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NEW CONSTRUCTIONS FOR SPECIAL MAGIC SQUARES

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Abstract: Systematic methods are presented for constructing special magic squares of arbitrarily large size, including regular (or associative), pandiagonal, and most-perfect magic squares as well as ultra-magic squares (regular and pandiagonal). The methods of matrix analysis are employed to define these special magic squares and develop novel methods of constructing them.

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1. Introduction

Magic squares have a continuing appeal to both professional and recreational mathematicians. They have found application in physics, computer science, image processing, and cryptography. Numerous methods of constructing magic squares have been developed over the past several hundred years. A review and historical account of this subject can be found in books by Ollerenshaw and Brée [7], Pasles [9], and Pickover [10].

In the present paper we develop novel methods for systematic construction of special magic squares of arbitrarily large size, including regular (or associative), pandiagonal (Nasik, perfect, or diabolic), and most-perfect magic squares as well as ultra-magic squares (regular and pandiagonal). These special magic squares are defined by matrix equations in the next section of the paper.

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Most of our construction methods involve the representation of a magic square as the linear combination of two orthogonal auxiliary squares. This method originates with Euler [2] and Benjamin Franklin (see Pasles [9]). The auxiliary squares are generated by analogy with knight's moves on a chess board using specific arrangements of their integer elements. Another construction method involves reflections of certain rows and columns of a serial square.

After defining the various special magic squares, the next two sections of the paper develop construction methods for odd-order and doubly-even order special magic squares. A few numerical examples are presented and the paper concludes with a tabular summary of our construction methods.

2. Definitions

We begin by defining a number of special matrices and several classes of special magic squares that will be constructed in what follows. All matrices considered here have real integer elements (for simplicity) and are square unless otherwise indicated.

Let u be the $1 \times n$ *unity column vector* with all elements 1, U - the $n \times n$ (*order* n) *unity matrix* with all elements 1, I - the *identity matrix*, and R - the *reflection matrix* with 1's on the cross (right-to-left) diagonal and all other elements 0. It should be noted that other authors use various other symbols for our u, U, and R. In matrix notation R and U satisfy the following identities:

$$R^{-1} = R^T = R, \quad \mathrm{tr}\,[U] = n, \quad U^i = n^{i-1}U, \; (i = 1, 2, \ldots)\,, \tag{1}$$

where R^T denotes the transpose of R and $\mathrm{tr}\,[U]$ denotes the trace of U. The matrix product RA reflects the elements of a matrix A about its horizontal centerline, AR reflects the elements of A about its vertical centerline, A^TR rotates the elements of A a quarter turn clockwise about its center, RA^T rotates the elements of A a quarter turn counter-clockwise, and RAR rotates the elements of A a half turn. These matrices together with A^T, RA^TR, and A itself constitute the eight *phases* (variants) of A as discussed by Loly, et.al. [4].

A *permutation matrix* P has a single 1 in all rows and columns and 0 for all other elements. It has the following properties:

$$UP = PU = U, \quad P^{-1} = P^T, \quad \det P = \pm 1\,. \tag{2}$$

The matrix operation PA interchanges rows of A and AP interchanges columns of A. Note that I and R are permutation matrices.

The matrix M is a *magic square* if the sum of the elements in each of its rows, each of its columns, the main diagonal, and the cross diagonal all equal the *index* m, i.e., if

$$Mu = \left(u^T M\right)^T = mu \quad \text{or} \quad MU = UM = mU, \tag{3}$$

$$\text{tr}\,[M] = \text{tr}\,[RM] = m. \tag{4}$$

The order-n matrix A is *natural* (or classical) if its elements are integers in the numerical sequence $0, 1, \ldots, n^2 - 1$. In what follows, M denotes a natural magic square matrix unless otherwise indicated. The index for a natural magic square is

$$m = n(n^2 - 1)/2\,. \tag{5}$$

The eight phases of M also are magic and natural as is easily verified. Subscripts are used to denote special classes and the order of M.

A magic square M_R is *regular* (or associative) if pairs of its elements that are symmetrically positioned with respect to its center add to the same *regularity index* r, i.e., if

$$M_R + RM_RR = rU. \tag{6}$$

This requires M_R of odd order to have $r/2$ as its center element. On taking the trace of (6) and noting (4) and (5), we have

$$r = 2mn^{-1} = n^2 - 1\,. \tag{7}$$

It is known that all order-3 magic squares are regular [4]. It follows from (6) and (7) that M_R satisfies the diagonal sum condition (4).

