International Journal of Pure and Applied Mathematics
Volume 78 No. 2 2012, 273-287
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

VARIANTS OF THE SHOOTING METHOD FOR COMPUTING THE PARETO SET IN THE CRITERION SPACE FOR THE BICRITERIA LINEAR PROGRAMMING PROBLEM

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Abstract: Based on the fact that the Pareto set of a bicriteria linear programming problem is a polygonal line we propose variants of the shooting method to compute this set.

AMS Subject Classification: 90C05, 90C29, 90C31
Key Words: bicriteria linear programming problem, Pareto set, criterion space, shooting method, normalization, rotation

1. Introduction

In this paper we adress the problem of the computation of the Pareto set in the criterion space for the bicriteria linear programming problem using variants of the standard shooting method. This method is based on the weighted-sums approach which as already been used in [1, 3, 4, 7, 6, 8, 9, 10, 12].

Received: May 5, 2012

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We consider the standard form of the bicriteria linear programming problem (see [11])

$$(P)\quad \begin{cases} \text{Maximize} & z_1(x) & = & c_1 x \\ \text{Maximize} & z_2(x) & = & c_2 x \\ & & & \\ \text{subject to} & x & \in & \mathcal{S}. \end{cases}$$

In this problem x, and the c_k^T's $(k = 1, 2)$, are column vectors in $\mathbb{R}^n$. The feasible set $\mathcal{S}$ in $\mathbb{R}^n$, that we assume bounded in this paper, is defined by

$$\mathcal{S} = \left\{ x \in \mathbb{R}^n \;\middle|\; Ax = b \text{ and } x \geq 0 \right\},$$

where A is a (m, n)-matrix, and b is a column vector in $\mathbb{R}^m$. Let us define the $(2, n)$-matrix C by

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The feasible set in the criterion space $\mathbb{R}^2$ is then

$$\mathcal{S}_c = \left\{ z \in \mathbb{R}^2 \mid z = Cx \text{ for } x \in \mathcal{S} \right\} = C\mathcal{S}.$$

It is well-known that $\mathcal{S}$ and $\mathcal{S}_c$ are polyhedral sets in $\mathbb{R}^n$ and $\mathbb{R}^2$ respectively. We assume that the cost c_1 and c_2 are linearly independant (it follows that the null vector condition does not hold, see [11]).

In Section 2 we review some results concerning the Pareto set or the efficiency set. In Section 3 we present the main results concerning the weighted-sums approach and its consequence on the geometric structure of the Pareto set in the criterion space. It is the basis of the shooting methods presented in Section 4. In Section 5 a simple example illustrates the methods.

2. Efficiency and Pareto Sets

A feasible solution $\bar{x} \in \mathcal{S}$ is efficient iff there does not exist another feasible solution $x \in \mathcal{S}$ such that $z(x) = Cx \geq C\bar{x} = z(\bar{x})$ and $z(x) = Cx \neq C\bar{x} = z(\bar{x})$, otherwise $\bar{x} \in \mathcal{S}$ is inefficient. Let $C^{\geq}$ be the semi-positive polar cone generated by the cost matrix C

$$C^{\geq} = \{ y \in \mathbb{R}^n \mid Cy \geq 0 \text{ and } Cy \neq 0 \} \cup \{ 0 \in \mathbb{R}^n \}.$$

The domination set at $\bar{x} \in \mathcal{S}$ is defined by $D(\bar{x}) = \{\bar{x}\} \oplus C^{\geq}$, which is the set addition of $\{\bar{x}\}$ and $C^{\geq}$.

Theorem 1. (see Steuer [11]) *Let $D(\bar{x})$ be the domination set at $\bar{x} \in \mathcal{S}$. Then $\bar{x}$ is efficient iff $D(\bar{x}) \cap \mathcal{S} = \{\bar{x}\}$.*

The set of all efficient feasible points of (P) is its efficiency set or its Pareto set in the decision space. It will be noted $\mathcal{E}$.

Similarly for the criterion space, let $\bar{z}$ in $\mathcal{S}_c$ and let us consider the semi-positive polar cone $C_c^{\geq} = [0, +\infty) \times [0, +\infty) = \mathbb{R}_+^2$. The domination set in the criterion space is $D_c(\bar{z}) = \{\bar{z}\} \oplus C_c^{\geq}$. The set of all efficient points of $\mathcal{S}_c$ in the criterion space is its efficiency set or its Pareto set in the criterion space. It will be noted $\mathcal{E}_c$.

