ONCE MORE ON THE GRACEFULNESS OF
THE DIGRAPHS $n - \vec{C}_m$

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Abstract: A digraph $D(V, E)$ is said to be graceful if there exists an injection $f : V(D) \rightarrow \{0, 1, \ldots, |E|\}$ such that the induced function $f' : E(D) \rightarrow \{1, 2, \ldots, |E|\}$ which is defined by $f'(u, v) = [f(v) - f(u)] \pmod{|E| + 1}$ for every directed edge $(u, v)$ is a bijection. Here, $f$ is called a graceful labeling (graceful numbering) of digraph $D(V, E)$, and $f'$ is called the induced edge’s graceful labeling of digraph $D(V, E)$. In this paper we discuss the gracefulness of the digraph $n - \vec{C}_m$ and prove the digraph $n - \vec{C}_{19}$ is graceful for even $n$ with more regular labeling than [9].

Key Words: digraph, directed cycles, graceful graph, graceful labeling

1. Introduction

A graph $G(V, E)$ is said to be graceful if there exists an injection $f : V(G) \rightarrow \{0, 1, \ldots, |E|\}$ such that the induced function $f' : E(G) \rightarrow \{1, 2, \ldots, |E|\}$ which is defined by $f'(u, v) = |f(u) - f(v)|$ for every edge $(u, v)$ is a bijection. Here,
$f$ is called a graceful labeling (graceful numbering) of $G$, while $f'$ is called the induced edge’s graceful labeling of $G$. A digraph $D(V,E)$ is said to be graceful if there exists an injection $f: V(D) \to \{0,1,\cdots,|E|\}$ such that the induced function $f': E(D) \to \{1,2,\cdots,|E|\}$ which is defined by $f'(u,v) = [f(v) - f(u)] \pmod{|E| + 1}$ for every directed edge $(u,v)$ is a bijection, where $[v] \pmod{n}$ denotes the least positive residue of $v$ modulo $n$. And, for any integers $a \leq b$, let $[a,b]$ denote the set of all consecutive integers from $a$ to $b$. Let $\bar{C}_m$ denote the directed cycle on $m$ vertices, $n - \bar{C}_m$ denote the graph obtained from any $n$ copies of $\bar{C}_m$ which have just one common edge. Digraph $n - \bar{C}_m$ has a potential application in the password and network. In [3], Ma has showed that $n - \bar{C}_3$ is a graceful graph. Xu has proved that $n - \bar{C}_m$ is a graceful digraph for $m = 4, 6, 8, 10$ and even $n$ in [5]. In [6], Jirimutu put forward a conjecture and a problem as following:

**Conjecture 1.** For any positive even $n$ and any integer $m \geq 14$, the digraph $n - \bar{C}_m$ is graceful.

**Problem 1.** For any positive odd $n$ and any odd $m \geq 14$, whether the digraph $n - \bar{C}_m$ is graceful?

After, Zhao has proved that $n - \bar{C}_m$ is a graceful digraph for $m = 15$ and even $n$ in [7], and for $m = 17$ and even $n$ in [8], and for $m = 19$ and even $n$ in [9], respectively. In this paper we discuss the gracefulness of the digraph $n - \bar{C}_m$ and prove the digraph $n - \bar{C}_{19}$ is graceful for even $n$ with more regular labeling than[9].

**Lemma 1.** For any positive integer $n$, and $m \geq 3$, the necessary condition of the digraph $n - \bar{C}_m$ to be graceful is $nm \equiv 0 \pmod{2}$.

**Lemma 2.** If $nm \equiv 1 \pmod{2}$, then the digraph $n - \bar{C}_m$ is not graceful.

**Lemma 3.** If the digraph $n - \bar{C}_m$ is graceful, then $f(v_0) = 0$ and $f(v_{m-1}) = \frac{q+1}{2}$ ($v_0$ and $v_{m-1}$ are two vertices of common edge).

### 2. Main Results

**Theorem 1.** For any positive even $n$, the digraph $n - \bar{C}_{19}$ is graceful.

**Proof.** Let $\bar{C}_{19}^1, \bar{C}_{19}^2, \cdots, \bar{C}_{19}^n$ denote the $n$ directed cycles of $n - \bar{C}_{19}$. Two vertices of common edge of $n - \bar{C}_{19}$ are denoted by $v_0$ and $v_{18}$, respectively. Other 17 vertices of $n - \bar{C}_{19}$ is denoted by $v_j^i$ for $j = [1, 17]$ and $i = [1, n]$. For convenience, we put $v_0^1 = v_0^2 = \cdots = v_0^n = v_0$, $v_{18}^1 = v_{18}^2 = \cdots = v_{18}^n = v_{18}$, and
take subscripts $j$’s modulo 19. Based on such notations, we define the vertex label $f$ of $n - \overline{C}_{19}$ as follows:

\[
f(v_0) = 0, \ f(v_{18}) = 9n + 1
\]

\[
f(v_j^i) = \begin{cases} 
\frac{j-1}{2}n + i & j = 1, 3, 5, i = 1, 2, \cdots, n \\
(2j + \lfloor \frac{j}{2} \rfloor)n + 1 - i & j = 2, 4, i = 1, 2, \cdots, n \\
11n + 2 - i & j = 6, i = 1, 2, \cdots, n \\
4n + i & j = 7, i = 1, 2, \cdots, n \\
15n + 3 - i & j = 8, i = 1, 2, \cdots, n \\
(j + 4)n + \frac{j-7}{2} + i & j = 9, 11, i = 1, 2, \cdots, n \\
17n + 3 - i & j = 10, i = 1, 2, \cdots, n \\
12n + 2 - i & j = 12, i = 1, 2, \cdots, n \\
3n + i & j = 13, i = 1, 2, \cdots, n \\
7n + 1 - i & j = 14, i = 1, 2, \cdots, n \\
12n + 1 + i & j = 15, i = 1, 2, \cdots, n \\
7n + i & j = 17, i = 1, 2, \cdots, n
\end{cases}
\]

\[
f(v_j^i) = \begin{cases} 
10n + 1 - i & j = 16, i = 1, 2, \cdots, \frac{n}{2} \\
18n + 3 - i & j = 16, i = \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n
\end{cases}
\]

Firstly, we show that $f$ is an injective mapping from $V(n - \overline{C}_{19})$ into $[0, 18n + 1]$.

For $j \in [0, 18]$, put $S_j = \{ f(v_j^i) | i \in [1, n] \}$, and set $S_{j, 1} = \{ f(v_j^i) | i \in [1, \frac{n}{2}] \}, S_{j, 2} = \{ f(v_j^i) | i \in [\frac{n}{2} + 1, n] \}$. Then:

- $S_0 = \{ f(v_0) \} = \{ 0 \}$,
- $S_1 = \{ f(v_1^i) \} = \{ 1, 2, \cdots, n \}$,
- $S_3 = \{ f(v_3^i) \} = \{ n + 1, n + 2, \cdots, 2n \}$,
- $S_5 = \{ f(v_5^i) \} = \{ 2n + 1, 2n + 2, \cdots, 3n \}$,
- $S_{13} = \{ f(v_{13}^i) \} = \{ 3n + 1, 3n + 2, \cdots, 4n \}$,
- $S_7 = \{ f(v_7^i) \} = \{ 4n + 1, 4n + 2, \cdots, 5n \}$,
- $S_2 = \{ f(v_2^i) \} = \{ 6n, \cdots, 5n + 2, 5n + 1 \}$,
- $S_{14} = \{ f(v_{14}^i) \} = \{ 7n, \cdots, 6n + 2, 6n + 1 \}$,
- $S_{17} = \{ f(v_{17}^i) \} = \{ 7n + 1, 7n + 2, \cdots, 8n \}$,
\[ S_4 = \{ f(v^4_i) \} = \{ 9n, \ldots, 8n + 2, 8n + 1 \}, \]
\[ S_{18} = \{ f(v^1_{18}) \} = \{ 9n + 1 \}, \]
\[ S_{16,1} = \{ f(v^i_{16}) \} = \{ 10n, \ldots, 9n + \frac{n}{2} + 2, 9n + \frac{n}{2} + 1 \}, \]
\[ S_6 = \{ f(v^6_i) \} = \{ 11n + 1, \ldots, 10n + 3, 10n + 2 \}, \]
\[ S_{12} = \{ f(v^i_{12}) \} = \{ 12n + 1, \ldots, 11n + 3, 11n + 2 \}, \]
\[ S_{15} = \{ f(v^i_{15}) \} = \{ 12n + 2, 12n + 3, \ldots, 13n + 1 \}, \]
\[ S_9 = \{ f(v^9_i) \} = \{ 13n + 2, 13n + 3, \ldots, 14n + 1 \}, \]
\[ S_8 = \{ f(v^8_i) \} = \{ 15n + 2, \ldots, 14n + 4, 14n + 3 \}, \]
\[ S_{11} = \{ f(v^i_{11}) \} = \{ 15n + 3, 15n + 4, \ldots, 16n + 2 \}, \]
\[ S_{10} = \{ f(v^i_{10}) \} = \{ 17n + 2, \ldots, 16n + 4, 16n + 3 \}, \]
\[ S_{16,2} = \{ f(v^i_{16}) \} = \{ 17n + \frac{n}{2} + 2, \ldots, 17n + 4, 17n + 3 \}. \]

It is obvious that \( S_i \cap S_j = \emptyset \) for \( i, j \in [0, 18] \), \( i \neq j \), which yields that \( f \) is an injection from \( V(n - \tilde{C}_19) \) into \([0, 18n + 1]\).