A magic square M_P is *pandiagonal* (Nasik, perfect, or diabolic) if all its broken diagonals (of n elements) in both directions sum to the magic index m. These conditions, including (4), can be expressed as

$$\text{tr}\left[K^i M_P\right] = \text{tr}\left[K^i R M_P\right] = m, \quad (i = 1, 2, \ldots, n)\,, \tag{8}$$

where K is the order-n permutation matrix that has all elements 0 except $K_{1n} = 1$ (upper right corner) and $K_{i,i-1} = 1\,,\ i = 2, 3, \cdots, n$ (diagonal below the main diagonal), e.g., for $n = 4$

$$K = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{9}$$

The operation KM shifts rows of M down one (and bottom row to top) while MK shifts columns of M one to the left (and first column to last). Power operations K^iM and MK^i give rise to repeated shifts. The following identities can be easily verified:

$$K^n = I, \quad K^{-i} = K^{n-i} = RK^iR, \ (i = 1, 2, \ldots, n),$$
$$\sum_{i=1}^{n} K^i = \sum_{i=1}^{n} K^{-i} = U. \tag{10}$$

It follows from the shifting properties of K that M_P satisfies

$$\sum_{i=1}^{n} K^i M_P K^i = mU \quad \text{and} \quad \sum_{i=1}^{n} K^{-i} M_P K^i = mU \tag{11}$$

which may be taken as an alternate condition for pandiagonality.

The term *panmagic* is applied to a pandiagonal magic square. An *ultramagic* square M_U is both panmagic and regular. It is known that there are no order-2 magic squares, no order-3 panmagic squares, and no order-4 ultramagic squares [10]. Furthermore, Rosser and Walker [12] proved that there are no natural, panmagic squares of singly-even order $(n = 6, 10, \ldots)$. An alternate derivation of the identity used in their proof is given by Nordgren [5] where analysis of spectra and matrix powers of special magic square matrices is presented. Also, it follows from a known transformation (to be discussed below) that there are no regular, natural magic squares of singly-even order either.

A *most-perfect* (*MP*) magic square M_M is of doubly-even order $(n = 4, 8, \ldots)$ and has the following additional properties:

1) two elements that are $n/2$ elements apart along all diagonals (including broken ones in both directions) sum to r, i.e.,

$$M_M + K^{\frac{n}{2}} M_M K^{\frac{n}{2}} = rU, \tag{12}$$

2) the elements of all 2 by 2 subsquares (including broken top-bottom, broken left-right, and the four corners) sum to the constant $2r$, i.e.,

$$(I + K) M_M (I + K) = 2rU. \tag{13}$$

As noted by Ollerenshaw and Brée [7] and verified in [5], it follows from (12) that M_M is pandiagonal and (13) ensures that M_M is magic. For an order-4 panmagic square, the pandiagonality conditions (11) with (10) lead to the known fact [10] that all such squares are MP. A method of constructing and counting the number of MP magic squares is given in [7].

The matrix definitions of regular, pandiagonal, and most-perfect squares, namely (6), (11), (12), and (13), provide convenient means of verifying the special properties of the magic squares to be constructed in what follows. Furthermore, from these equations it can be shown that the phases of a matrix retain each of its special properties. This is an important fact for our construction methods.

Next, we present the matrix form of a known transformation. A regular magic square M_R of even order n can be transformed to a panmagic square M_P by the transformation

$$M_P = W M_R W, \tag{14}$$

where the permutation matrix W is expressed in terms of order-$n/2$ submatrices $\hat{R}$, $\hat{I}$, and $\hat{O}$ as

$$W = \begin{bmatrix} \hat{R} & \hat{O} \\ \hat{O} & \hat{I} \end{bmatrix} \tag{15}$$

in which $\hat{O}$ is the matrix with all elements zero. It can be verified that W satisfies the identities

$$W R W = K^{\frac{n}{2}} \quad \text{and} \quad W^{-1} = W. \tag{16}$$

Using the regularity condition (6) and (16), it follows that M_P from (14) satisfies the first most-perfect condition (12) and hence it is panmagic. Also, if a panmagic square satisfies (12), then it can be transformed to a regular magic square by the transformation