Theorem 2. *Let $D(\bar{z})$ be the domination set at $\bar{z} \in \mathcal{S}_c$. Then $\bar{z}$ is efficient iff $D(\bar{z}) \cap \mathcal{S}_c = \{\bar{z}\}$.*

From these two results, the link between $\mathcal{E}$ and $\mathcal{E}_c$ is given by the next theorem.

Theorem 3. $\mathcal{E}_c = C\mathcal{E}$.

3. Weighted-Sums Approach

In the weighted-sums approach, we replace the bicriteria linear programming problem by a single criteria linear programming problem. Let us consider two weight functions $\mathrm{w}_1(\lambda)$ and $\mathrm{w}_2(\lambda)$, well-defined for $\lambda \in [\underline{\Lambda}, \overline{\Lambda}]$, and introduce the following single criteria problem (P_λ).

$$(P_\lambda) \quad \begin{cases} \text{Maximize} & z_\lambda(x) = \mathrm{w}_1(\lambda) z_1(x) + \mathrm{w}_2(\lambda) z_2(x) \\ & \qquad\;\;\, = (\mathrm{w}_1(\lambda), \mathrm{w}_2(\lambda)) C x \\ \text{subject to} & \qquad\; x \in \mathcal{S}. \end{cases}$$

Under the condition that the cone generated by the vectors $\mathrm{w}_1(\lambda) c_1$ and $\mathrm{w}_2(\lambda) c_2$, defined by

$$\mathcal{C}_{\mathrm{w}} = \left\{ \mu \left[\mathrm{w}_1(\lambda) c_1 + \mathrm{w}_2(\lambda) c_2 \right] \mid \mu > 0 \text{ and } \lambda \in (\underline{\Lambda}, \overline{\Lambda}) \right\},$$

is equal to the criterion cone

$$\mathcal{C} = \{ \mu_1 c_1 + \mu_2 c_2 \mid \mu_1 > 0 \text{ and } \mu_2 > 0 \},$$

we have the following results.

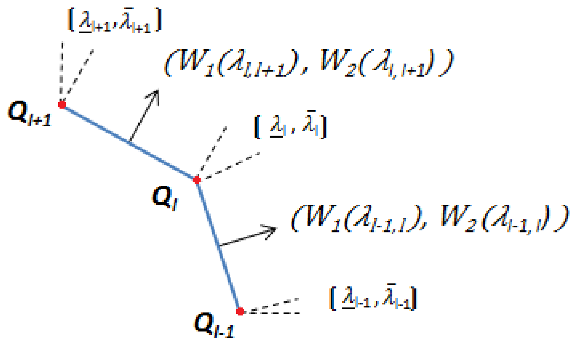

Figure 1: Polygonal line

Theorem 4. (see Steuer [11]) *If the cost vectors c_1 and c_2 are linearly independant, then*

$$\mathcal{E} = \bigcup_{\lambda \in [\underline{\Lambda}, \overline{\Lambda}]} \arg\max_{x \in \mathcal{S}} z_\lambda(x).$$

From this result and the fact the criterion space is $\mathbb{R}^2$, we obtain the next result.

Theorem 5. (see Dubeau and Kadri [5]) *$\mathcal{E}_c$ is a polygonal line.*

The polygonal line $\mathcal{E}_c$ is decomposed into L segments $[Q_{l-1}, Q_l]$ such that $\mathcal{E}_c = \bigcup_{l=1}^{L} [Q_{l-1}, Q_l]$. To each segment $[Q_{l-1}, Q_l]$ we can associate a real number $\lambda_{l-1,l}$ such that $(\mathrm{w}_1(\lambda_{l-1,l}), \mathrm{w}_2(\lambda_{l-1,l}))$ is orthogonal to the segment $[Q_{l-1}, Q_l]$. Also to each extremity Q_l we can associate an interval $[\underline{\lambda}_l, \overline{\lambda}_l]$ such that

$$\begin{cases} \underline{\lambda}_0 = 0 \text{ and } \underline{\lambda}_l = \lambda_{l-1,l} \text{ for } l = 1, ..., L \\ \overline{\lambda}_L = 1 \text{ and } \overline{\lambda}_l = \lambda_{l,l+1} \text{ for } l = 0, ..., L-1. \end{cases}$$

Moreover the $\lambda_{l-1,l}$ are ordered as follows

$$\underline{\Lambda} < \lambda_{0,1} < .... < \lambda_{l-1,l} < \lambda_{l,l+1} < .... < \lambda_{L-1,L} < \overline{\Lambda}$$

The Figure 1 illustrates the polygonal line and its decomposition.

