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ON THE STRUCTURE OF THE ZEROES OF (h, q)-GENOCCHI POLYNOMIALS

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Abstract: In this paper we observe the behavior of complex roots of the (h, q)-Genocchi polynomials $G_{n,q}^{(h)}(x)$, using numerical investigation. Finally, we give a table for the solutions of the (h, q)-Genocchi polynomials $G_{n,q}^{(h)}(x)$.

AMS Subject Classification: 11B68, 11S40, 11S80
Key Words: Genocchi numbers and polynomials, (h, q)-Genocchi numbers and polynomials

1. Introduction

It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the (h, q)- Genocchi polynomials $G_{n,q}^{(h)}(x)$. In Section 2, we study (h, q)-Genocchi polynomials $G_{n,q}^{(h)}(x)$. In Section 3, we describe the beautiful zeros of the (h, q)-Genocchi polynomials $G_{n,q}^{(h)}(x)$ using a numerical investigation. We also display distribution and structure of the zeros of the (h, q)-Genocchi polynomials $G_{n,q}^{(h)}(x)$ by using computer.

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Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. First, we introduce the Genocchi numbers and polynomials. The Euler numbers E_n are defined by the generating function:

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \text{ cf. } [1,\ 2,\ 3,\ 4,\ 5] \tag{1.1}$$

where we use the technique method notation by replacing E^n by $G_n (n \geq 0)$ symbolically. We consider the Euler polynomials $G_n(x)$ as follows:

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.2}$$

2. (h, q)-Genocchi Numbers and Polynomials

Let q be a complex number with $|q| < 1$ and $h \in \mathbb{Z}$. The (h, q)-Genocchi numbers $G_{n,q}^{(h)}$ and polynomials $G_{n,q}^{(h)}(x)$, are defined by means of the following generating function:

$$F_q^{(h)}(t) = \frac{[2]_q t}{q^h e^t + 1} = \sum_{n=0}^{\infty} G_{n,q}^{(h)} \frac{t^n}{n!}, \tag{2.1}$$

$$F_q^{(h)}(x, t) = \frac{[2]_q t}{q^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{2.2}$$

Here, $[x]_q$ is a q-extension of x which is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

Note that $\lim_{q \to 1} [x]_q = x$. Setting $q \to 1$, above (2.1) and (2.2) will become the corresponding definitions of the Genocchi numbers G_n and Genocchi numbers $G_n(x)$.

Because

$$\frac{\partial}{\partial x} F_q^{(h)}(x, t) = \sum_{n=0}^{\infty} \frac{d}{dx} G_{n,q}^{(h)}(x) \frac{t^n}{n!},$$

it follows the important relation

$$\frac{d}{dx}G_{n,q}^{(h)}(x) = nG_{n-1,q}^{(h)}(x).$$

Thus we obtain the following theorem.

Theorem 1. *For any positive integer n, we have*

$$\int_a^x G_{n-1,q}^{(h)}(t)dt = \frac{1}{n}(G_{n,q}^{(h)}(x) - G_{n,q}^{(h)}(a)).$$

From (2.2), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(q^h G_{n,q}^{(h)}(x+1) + G_{n,q}^{(h)}(x)\right)\frac{t^n}{n!} \\
&= [2]_q t\sum_{n=0}^{\infty}(-1)^n q^{h(n+1)}e^{(n+1+x)t} + [2]_q t\sum_{n=0}^{\infty}(-1)^n q^{hn}e^{(n+x)t} \qquad (2.3)\\
&= \sum_{n=0}^{\infty}[2]_q n x^{n-1}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{n!}$ on both sides of (2.3), we get the following theorem.

Theorem 2. *For any positive integer n, we have*

$$q^h G_{n,q}^{(h)}(x+1) + G_{n,q}^{(h)}(x) = n[2]_q x^{n-1}.$$

By using (2.2), after some elementary calculations, we have the following theorem.