$$M_R = W M_P W, \tag{17}$$

as can be shown using (16). Thus, any order-4 panmagic squares can be transformed to a regular magic square by (17). For example, for $n = 4$, (14) gives

$$W M_R W = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 13 & 11 & 6 \\ 7 & 10 & 12 & 1 \\ 14 & 3 & 5 & 8 \\ 9 & 4 & 2 & 15 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 7 & 12 & 1 \\ 13 & 0 & 11 & 6 \\ 3 & 14 & 5 & 8 \\ 4 & 9 & 2 & 15 \end{bmatrix} = M_P. \tag{18}$$

The transformation represented by (14) was known to Planck [11] who called it the "A-D method." Also, for order-8 magic squares, this transformation was used by Setsuda as posted on the Suzuki website [13]. However, the convenient matrix formulation (14) and the first most-perfect condition (12) on M_P are believed to be new.

Any matrix M can be written in auxiliary form as

$$M = nM_B + M_A, \tag{19}$$

where

$$M_A = M\,(\mathrm{mod} n) \quad \text{and} \quad M_B = (M - M_A)\,/n. \tag{20}$$

are *auxiliary* squares. This form of M originates with Euler [2] and Benjamin Franklin (see Pasles [9]). It is useful for constructing magic squares of various types as done by Euler [2], Franklin, Frost [3], Ollerenshaw [8], and others. We use it extensively in our construction methods.

Two matrices are said to be *orthogonal* if all ordered pairs of numbers in the same position in the two matrices are different. The auxiliary form (19) can be used to construct magic squares when M_A and M_B are orthogonal unnatural magic squares with elements $0, 1, \ldots, n-1$, each repeated n times. A *Latin square* has elements $0, 1, \ldots, n-1$ in each row and each column. If M_A and M_B are orthogonal Latin squares that satisfy the diagonal sum condition (4) with $m = n(n-1)/2$, then M from (19) is a magic square. However, M may be a magic square even when M_A and M_B are not Latin squares; for examples see [2], [8], and (33) below. Also, (19) is the basis of the well-known composite method of constructing magic squares as discussed by Pickover [10] and Ollerenshaw [8]. It is not difficult to show that if M_A and M_B are regular, pandiagonal, or most-perfect, then so is M from (19).

3. Odd-Order Squares

Here we develop systematic methods of constructing regular, pandiagonal, and ultra-magic squares of any odd order $n \geq 5$.

First, we recall that a regular magic square of any odd order can be constructed by de la Loubère's method [10] as follows: Starting with 0 in the center cell of the top row, place successive integers in cells that are diagonally up and to the right (with wraparound); if this cell is occupied put the integer in the cell below instead. For $n = 5$ this construction results in

$$M_{R5} = \begin{bmatrix} 16 & 23 & 0 & 7 & 14 \\ 22 & 4 & 6 & 13 & 15 \\ 3 & 5 & 12 & 19 & 21 \\ 9 & 11 & 18 & 20 & 2 \\ 10 & 17 & 24 & 1 & 8 \end{bmatrix} = 5M_{B5} + M_{A5}$$

$$= 5 \begin{bmatrix} 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 & 2 \\ 0 & 2 & 4 & 1 & 3 \end{bmatrix}. \tag{21}$$

where the orthogonal, regular, Latin, auxiliary squares M_{A5} and M_{B5} are obtained from (19). The form of these auxiliary squares can be applied to the construction of higher odd-order regular magic squares. One starts with the integers $0, 1, \ldots, n-1$ in the center column and replicates them using the moves of a knight or bishop on an n by n chess board - down two cells and left one cell for M_{An} and down one cell and left one cell for M_{Bn}. The auxiliary form (21) shows why de la Loubère's method produces a regular magic square.

The chess moves used in the construction of auxiliary Latin squares such as M_{A5} and M_{B5} in (21) can be characterized by the number of rows r that the "super knight" moves down and the number of columns c that it moves right, the combined move being written as $[r, c]$. The knight returns to its starting cell after n moves and the n cells visited by the knight is called a *path* and characterized by $[r, c]$, e.g., the knight's path for M_{A5} of (21) is $[2, -1]$.