The intervals $[\underline{\lambda}_l, \overline{\lambda}_l]$ are obtained from the optimality conditions for the problem (P_λ). Indeed, let us consider a partition of the matrix A in two submatrices B and N, $A = [B, N]$, where B is an invertible matrix, and let $x = (x_B, x_N)$, we can write $x_B = B^{-1}[b - Nx_N]$. Also let $C = [C_B, C_N]$ the corresponding partition for C. Then we can write

$$\begin{aligned} z_\lambda(x) &= (\mathrm{w}_1(\lambda), \mathrm{w}_2(\lambda))Cx \\ &= (\mathrm{w}_1(\lambda), \mathrm{w}_2(\lambda))[C_B x_B + C_N x_N] \\ &= (\mathrm{w}_1(\lambda), \mathrm{w}_2(\lambda))C_B B^{-1}b + \\ &\quad\ (\mathrm{w}_1(\lambda), \mathrm{w}_2(\lambda))[C_N - C_B B^{-1}N]x_N. \end{aligned}$$

The basic feasible solution ($x_B = B^{-1}b \geq 0$, $x_N = 0$) is optimal for (P_λ) under the condition

$$(\mathrm{w}_1(\lambda), \mathrm{w}_2(\lambda))[C_N - C_B B^{-1}N] \leq 0,$$

which is a system of $n - m$ inequalities for λ

$$\mathrm{w}_1(\lambda)\bar{c}_{1N} + \mathrm{w}_2(\lambda))\bar{c}_{2N} \leq 0$$

where

$$\begin{cases} \bar{c}_{1N} = c_{1N} - c_{1B}B^{-1}N, \\ \bar{c}_{2N} = c_{2N} - c_{2B}B^{-1}N. \end{cases}$$

Under certain conditions on the weight functions, we can solve this system of inequalities. Hence as long as $\lambda \in [\underline{\lambda}, \overline{\lambda}] \cap [\underline{\Lambda}, \overline{\Lambda}]$, the basic feasible solution stay optimal and

$$(x_B = B^{-1}b \geq 0 \ , \ x_N = 0) \in \arg\max_{x \in \mathcal{S}} z_\lambda(x).$$

4. The Shooting Methods

We propose a shooting methods to find the extremities of each segment of the polygonal line $\mathcal{E}_c$. We have seen that to each Q_l ($l = 0, ...L$) of $\mathcal{E}_c$ we can associate an interval $[\underline{\lambda}_l, \overline{\lambda}_l]$. The idea is to determine the extremities and deduce the interval $[\underline{\lambda}_l, \overline{\lambda}_l]$ for which Q_l correspond to the optimum of (P_λ) for all $\lambda \in [\underline{\lambda}_l, \overline{\lambda}_l]$. The values $\underline{\lambda}_l$ and $\overline{\lambda}_l$ are obtained from the optimality conditions using the reduced costs of the two criteria.

Shooting Method

Step 0 Initialization

- Select the weight functions for the auxiliairy problem (P_λ)
- List of intervals not already visited :
  $\mathcal{P} = \{([\underline{\Lambda}, \overline{\Lambda}], \Delta = \overline{\Lambda} - \underline{\Lambda})\}$.
- List of intervals visited : $\mathcal{Q} = \emptyset$.