Theorem 3. *For any positive integer n, we have*

$$G_{n,q^{-1}}^{(h)}(1-x) = (-1)^{n-1}q^{h-1}G_{n,q}^{(h)}(x).$$

Here is the list of the first q-Genocchi polynomials $G_{n,q}(x)$.

$$\begin{aligned}
&G_{0,q}^{(h)}(x) = 0, \quad G_{1,q}^{(h)}(x) = \frac{[2]_q}{1+q^h},\\
&G_{2,q}^{(h)}(x) = -\frac{2[2]_q q^h}{(1+q^h)^2} + \frac{2[2]_q x}{1+q^h},\\
&G_{3,q}^{(h)}(x) = \frac{6[2]_q q^{2h}}{(1+q^h)^3} - \frac{3[2]_q q^h}{(1+q^h)^2} - \frac{6[2]_q q^h x}{(1+q^h)^2} + \frac{3[2]_q x^2}{1+q^h}.
\end{aligned}$$

3. Distribution and Structure of the Zeros

In this section, we investigate the zeros of (h,q)-Genocchi polynomials $G_{n,q}^{(h)}(x)$ by using computer. Let q be a complex number with $0 < q < 1$ and $h \in \mathbb{N}$. We plot the zeros of $G_{n,q}^{(h)}(x), x \in \mathbb{C}$ (see Figure 1).

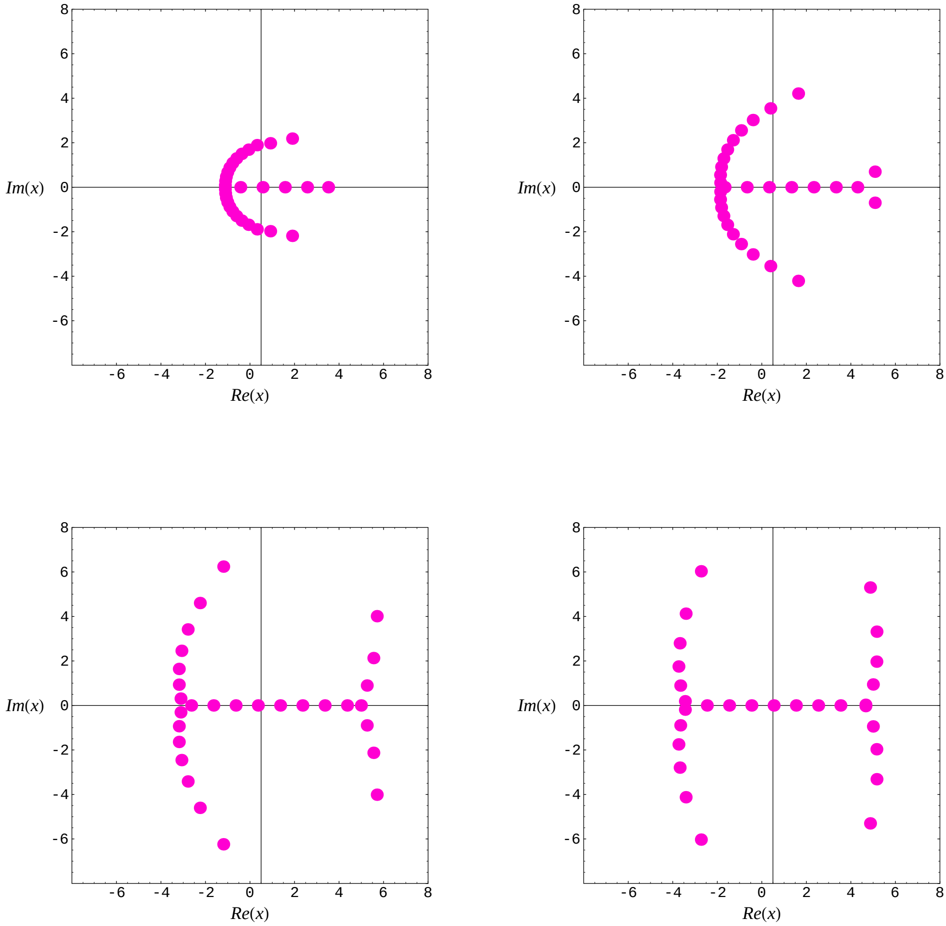

Figure 1: Zeros of $G_{30,q}(x)$

In Figure 1 (top-left), we choose $n = 30, q = 1/10$, and $h = 3$. In Figure 1(top-right), we choose $n = 30, q = 3/10$, and $h = 3$. In Figure 1 (bottom-left), we choose $n = 30, q = 7/10$, and $h = 3$. In Figure 1 (bottom-right), we choose $n = 30, q = 9/10$, and $h = 3$.