A criterion attributed to Brée by Ollerenshaw [8] states that two paths $[r, c]$ and $[r', c']$ starting at the same elemental cell of M do not intersect again if

$$b = \left| rc' - r'c \right| \tag{22}$$

is not a factor (other than 1) of the order n. As shown in [8] and Appendix A, it follows that if auxiliary Latin squares M_A and M_B are constructed from paths that satisfy (22), then M_A and M_B are orthogonal. For example, in de la Loubère's construction, the paths for M_{An} and M_{Bn} are $[2, -1]$ and $[1, -1]$, respectively, and (22) gives $b = 1$ which makes them orthogonal. Brée's criterion is proved and discussed further in Appendix A. In particular, we show that if M is formed by a knight's path $[\pm 2, \pm 1]$ or $[\pm 1, \pm 2]$, then the phases of M correspond to other knight's paths which give $b = 3, 4$, or 5 from (22). Thus, appropriate phases can be selected for orthogonal M_A and M_B formed by knight's paths unless n is a multiple of $60 = 3 \times 4 \times 5$. This case is discussed further in Section 4 and Appendix A. Since the phases of the auxiliary matrix M_A retain its special properties, they can be used for M_B in (19) provided that M_A and M_B are orthogonal.

Panmagic squares of odd order have been constructed from auxiliary squares by Ollerenshaw [8]. When $n \geq 5$ is not a multiple of 3 her approach produces an ultra-magic square upon starting M_{An} with $0, 1, \ldots n-1$ in the center column (different from [8]) and replicating via knight's paths of $[2, -1]$ for M_{An} and $[2, 1]$ for $M_{Bn} = M_{An}R$. Since (22) gives $b = 4$ for these paths and n is odd, M_{An} and M_{Bn} are orthogonal by Brée's criterion. Furthermore, it is not difficult to show that M_{An} and M_{Bn} are regular and pandiagonal Latin squares. Thus M_n from (19) is an ultra-magic square. For example, an ultra-magic square of order-5 constructed by this method is

$$\begin{aligned} M_{U5} = 5 &\begin{bmatrix} 4 & 2 & 0 & 3 & 1 \\ 0 & 3 & 1 & 4 & 2 \\ 1 & 4 & 2 & 0 & 3 \\ 2 & 0 & 3 & 1 & 4 \\ 3 & 1 & 4 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 & 2 \\ 0 & 2 & 4 & 1 & 3 \end{bmatrix} \\ = 5M_{B5} + M_{A5} = &\begin{bmatrix} 21 & 13 & 0 & 17 & 9 \\ 2 & 19 & 6 & 23 & 10 \\ 8 & 20 & 12 & 4 & 16 \\ 14 & 1 & 18 & 5 & 22 \\ 15 & 7 & 24 & 11 & 3 \end{bmatrix}. \end{aligned} \tag{23}$$

In this construction, the left-sloping diagonal elements of M_{An} (with wrap around) always are in numerical sequence. The right-sloping diagonal elements advance by 3 (mod$n$) each row down , e.g., $1, 4, 2, 0, 3$ on the main diagonal of M_{A5}. Thus, when n is not a multiple of 3, the right-sloping diagonals of M_{An} consist of the integers $0, 1, \ldots, n-1$, whence M_{An} is pandiagonal and so is $M_{Bn} = M_{An}R$. Also, M_{An} is regular due to the knight's path $[2, -1]$ with $0, 1, \ldots n-1$ in the center column. Since $M_{Bn} = M_{An}R$ is a phase of M_{An}, it is also pandiagonal and regular. As noted above, M_{An} and M_{Bn} are orthogonal. Therefore, M_{Un} from (19) is an ultra-magic square. Also, M_{Un} can be constructed directly by starting with 0 as the center element of its top row and sequencing the integer elements in a knight's move of up two, left one, and down one if blocked. This method was used by Andrews [1] to construct an ultra-magic square similar to M_{U5} in (23) but he did not generalize this method to higher odd-order ultra-magic squares.