Step 1 Repeat until $\mathcal{P} = \emptyset$:

- Choose the longest interval in $\mathcal{P}$, $[\underline{\lambda}^*, \overline{\lambda}^*]$.
- Set $\lambda = \frac{1}{2}[\underline{\lambda}^* + \overline{\lambda}^*]$.
- Solve (P_λ) and compute $\underline{\lambda}$ and $\overline{\lambda}$ from the optimality conditions as indicated in Steps A and B below.
- Insert in $\mathcal{P}$: $\begin{cases} ([\underline{\lambda}^*, \underline{\lambda}], \ \ \Delta = \underline{\lambda} - \underline{\lambda}^*) \ \text{if}\ \Delta \neq 0 \\ ([\overline{\lambda}, \overline{\lambda}^*], \ \ \Delta = \overline{\lambda}^* - \overline{\lambda}) \ \text{if}\ \Delta \neq 0. \end{cases}$
- Insert in $\mathcal{Q}$: $([\underline{\lambda}, \overline{\lambda}], \ \ \Delta = \overline{\lambda} - \underline{\lambda})$.

This algorithm terminates in a finite number of steps because it moves on extremities of segments of $\mathcal{E}_c$, and there are a finite number of such extremities. This algorithm has been programmed in *MATLAB*, see [2].

Example 6. The standard weighted-sums approach. The weight functions are $\mathrm{w}_1(\lambda) = 1 - \lambda$ and $\mathrm{w}_2(\lambda) = \lambda$, and $[\underline{\Lambda}, \overline{\Lambda}] = [0, 1]$. We have

$$z^s_\lambda(x) = (1 - \lambda)z_1(x) + \lambda z_2(x) = c_s(\lambda)x$$

where the cost vector for (P_λ) is

$$c_s(\lambda) = (1 - \lambda)c_1 + \lambda c_2.$$

Then $\underline{\lambda}$ and $\overline{\lambda}$ are defined as follows.

Step A For each $i \in N$

- $\bar{c}_{1N_i} \leq 0$ and $\bar{c}_{2N_i} \leq 0$, then set $\underline{\lambda}_i = \underline{\Lambda}$ and $\overline{\lambda}_i = \overline{\Lambda}$
- $\bar{c}_{1N_i} > 0$ and $\bar{c}_{2N_i} \leq 0$, then set $\underline{\lambda}_i = \frac{\bar{c}_{1N_i}}{\bar{c}_{1N_i} - \bar{c}_{2N_i}}$ and $\overline{\lambda}_i = \overline{\Lambda}$
- $\bar{c}_{1N_i} \leq 0$ and $\bar{c}_{2N_i} > 0$, then set $\underline{\lambda}_i = \underline{\Lambda}$ and $\overline{\lambda}_i = \frac{\bar{c}_{1N_i}}{\bar{c}_{1N_i} - \bar{c}_{2N_i}}$

- $\bar{c}_{1N_i} > 0$ and $\bar{c}_{2N_i} > 0$ cannot occurs for an optimal solution of (P_λ).

Step B $\underline{\lambda}$ and $\overline{\lambda}$ are defined as follows:

- $\underline{\lambda}' = \max\left\{\underline{\lambda}_i \mid i \in N\right\}$ and $\underline{\lambda} = \max\left\{\underline{\Lambda}, \underline{\lambda}'\right\}$
- $\overline{\lambda}' = \min\left\{\overline{\lambda}_i \mid i \in N\right\}$ and $\overline{\lambda} = \min\left\{\overline{\Lambda}, \overline{\lambda}'\right\}$

Example 7. The standard weighted-sums approach with normalized cost vectors. The normalized cost vectors are $\hat{c}_1 = c_1/\left\|c_1\right\|$ and $\hat{c}_2 = c_2/\left\|c_2\right\|$. The weight functions are

$$\mathrm{w}_1(\lambda) = \frac{1-\lambda}{\|c_1\|} \quad \text{and} \quad \mathrm{w}_2(\lambda) = \frac{\lambda}{\|c_2\|},$$

and $[\underline{\Lambda}, \overline{\Lambda}] = [0, 1]$. We have

$$z_\lambda^u(x) = \frac{1-\lambda}{\|c_1\|} z_1(x) + \frac{\lambda}{\|c_2\|} z_2(x) = c_u(\lambda)x$$

where the cost vector for (P_λ) is

$$c_u(\lambda) = (1-\lambda)\hat{c}_1 + \lambda\hat{c}_2 = \frac{1-\lambda}{\|c_1\|} c_1 + \frac{\lambda}{\|c_2\|} c_2.$$

We set

$$\hat{\bar{c}}_{1N_i} = \bar{c}_{1N_i}/\left\|c_1\right\| \quad \text{and} \quad \hat{\bar{c}}_{2N_i} = \bar{c}_{2N_i}/\left\|c_2\right\|$$

Then $\underline{\lambda}$ and $\overline{\lambda}$ are defined as follows.

