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IMPLICIT ITERATION SCHEME FOR THREE HEMICONTRACTIVE MAPPINGS

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Abstract: The purpose of this paper is to characterize conditions for the convergence of the implicit iterative scheme to the common fixed point of three ϕ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space.

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1. Introduction

Let K be a nonempty subset of an arbitrary Banach space X and X^* be its dual space. The symbols $D(T)$ and $F(T)$ stand for the domain and the set of fixed points of T (for a single-valued mapping $T : X \to X$, $x \in X$ is called a *fixed point* of T iff $Tx = x$). We denote by J the *normalized duality mapping* from X to 2^{X^*} defined by

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$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad \forall x \in X.$$

Let $T : D(T) \subset X \to X$ be a mapping.

Definition 1.1. (see [1], [4], [9], [18]) (1) T is said to be *strongly pseudo-contractive* if there exists a constant $t > 1$ such that for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle \leq t^{-1}\|x-y\|^2.$$

(2) T is said to be *strictly hemicontractive* if $F(T) \neq \emptyset$ and if there exists a constant $t > 1$ such that for all $x \in D(T)$ and $q \in F(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - q, j(x-q)\rangle \leq t^{-1}\|x-q\|^2.$$

(3) T is said to be *ϕ-strongly pseudocontractive* if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$$

(4) T is said to be *ϕ-hemicontractive* if $F(T) \neq \emptyset$ and if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $x \in D(T)$ and $q \in F(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$\operatorname{Re}\langle Tx - q, j(x-q)\rangle \leq \|x-q\|^2 - \phi(\|x-q\|)\|x-q\|.$$

Clearly, each strictly hemicontractive mapping is ϕ-hemicontractive.

Chidume [1] established that the Mann iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitz strongly pseudocontractive mapping from a bounded closed convex subset of L_p (or l_p) into itself. Afterwards, several authors generalized this result of Chidume in various directions [4]-[8], [13]-[15], [17] and [18].

In 2001, Xu and Ori [19] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \ldots, N\}$), with $\{\alpha_n\}$ a real sequence in $(0,1)$, and an initial point $x_0 \in K$:

$$\begin{aligned}
x_1 &= (1-\alpha_1)x_0 + \alpha_1 T_1 x_1,\\
x_2 &= (1-\alpha_2)x_1 + \alpha_2 T_2 x_2,\\
&\vdots\\
x_N &= (1-\alpha_N)x_{N-1} + \alpha_N T_N x_N,\\
x_{N+1} &= (1-\alpha_{N+1})x_N + \alpha_{N+1} T_{N+1} x_{N+1},\\
&\vdots,
\end{aligned}$$

which can be written in the following compact form:

$$x_n = (1 - \alpha_n)x_{n-1} + \alpha_n T_n x_n \quad \forall n \geq 1, \tag{1.1}$$

where $T_n = T_{n\,(mod\,N)}$ (here the $mod\,N$ function takes values in I). Xu and Ori [19] proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [10], Oslilike proved the following result.

Theorem 1.2. *Let X be a real Banach space and K be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be N strictly pseudocontractive from K to K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the following conditions:*

(i) $0 < \alpha_n < 1$,
(ii) $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$,
(iii) $\sum_{n=1}^{\infty}(1 - \alpha_n)^2 < \infty$.

For arbitrary $x_0 \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ by the implicit iterative process (1.1). *Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if* $\lim_{n\to\infty} \inf d(x_n, F) = 0$.

Remark 1.3. One can easily see that for $\alpha_n = 1 - \frac{1}{n^{\frac{1}{2}}}$, $\sum_{n=1}^{\infty}(1-\alpha_n)^2 = \infty$. Hence the results of Osilike [10] are needed to be improve.

The purpose of this paper is to characterize conditions for the convergence of the implicit iterative scheme to the common fixed point of three ϕ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space. Our results extend and improve most results in recent literature [1]-[8], [10]-[12], [14], [15], [17] and [18].

2. Preliminaries

The following results are now well known.

Lemma 2.1. (see [16]) *For all x, $y \in X$ and $j(x + y) \in J(x + y)$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2Re\,\langle y, j(x + y)\rangle\,.$$

Lemma 2.2. *Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ be a real sequence satisfying*

$$0 \leq \lambda_n \leq 1 \quad \textit{and} \quad \sum_{n=0}^{\infty} \lambda_n = \infty.$$

Suppose that there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ *with* $\phi(0) = 0$. *If there exists a positive integer* n_0 *such that*

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n \phi(\theta_{n+1})\theta_{n+1} + \sigma_n + \omega_n$$

for all $n \ge n_0$ *with* $\sigma_n \ge 0$, $\forall n \in \mathbb{N}$, $\sigma_n = o(\lambda_n)$ *and* $\sum_{n=0}^{\infty} \omega_n < \infty$, *then* $\lim_{n\to\infty} \theta_n = 0$.