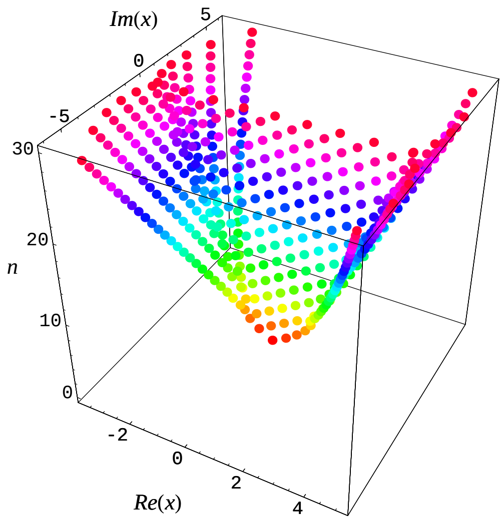

Figure 2: Stacks of zeros $G_{n,q}^{(h)}(x)$ for $1 \le n \le 30$

In Figure 2, we choose $q = 9/10$ and $h = 3$. In Figures 1-2, $G_{n,q}^{(h)}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry. This translates to the following open problem: Prove or disprove: $G_{n,q}^{(h)}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry. Our numerical results for numbers of real and complex zeros of $G_{n,q}^{(h)}(x)$ are displayed in Table 1. In Table 1, we choose $q = 1/10, 9/10$, and $h = 3$.

Table 1. Numbers of real and complex zeros of $G_{n,q}^{(h)}(x)$

degree n	$q = 1/10$		$q = 9/10$	
	real zeros	complex zeros	real zeros	complex zeros
2	1	0	1	0
3	2	0	2	0
4	1	2	3	0
5	2	2	4	0
6	1	4	3	2
7	2	4	2	4
8	1	6	3	4
9	2	6	4	4
10	3	6	5	4
11	2	8	4	6

We observe a remarkably regular structure of the complex roots of (h,q)-Genocchi polynomials $G_{n,q}^{(h)}(x)$. We hope to verify a remarkably regular structure of the complex roots of (h,q)-Genocchi polynomials $G_{n,q}^{(h)}(x)$(Table 1). Next, we calculated an approximate solution satisfying $G_{n,q}^{(h)}(x), x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $G_{n,q}^{(h)}(x) = 0, q = 1/10$

degree n	x
2	0.0009990
3	−0.030592, 0.032590
4	0.110829
5	−0.15026, 0.21557
6	0.33398
7	−0.2354, 0.45990
8	0.59016

Plots of real zeros of $G_{n,q}^{(h)}((x)$ for $1 \leq n \leq 30, q = 1/10, 9/10$ structure are presented (see Figure 3).

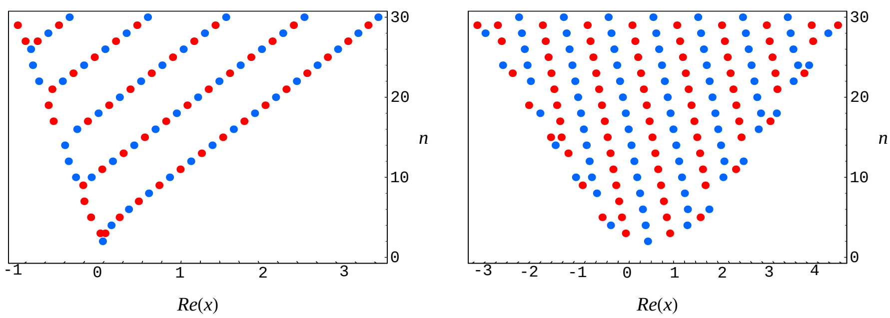

Figure 3: Real zeros of $G_{n,q}^{(h)}((x)$ for $q = 1/10, 9/10, 1 \leq n \leq 30$

We display the plot of real zeros of $G_{n,q}^{(h)}((x)$ and $1 \leq n \leq 30$ (see Figure 4).