Step A For each $i \in N$

- $\hat{\bar{c}}_{1N_i} \leq 0$ and $\hat{\bar{c}}_{2N_i} \leq 0$, then set $\underline{\lambda}_i = \underline{\Lambda}$ and $\overline{\lambda}_i = \overline{\Lambda}$
- $\hat{\bar{c}}_{1N_i} > 0$ and $\hat{\bar{c}}_{2N_i} \leq 0$, then set $\underline{\lambda}_i = \frac{\hat{\bar{c}}_{1N_i}}{\hat{\bar{c}}_{1N_i} - \hat{\bar{c}}_{2N_i}}$ and $\overline{\lambda}_i = \overline{\Lambda}$
- $\hat{\bar{c}}_{1N_i} \leq 0$ and $\hat{\bar{c}}_{2N_i} > 0$ then set $\underline{\lambda}_i = \underline{\Lambda}$ and $\overline{\lambda}_i = \frac{\hat{\bar{c}}_{1N_i}}{\hat{\bar{c}}_{1N_i} - \hat{\bar{c}}_{2N_i}}$
- $\hat{\bar{c}}_{1N_i} > 0$ and $\hat{\bar{c}}_{2N_i} > 0$ cannot occurs for an optimal solution of (P_λ).

Step B $\underline{\lambda}$ and $\overline{\lambda}$ are defined as follows:

- $\underline{\lambda}' = \max\left\{\underline{\lambda}_i \mid i \in N\right\}$ and $\underline{\lambda} = \max\left\{\underline{\Lambda}, \underline{\lambda}'\right\}$
- $\overline{\lambda}' = \min\left\{\overline{\lambda}_i \mid i \in N\right\}$ and $\overline{\lambda} = \min\left\{\overline{\Lambda}, \overline{\lambda}'\right\}$

Example 8. Rotation of a normalized weighted cost vector. We consider the two row cost vectors c_1 and c_2 and the angle θ between them given by

$$\theta = \arccos\left(\frac{c_1 c_2^T}{\|c_1\| \ \|c_2\|}\right).$$

Using the normalized cost vectors $\hat{c}_1 = c_1/\|c_1\|$ and $\hat{c}_2 = c_2/\|c_2\|$, we define the following two orthonormal vectors

$$u = \frac{\hat{c}_2 + \hat{c}_1}{\|\hat{c}_2 + \hat{c}_1\|} = \frac{\hat{c}_2 + \hat{c}_1}{2\cos(\theta/2)} \quad \text{and} \quad v = \frac{\hat{c}_2 - \hat{c}_1}{\|\hat{c}_2 - \hat{c}_1\|} = \frac{\hat{c}_2 - \hat{c}_1}{2\sin(\theta/2)}.$$

The cost vector for (P_λ) is then obtained by rotation of u by an angle λ from $\lambda = -\theta/2$, to get $\hat{c}_1$, up to $\lambda = \theta/2$, to get $\hat{c}_2$. We obtain a normalized cost vector

$$c_r(\lambda) = \frac{\sin(\frac{\theta}{2} - \lambda)}{\sin(\theta)}\hat{c}_1 + \frac{\sin(\frac{\theta}{2} + \lambda)}{\sin(\theta)}\hat{c}_2 = \frac{\sin(\frac{\theta}{2} - \lambda)}{\|c_1\|\sin(\theta)}c_1 + \frac{\sin(\frac{\theta}{2} + \lambda)}{\|c_2\|\sin(\theta)}c_2.$$

Then the weight functions are

$$\mathrm{w}_1(\lambda) = \frac{\sin(\frac{\theta}{2} - \lambda)}{\|c_1\|\sin(\theta)} \quad \text{and} \quad \mathrm{w}_2(\lambda) = \frac{\sin(\frac{\theta}{2} + \lambda)}{\|c_2\|\sin(\theta)},$$

and $[\underline{\Lambda}, \overline{\Lambda}] = [-\theta/2, \theta/2]$. We have

$$z_\lambda^r(x) = c_r(\lambda)x = \frac{\sin(\frac{\theta}{2} - \lambda)}{\|c_1\|\sin(\theta)}z_1(x) + \frac{\sin(\frac{\theta}{2} + \lambda)}{\|c_2\|\sin(\theta)}z_2(x).$$

Let us observe that

$$\frac{\mathrm{w}_1(\lambda)}{\mathrm{w}_2(\lambda)} = \frac{1 - \frac{\tan(\lambda)}{\tan(\theta/2)}}{1 + \frac{\tan(\lambda)}{\tan(\theta/2)}}.$$

Then $\underline{\lambda}$ and $\overline{\lambda}$ are defined as follows.