3. Main Results

Now we prove our main results.

Theorem 3.1. *Let* K *be a nonempty convex subset of an arbitrary Banach space* X *and* $T_i : K \to K$ $(i = 1, 2, 3)$ *be three uniformly continuous and* ϕ*-hemicontractive mappings. Let* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n^i\}_{n=1}^{\infty}$ $(i = 1, 2, 3)$ *be real sequences in* $[0, 1]$ *satisfying the following conditions:*

(i) $\sum_{i=1}^{3} \beta_n^i + \alpha_n = 1$,

(ii) $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$,

(iii) $\lim_{n\to\infty} (1 - \alpha_n) = 0$.

Suppose that $\{x_n\}_{n=1}^{\infty}$ *is the sequence generated from arbitrary* $x_0 \in K$ *by*

$$x_n = \alpha_n x_{n-1} + \sum_{i=1}^{3} \beta_n^i T_i x_n, \quad \forall n \ge 1. \tag{3.1}$$

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to the common fixed point* q *of* T_i $(i = 1, 2, 3)$.

(b) $\lim_{n\to\infty} T_i x_n = q$ $(i = 1, 2, 3)$.

(c) $\{T_i x_n\}_{n=1}^{\infty}$ $(i = 1, 2, 3)$ *are bounded.*

Proof. Since each T_i $(i = 1, 2, 3)$ is ϕ-hemicontractive, it follows that $\bigcap_{i=1}^{3} F(T_i)$ is a singleton. Let $\bigcap_{i=1}^{3} F(T_i) = \{q\}$ for some $q \in K$.

Suppose that $\lim_{n\to\infty} x_n = q$, then the uniform continuity of T_i $(i = 1, 2, 3)$ yields that

$$\lim_{n\to\infty} T_i x_n = q \quad (i = 1, 2, 3).$$

Therefore $\{T_i x_n\}_{n=1}^{\infty}$ $(i = 1, 2, 3)$ are bounded.

Put

$$M_1 = \|x_0 - q\| + \sum_{i=1}^{3} \sup_{n\ge1} \|T_i x_n - q\| . \tag{3.2}$$

It is clear that $\|x_0 - q\| \le M_1$. Let $\|x_{n-1} - q\| \le M_1$. Next we will prove that $\|x_n - q\| \le M_1$.

Consider

$$\begin{aligned}
&\|x_n - q\| \\
&= \left\| \alpha_n x_{n-1} + \sum_{i=1}^{3} \beta_n^i T_i x_n - q \right\| \\
&= \left\| \alpha_n (x_{n-1} - q) + \sum_{i=1}^{3} \beta_n^i (T_i x_n - q) \right\| \\
&\le \alpha_n \|x_{n-1} - q\| + \sum_{i=1}^{3} \beta_n^i \|T_i x_n - q\| \\
&\le \alpha_n M_1 + \sum_{i=1}^{3} \beta_n^i \|T_i x_n - q\| \\
&= \alpha_n \left(\|x_0 - q\| + \sum_{i=1}^{3} \sup_{n\ge1} \|T_i x_n - q\| \right) + \sum_{i=1}^{3} \beta_n^i \|T_i x_n - q\| \\
&\le \|x_0 - q\| + \left(\alpha_n \sum_{i=1}^{3} \sup_{n\ge1} \|T_i x_n - q\| + \sum_{i=1}^{3} \beta_n^i \|T_i x_n - q\| \right) \\
&\le \|x_0 - q\| \\
&\quad + \left(\left(1 - \sum_{i=1}^{3} \beta_n^i \right) \sum_{i=1}^{3} \sup_{n\ge1} \|T_i x_n - q\| + \sum_{i=1}^{3} \beta_n^i \sup_{n\ge1} \|T_i x_n - q\| \right) \\
&= \|x_0 - q\| + \sum_{i=1}^{3} \sup_{n\ge1} \|T_i x_n - q\| \\
&= M_1.
\end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - q\}_{n=1}^{\infty}$ is bounded. Thus there is a constant $M > 0$ satisfying

$$M = \max \left\{ \sup_{n\ge1} \{\|T_i x_n - q\|^2\}_{i=1}^{3} \right\} + \sup_{n\ge1} \|x_n - q\| < \infty. \tag{3.3}$$