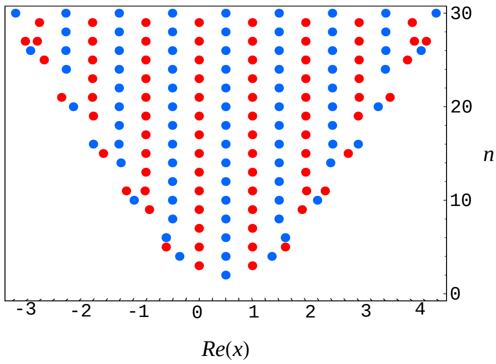

Figure 4: Real zeros of $G_{n,q}^{(h)}((x)$ for $q \to 1$

Observe that the structure of the zeros of the Genocchi polynomials $G_n(x)$ resembles the structure of the zeros of the (h,q)-Genocchi polynomials $G_{n,q}^{(h)}((x)$ as $q \to 1$(see Figures 3, 4). We shall consider the more general open problem. How many roots does $G_{n,q}^{(h)}((x)$ have in general? Prove or disprove: $G_{n,q}^{(h)}((x)$ has $n-1$ distinct solutions. Find the numbers of complex zeros $C_{G_{n,q}^{(h)}((x)}$ of $G_{n,q}^{(h)}((x), Im(x) \neq 0$. Prove or give a counterexample: *Conjecture*: Since $n-1$ is the degree of the polynomial $G_{n,q}^{(h)}((x)$, the number of real zeros $R_{G_{n,q}^{(h)}((x)}$ lying on the real plane $Im(x) = 0$ is then $R_{G_{n,q}(x)} = n - 1 - C_{G_{n,q}(x)}$, where $C_{G_{n,q}^{(h)}((x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{G_{n,q}^{(h)}((x)}$ and $C_{G_{n,q}^{(h)}((x)}$.

The plot shows $G_{n,q}^{(h)}(x)$ for real $-9/10 \leq q \leq 9/10$ and $-5 \leq x \leq 5$, with the zero contour indicated in black (see Figure 5). In Figure 5 (top-left), we choose $n = 2$ and $h = 3$. In Figure 5 (top-right), we choose $n = 3$ and $h = 3$. In Figure 5 (bottom-left), we choose $n = 4$ and $h = 3$. In Figure 5 (bottom-right), we choose $n = 5$ and $h = 3$.

We plot the $G_{n,q}^{(h)}(x)$, respectively (see Figures 1-5). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the $G_{n,q}^{(h)}(x)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the (h,q)-Genocchi polynomials $G_{n,q}^{(h)}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [1], [2], [3], [4], [5].

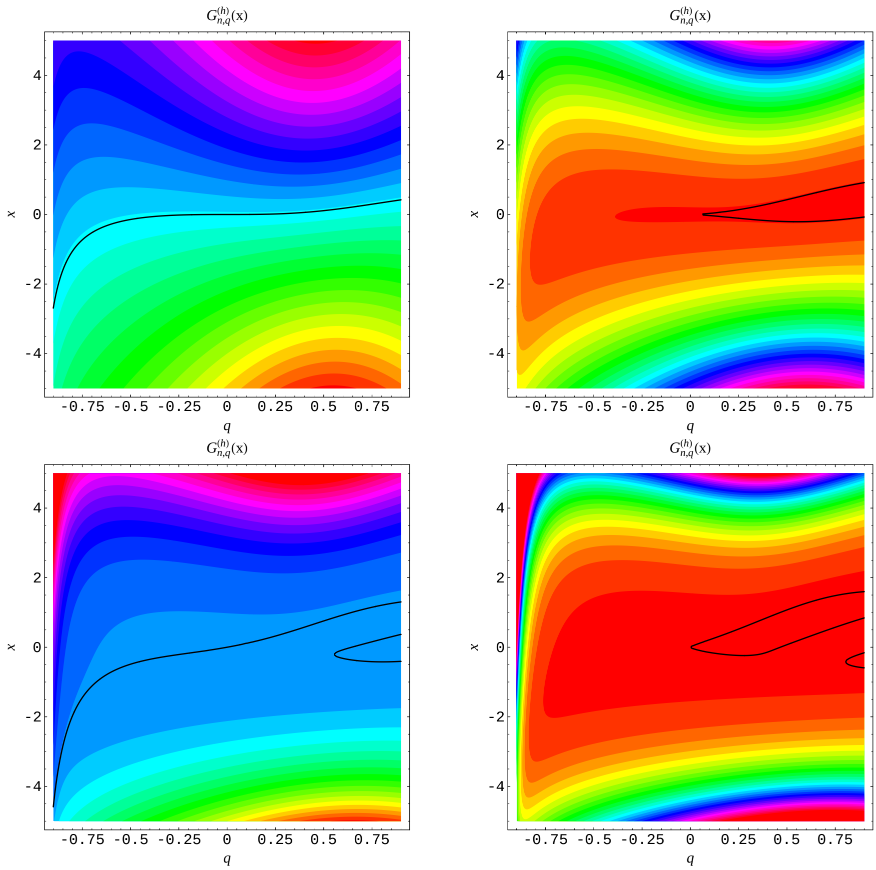

Figure 5: Zero contour of $G_{n,q}^{(h)}(x)$

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