Step A For each $i \in N$

- $\hat{\hat{c}}_{1N_i} \leq 0$ and $\hat{\hat{c}}_{2N_i} \leq 0$, then set $\underline{\lambda}_i = \underline{\Lambda}$ and $\overline{\lambda}_i = \overline{\Lambda}$
- $\hat{\hat{c}}_{1N_i} > 0$ and $\hat{\hat{c}}_{2N_i} \leq 0$, then we consider $\frac{\mathrm{w}_1(\lambda)}{\mathrm{w}_2(\lambda)} \leq -\frac{\hat{\hat{c}}_{2N_i}}{\hat{\hat{c}}_{1N_i}} = \rho_i$ with $\rho_i \geq 0$
  and set $\underline{\lambda}_i = \arctan\left(\frac{1-\rho_i}{1+\rho_i}\tan(\theta/2)\right)$ and $\overline{\lambda}_i = \overline{\Lambda}$

Example	Cost vector	λ
Ex 6	$c_s(\lambda) = (1-\lambda)c_1 + \lambda c_2$	λ
Ex 7	$c_u(\lambda) = \frac{1-\lambda}{\|c_1\|}c_1 + \frac{\lambda}{\|c_2\|}c_2$	$\lambda^u = \frac{\lambda\|c_2\|}{(1-\lambda)\|c_1\|+\lambda\|c_2\|}$
Ex 8	$c_r(\lambda) = \frac{\sin(\frac{\theta}{2}-\lambda)}{\|c_1\|\sin(\theta)}c_1 + \frac{\sin(\frac{\theta}{2}+\lambda)}{\|c_2\|\sin(\theta)}c_2$	$\lambda^r = \arctan\left(-\frac{(1-\lambda)\|c_1\|-\lambda\|c_2\|}{(1-\lambda)\|c_1\|+\lambda\|c_2\|}\tan(\frac{\theta}{2})\right)$

Table 1: Relations between the parameters λ

- $\hat{\bar{c}}_{1N_i} \leq 0$ and $\bar{\bar{c}}_{2N_i} > 0$, then we consider $\frac{w_2(\lambda)}{w_1(\lambda)} \leq -\frac{\hat{\bar{c}}_{1N_i}}{\hat{\bar{c}}_{2N_i}} = \rho_i$ with $\rho_i \geq 0$
  and set $\underline{\lambda}_i = \underline{\Lambda}$ and $\overline{\lambda}_i = -\arctan\left(\frac{1-\rho_i}{1+\rho_i}\tan(\theta/2)\right)$
- $\hat{\bar{c}}_{1N_i} > 0$ and $\hat{\bar{c}}_{2N_i} > 0$ cannot occurs for an optimal solution of (P_λ).

Step B $\underline{\lambda}$ and $\overline{\lambda}$ are defined as follows:

- $\underline{\lambda}' = \max\left\{\underline{\lambda}_i \mid i \in N\right\}$ and $\underline{\lambda} = \max\left\{\underline{\Lambda}, \underline{\lambda}'\right\}$
- $\overline{\lambda}' = \min\left\{\overline{\lambda}_i \mid i \in N\right\}$ and $\overline{\lambda} = \min\left\{\overline{\Lambda}, \overline{\lambda}'\right\}$

Remark 9. Each method uses a weighted cost vector. The relations between the different parameters λ for cost vectors pointing in the same direction are given in Table 1. The cost vectors are sketched in Figure 2