Secondly, we show the induced edges labeling \( f' \) is a bijection from \( E(n - \tilde{C}_19) \) onto \([1, 8n + 1]\). Set \( [f(v^i_j) - f(v^i_{j-1})] = f(v^i_j) - f(v^i_{j-1}) \pmod{18n + 2} \). Denote \( B_j = B_{j,1} \cup B_{j,2} \), where

\[
\begin{align*}
B_{j,1} &= \{ [f(v^i_j) - f(v^i_{j-1})]|j \in [0, 18]; i \in [1, \frac{n}{2}] \} \\
B_{j,2} &= \{ [f(v^i_j) - f(v^i_{j-1})]|j \in [0, 18]; i \in [\frac{n}{2} + 1, n] \};
\end{align*}
\]

and let \( B = \bigcup_{j=1}^{18} B_j \). Then, in order to prove that \( f' \) is a bijection, it suffices to show \( B = [1, 18n + 1] \), or \([1, 18n + 1] \subseteq B \) equivalently.

(1) For \( j = 1, 18, i \in [1, n] \). We have
\[ B_1 \cup B_{18} = \{ 1, 2, \ldots, 2n \} = [1, 2n]. \]

(2) For \( j = 10, 14, i \in [1, n] \). We have
\[ B_{10} \cup B_{14} = \{ 2n + 1, 2n + 2, \ldots, 4n \} = [2n + 1, 4n]. \]

which and (1) imply \( [1, 4n] \subseteq B \).

(3) For \( j = 2, i \in [1, n]; j = 16, i \in [\frac{n}{2} + 1, n] \), and \( j = 15, i \in [1, \frac{n}{2}] \). We have
\[ B_2 \cup B_{16,2} \cup B_{15,1} = \{ 4n + 1, 4n + 2, \ldots, 6n \} = [4n + 1, 6n]. \]
which and (2) imply $[1, 6n] \subseteq B$.

(4) For $j = 4, i \in [1, n]; j = 15, 6, i \in [\frac{n}{2} + 1, n]$. We have

$$B_4 \cup B_{15,2} \cup B_{6,2} = \{6n + 1, 6n + 2, \cdots, 8n\} = [6n + 1, 8n].$$

which and (3) imply $[1, 8n] \subseteq B$.

(5) For $j = 6, i \in [1, \frac{n}{2}], j = 17, i \in [\frac{n}{2} + 1, n]$, and $j = 0, i \in [1, n]$. We have

$$B_{6,1} \cup B_{17,2} \cup B_0 = \{8n + 1, 8n + 2, \cdots, 9n + 1\} = [8n + 1, 9n + 1].$$

which and (4) imply $[1, 9n + 1] \subseteq B$.

(6) For $j = 13, 8, i \in [1, n]$. We have

$$B_{13} \cup B_8 = \{9n + 2, 9n + 3, \cdots, 11n + 1\} = [9n + 2, 11n + 1].$$

which and (5) imply $[1, 11n + 1] \subseteq B$.

(7) For $j = 5, 7, i \in [1, n]$. We have

$$B_5 \cup B_7 = \{11n + 2, 11n + 3, \cdots, 15n + 1\} = [11n + 2, 13n + 1].$$

which and (6) imply $[1, 13n + 1] \subseteq B$.

(8) For $j = 12, 3, i \in [1, n]$. We have

$$B_{12} \cup B_3 = \{13n + 2, 13n + 3, \cdots, 15n + 1\} = [13n + 2, 15n + 1].$$

which and (7) imply $[1, 15n + 1] \subseteq B$.

(9) For $j = 16, 17, i \in [1, \frac{n}{2}]$, and $j = 11, 9, i \in [1, n]$. We have

$$B_{16,1} \cup B_{17,1} \cup B_{11} \cup B_9 = \{15n + 2, 15n + 3, \cdots, 18n + 1\} = [15n + 2, 18n + 1],$$

which and (8) imply $[1, 18n + 1] \subseteq B$.

So $f'$ is a bijection, then $n - \tilde{C}_{19}$ is graceful for any positive even $n$. This completes the proof.
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