By virtue of Lemma 2.1 and (3.1), we infer that

$$\begin{aligned}
\|x_n - q\|^2 &= \left\|\alpha_n x_{n-1} + \sum_{i=1}^{3} \beta_n^i T_i x_n - q\right\|^2 \\
&= \left\|\alpha_n (x_{n-1} - q) + \sum_{i=1}^{3} \beta_n^i (T_i x_n - q)\right\|^2 \\
&\leq \alpha_n^2 \|x_{n-1} - q\|^2 + 2\sum_{i=1}^{3} \beta_n^i \mathrm{Re}\, \langle T_i x_n - q, j(x_n - q)\rangle \\
&\leq \alpha_n^2 \|x_{n-1} - q\|^2 + 2\sum_{i=1}^{3} \beta_n^i \|x_n - q\|^2 \\
&\quad - 2\sum_{i=1}^{3} \beta_n^i \phi(\|x_n - q\|) \|x_n - q\| \\
&= \alpha_n^2 \|x_{n-1} - q\|^2 + 2(1 - \alpha_n) \|x_n - q\|^2 \\
&\quad - 2(1 - \alpha_n)\phi(\|x_n - q\|) \|x_n - q\| .
\end{aligned} \tag{3.4}$$

Consider

$$\begin{aligned}
\|x_n - q\|^2 &= \left\|\alpha_n x_{n-1} + \sum_{i=1}^{3} \beta_n^i T_i x_n - q\right\|^2 \\
&= \left\|\alpha_n (x_{n-1} - q) + \sum_{i=1}^{3} \beta_n^i (T_i x_n - q)\right\|^2 \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + \sum_{i=1}^{3} \beta_n^i \|T_i x_n - q\|^2 \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + \sum_{i=1}^{3} \beta_n^i \sup_{n\geq 1} \|T_i x_n - q\|^2 \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + \sum_{i=1}^{3} \beta_n^i \max\left\{\sup_{n\geq 1}\{\|T_i x_n - q\|^2\}_{i=1}^{3}\right\} \\
&\leq \|x_{n-1} - q\|^2 + M(1 - \alpha_n),
\end{aligned} \tag{3.5}$$

where the first inequality holds by the convexity of $\|\cdot\|^2$.

Substituting (3.5) in (3.4), we get

$$\begin{aligned}\|x_n - q\|^2 &\le (\alpha_n^2 + 2(1-\alpha_n))\,\|x_{n-1} - q\|^2 + 2M(1-\alpha_n)^2 \\ &\quad - 2(1-\alpha_n)\phi(\|x_n - q\|)\,\|x_n - q\| \\ &= (1 + (1-\alpha_n)^2)\,\|x_{n-1} - q\|^2 + 2M(1-\alpha_n)^2 \\ &\quad - 2(1-\alpha_n)\phi(\|x_n - q\|)\,\|x_n - q\| \\ &\le \|x_{n-1} - q\|^2 + M(M+2)(1-\alpha_n)^2 \\ &\quad - 2(1-\alpha_n)\phi(\|x_n - q\|)\,\|x_n - q\| \\ &= \|x_{n-1} - q\|^2 + (1-\alpha_n)l_n \\ &\quad - 2(1-\alpha_n)\phi(\|x_n - q\|)\,\|x_n - q\|\,,\end{aligned} \tag{3.6}$$

where

$$l_n = M(M+2)(1-\alpha_n) \to 0 \tag{3.7}$$

as $n \to \infty$.

Denote

$$\theta_n = \|x_{n-1} - p\|, \quad \lambda_n = 2(1-\alpha_n) \quad \text{and} \quad \sigma_n = (1-\alpha_n)l_n.$$

Condition (ii) assures the existence of a rank $n_0 \in \mathbb{N}$ such that $\lambda_n = 2(1-\alpha_n) \le 1$ for all $n \ge n_0$. Now with the help of (iii), (3.7) and Lemma 2.2, we obtain from (3.6) that

$$\lim_{n\to\infty} \|x_n - q\| = 0.$$

This completes the proof. □

Corollary 3.2. Let K be a nonempty convex subset of an arbitrary Banach space X and $T_i : K \to K$ $(i = 1, 2, 3)$ be three Lipschitz and ϕ-hemicontractive mappings. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n^i\}_{n=1}^{\infty}$ $(i = 1, 2, 3)$ be real sequences in $[0, 1]$ satisfying the conditions (i)-(iii). Suppose that $\{x_n\}_{n=1}^{\infty}$ is the sequence generated by (3.1).

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the common fixed point q of T_i $(i = 1, 2, 3)$.

(b) $\lim_{n\to\infty} T_i x_n = q$ $(i = 1, 2, 3)$.

(c) $\{T_i x_n\}_{n=1}^{\infty}$ $(i = 1, 2, 3)$ are bounded.

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