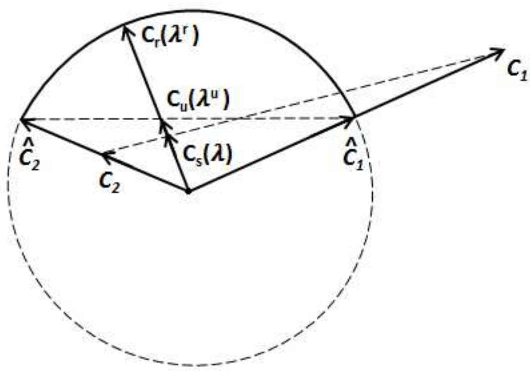

Figure 2: Weighted cost vectors

5. Numerical Examples

Let us consider the following linear bicriteria problem :

$$
(\mathrm{P}) \left\{ \begin{array}{llcl}
\text{Maximize} & z_1 & = & x_1 + 6x_2 \\
\text{Maximize} & z_2 & = & 2x_1 - 4x_2 \\
 & & & \\
\text{subject to} & x_1 & \leq & 20 \\
 & x_2 & \leq & 15 \\
 & x_1 - 2x_2 & \leq & 11 \\
 & 2x_1 - x_2 & \leq & 34 \\
 & 2x_1 + x_2 & \leq & 48 \\
 & x_1 + x_2 & \leq & 29 \\
 & x_1 + 2x_2 & \leq & 40 \\
 & x_1 + 3x_2 & \leq & 52 \\
 & x_1 + 6x_2 & \leq & 91 \\
 & x_1 \geq 0 & , & x_2 \geq 0
\end{array} \right.
$$

The Pareto sets $\mathcal{E}$ and $\mathcal{E}_c$ for this problem are given in Figure 3 and Figure 4.

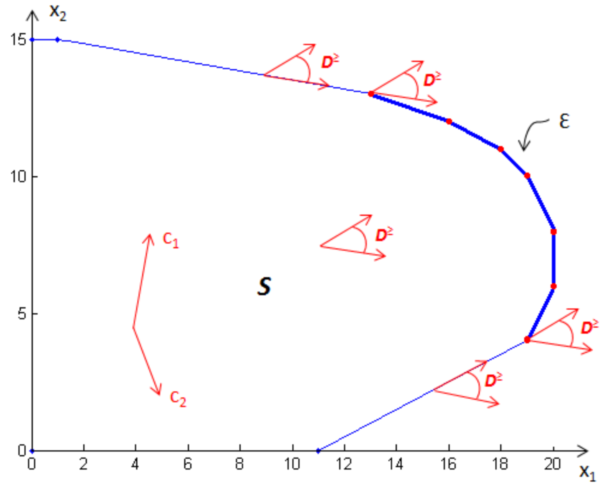

Figure 3: Feasible set $\mathcal{S}$ and Pareto set $\mathcal{E}$ in the decision space

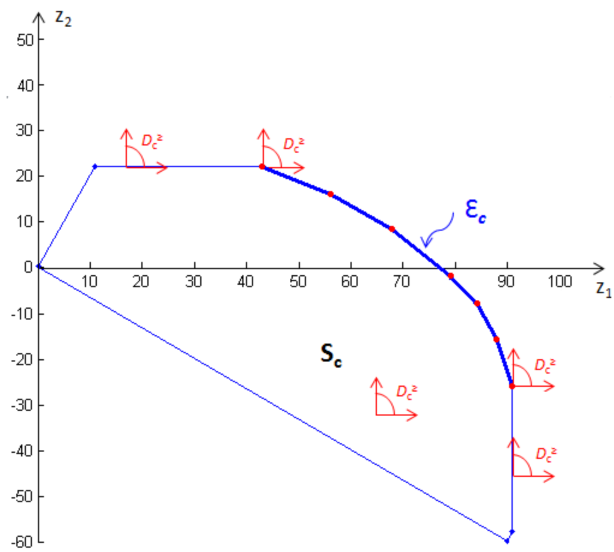

Figure 4: Feasible set $\mathcal{S}_c$ and Pareto set $\mathcal{E}_c$ in the criterion space

Using the standard shooting method of Example 6 we have

$$z_\lambda^s(x) = (1-\lambda, \lambda)\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} = (1-\lambda, \lambda)\begin{pmatrix} 1 & 6 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\underline{\lambda}$	$\overline{\lambda}$	λ	z^s_λ	z_1	z_2	x_1	x_2	Rank of detection
0	0.2308	0.2272	64.4091	91	-26	13	13	3
0.2308	0.3333	0.2820	58.6667	88	-16	16	12	6
0.3333	0.4545	0.3426	52.4755	84	-8	18	11	4
0.4545	0.5238	0.5000	38.5000	79	-2	19	10	1
0.5238	0.6000	0.5619	34.2857	68	8	20	8	7
0.6000	0.6842	0.6040	31.8396	56	16	20	6	5
0.6842	1.0000	0.7619	27.0000	43	22	19	4	2

Table 2: Results for the standard weighted-sums approach.

and we get the results of Table 2.

