n+1, \cdots, 11n+3, 11n+2\},\\
S_{15} &= \{f(v_{15}^i)\} = \{12n+2, 12n+3, \cdots, 13n+1\},\\
S_9 &= \{f(v_9^i)\} = \{13n+2, 13n+3, \cdots, 14n+1\},\\
S_8 &= \{f(v_8^i)\} = \{15n+2, \cdots, 14n+4, 14n+3\},\\
S_{11} &= \{f(v_{11}^i)\} = \{15n+3, 15n+4, \cdots, 16n+2\},\\
S_{10} &= \{f(v_{10}^i)\} = \{17n+2, \cdots, 16n+4, 16n+3\},\\
S_{16,2} &= \{f(v_{16}^i)\} = \{17n+\frac{n}{2}+2, \cdots, 17n+4, 17n+3\}.
\end{aligned}
$$

It is obvious that $S_i \cap S_j = \emptyset$ for $i, j \in [0, 18]$, $i \neq j$, which yields that f is an injection from $V(n - \vec{C}_{19})$ into $[0, 18n+1]$.

Secondly, we show the induced edges labeling f' is a bijection from $E(n - \vec{C}_{19})$ onto $[1, 8n+1]$. Set $[f(v_j^i) - f(v_{j-1}^i)] = f(v_j^i) - f(v_{j-1}^i) \pmod{(18n+2)}$. Denote $B_j = B_{j,1} \cup B_{j,2}$, where

$$
\begin{aligned}
B_{j,1} &= \{[f(v_j^i) - f(v_{j-1}^i)] | j \in [0, 18]; i \in [1, \tfrac{n}{2}]\}\\
B_{j,2} &= \{[f(v_j^i) - f(v_{j-1}^i)] | j \in [0, 18]; i \in [\tfrac{n}{2}+1, n]\},
\end{aligned}
$$

and let $B = \bigcup_{j=1}^{18} B_j$. Then, in order to prove that f' is a bijection, it suffices to show $B = [1, 18n+1]$, or $[1, 18n+1] \subseteq B$ equivalently.

(1) For $j = 1, 18, i \in [1, n]$. We have

$$B_1 \cup B_{18} = \{1, 2, \cdots, 2n\} = [1, 2n].$$

(2) For $j = 10, 14, i \in [1, n]$. We have

$$B_{10} \cup B_{14} = \{2n+1, 2n+2, \cdots, 4n\} = [2n+1, 4n].$$

which and (1) imply $[1, 4n] \subseteq B$.

(3) For $j = 2, i \in [1, n]$; $j = 16, i \in [\frac{n}{2}+1, n]$, and $j = 15, i \in [1, \frac{n}{2}]$. We have

$$B_2 \cup B_{16,2} \cup B_{15,1} = \{4n+1, 4n+2, \cdots, 6n\} = [4n+1, 6n].$$

which and (2) imply $[1, 6n] \subseteq B$.

(4) For $j = 4, i \in [1, n]; j = 15, 6, i \in [\frac{n}{2} + 1, n]$. We have

$$B_4 \cup B_{15,2} \cup B_{6,2} = \{6n + 1, 6n + 2, \cdots, 8n\} = [6n + 1, 8n].$$

which and (3) imply $[1, 8n] \subseteq B$.

(5) For $j = 6, i \in [1, \frac{n}{2}], j = 17, i \in [\frac{n}{2} + 1, n]$. and $j = 0, i \in [1, n]$. We have

$$B_{6,1} \cup B_{17,2} \cup B_0 = \{8n + 1, 8n + 2, \cdots, 9n + 1\} = [8n + 1, 9n + 1].$$

which and (4) imply $[1, 9n + 1] \subseteq B$.

(6) For $j = 13, 8, i \in [1, n]$. We have

$$B_{13} \cup B_8 = \{9n + 2, 9n + 3, \cdots, 11n + 1\} = [9n + 2, 11n + 1].$$

which and (5) imply $[1, 11n + 1] \subseteq B$.

(7) For $j = 5, 7, i \in [1, n]$. We have

$$B_5 \cup B_7 = \{11n + 2, 11n + 3, \cdots, 15n + 1\} = [11n + 2, 13n + 1].$$

which and (6) imply $[1, 13n + 1] \subseteq B$.

(8) For $j = 12, 3, i \in [1, n]$. We have

$$B_{12} \cup B_3 = \{13n + 2, 13n + 3, \cdots, 15n + 1\} = [13n + 2, 15n + 1].$$

which and (7) imply $[1, 15n + 1] \subseteq B$.

(9) For $j = 16, 17, i \in [1, \frac{n}{2}]$, and $j = 11, 9, i \in [1, n]$. We have

$$B_{16,1} \cup B_{17,1} \cup B_{11} \cup B_9 = \{15n + 2, 15n + 3, \cdots, 18n + 1\} \\ = [15n + 2, 18n + 1],$$

which and (8) imply $[1, 18n + 1] \subseteq B$.

So $f^{\prime}$ is a bijection, then $n - \vec{C}_{19}$ is graceful for any positive even n. This completes the proof.

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