Using the standard shooting method with normalized cost vectors of Example 7, we have

$$\hat{c}_1 = c_1/\sqrt{37} \quad \text{and} \quad \hat{c}_2 = c_2/\sqrt{20}.$$

and

$$\begin{aligned} z^u_\lambda(x) &= \left(\frac{1-\lambda}{\sqrt{37}}, \frac{\lambda}{\sqrt{20}}\right)\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \\ &= \left(\frac{1-\lambda}{\sqrt{37}}, \frac{\lambda}{\sqrt{20}}\right)\begin{pmatrix} 1 & 6 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

Hence we get the results of Table 3.

Using the normalized weighted cost vector of Example 8, we have

$$\theta = \arccos\left(\frac{-22}{\sqrt{37}\,\sqrt{20}}\right) = 2.512796 \text{ rad},$$

hence $[\underline{\Lambda}, \overline{\Lambda}] = [-1.256398, 1.256398]$ and

$$\begin{aligned} z^r_\lambda(x) &= \left(\frac{\sin(\frac{\theta}{2}-\lambda)}{\sqrt{37}\sin(\theta)}, \frac{\sin(\frac{\theta}{2}+\lambda)}{\sqrt{20}\sin(\theta)}\right)\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \\ &= \left(\frac{\sin(\frac{\theta}{2}-\lambda)}{\sqrt{37}\sin(\theta)}, \frac{\sin(\frac{\theta}{2}+\lambda)}{\sqrt{20}\sin(\theta)}\right)\begin{pmatrix} 1 & 6 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

We obtain the results of Table 4.

$\underline{\lambda}$	$\overline{\lambda}$	λ	z_λ^u	z_1	z_2	x_1	x_2	Rank of detection
0	0.1807	0.0903	13.0833	91	-26	13	13	5
0.1807	0.2688	0.2236	10.4329	88	-16	16	12	3
0.2688	0.3798	0.3580	8.2259	84	-8	18	11	4
0.3798	0.4471	0.4136	7.4320	79	-2	19	10	7
0.4471	0.5244	0.5000	6.4840	68	8	20	8	1
0.5244	0.6143	0.5694	6.0014	56	16	20	6	6
0.6143	1.0000	0.7622	5.4305	43	22	19	4	2

Table 3: Results for the standard weighted-sums with normalized cost vectors.

$\underline{\lambda}$	$\overline{\lambda}$	λ	z_λ^r	z_1	z_2	x_1	x_2	Rank of detection
-1.2564	-1.0998	-1.1072	16.3829	91	-26	13	13	6
-1.0998	-0.9580	-1.0289	17.2095	88	-16	16	12	7
-0.9580	-0.6363	-0.7855	19.5421	84	-8	18	11	3
-0.6363	-0.3145	-0.4755	21.2607	79	-2	19	10	5
-0.3145	0.1492	0.0000	20.9672	68	8	20	8	1
0.1492	0.6128	0.3810	18.0864	56	16	20	6	4
0.6128	1.2564	0.7028	14.0594	43	22	19	4	2

Table 4: Results for the rotation of a normalized weighted cost vector.

6. Conclusion

We have pointed out how to construct the Pareto set in the criterion space using a simple method based on the weighted-sums approach. We have suggested a shooting method for the bounded case to identify the Pareto set.

The method can easily be extended to cover the unbounded case. Using the optimality condition, we could find the values of λ which leave basic the current solution when the unboundedness of the problem is detected. Extension to more than two criteria seems to be possible, but remain to be developped.

Acknowledgments

This work has been partially supported by a NSERC (Natural Sciences and Engineering Research Council of Canada) individual discovery grant for the first author.

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