When n is a multiple of 3, the right-sloping diagonals of M_A repeat in sets of

3. For example, a general order-9 auxiliary square constructed by the foregoing method with the knight's path $[2,-1]$ can be written in symbolic form as

$$M_{A9} = \begin{bmatrix} b & d & f & h & a & c & e & g & i \\ c & e & g & i & b & d & f & h & a \\ d & f & h & a & c & e & g & i & b \\ e & g & i & b & d & f & h & a & c \\ f & h & a & c & e & g & i & b & d \\ g & i & b & d & f & h & a & c & e \\ h & a & c & e & g & i & b & d & f \\ i & b & d & f & h & a & c & e & g \\ a & c & e & g & i & b & d & f & h \end{bmatrix}. \tag{24}$$

The rows, columns, and left-sloping (broken) diagonals again contain all the elements $a, b, \ldots, i$, but the right-sloping ones do not. In order to make them add to the index $m = 36$, we require that

$$b + e + h = a + d + g = c + f + i = 12\,. \tag{25}$$

These equations together with the regularity conditions are satisfied by taking

$$M_{C9} = \begin{bmatrix} f & h & a \\ c & e & g \\ i & b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 4 & 3 \\ 6 & 8 & 7 \end{bmatrix} \tag{26}$$

which can be constructed by interchanging elements in the first and last rows of the *serpentine serial matrix*

$$M_{S9} = \begin{bmatrix} 0 & 1 & 2 \\ 5 & 4 & 3 \\ 6 & 7 & 8 \end{bmatrix}. \tag{27}$$

Then, (24) with (26) results in the regular and pandiagonal Latin square

$$M_{A9} = \begin{bmatrix} 8 & 7 & 1 & 0 & 2 & 5 & 4 & 3 & 6 \\ 5 & 4 & 3 & 6 & 8 & 7 & 1 & 0 & 2 \\ 7 & 1 & 0 & 2 & 5 & 4 & 3 & 6 & 8 \\ 4 & 3 & 6 & 8 & 7 & 1 & 0 & 2 & 5 \\ 1 & 0 & 2 & 5 & 4 & 3 & 6 & 8 & 7 \\ 3 & 6 & 8 & 7 & 1 & 0 & 2 & 5 & 4 \\ 0 & 2 & 5 & 4 & 3 & 6 & 8 & 7 & 1 \\ 6 & 8 & 7 & 1 & 0 & 2 & 5 & 4 & 3 \\ 2 & 5 & 4 & 3 & 6 & 8 & 7 & 1 & 0 \end{bmatrix}, \tag{28}$$

where the rows of M_{C9} form the center row of M_{A9}. As before, an ultra-magic square follows from (19) with $M_{B9} = M_{A9}R$ again being orthogonal to M_{A9}. This method can be generalized to any odd order n that is a multiple of 3. For example, for $n = 15$ one starts with a serpentine serial matrix modified in the first and last rows as before, namely

$$M_{C15} = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 4 & 3 \\ 6 & 7 & 8 \\ 11 & 10 & 9 \\ 12 & 14 & 13 \end{bmatrix} \qquad (29)$$

which is regular ($r = 14$) and all three columns add to $m/3 = 35$ as required for pandiagonality. Next, the rows of M_{C15} are used to form the center row of M_{A15} which is replicated in a knight's path $[2, -1]$ as before giving the regular and pandiagonal Latin square M_{A15} as follows:

$$\begin{bmatrix} 4 & 3 & 6 & 7 & 8 & 11 & 10 & 9 & 12 & 14 & 13 & 1 & 0 & 2 & 5 \\ 12 & 14 & 13 & 1 & 0 & 2 & 5 & 4 & 3 & 6 & 7 & 8 & 11 & 10 & 9 \\ 3 & 6 & 7 & 8 & 11 & 10 & 9 & 12 & 14 & 13 & 1 & 0 & 2 & 5 & 4 \\ 14 & 13 & 1 & 0 & 2 & 5 & 4 & 3 & 6 & 7 & 8 & 11 & 10 & 9 & 12 \\ 6 & 7 & 8 & 11 & 10 & 9 & 12 & 14 & 13 & 1 & 0 & 2 & 5 & 4 & 3 \\ 13 & 1 & 0 & 2 & 5 & 4 & 3 & 6 & 7 & 8 & 11 & 10 & 9 & 12 & 14 \\ 7 & 8 & 11 & 10 & 9 & 12 & 14 & 13 & 1 & 0 & 2 & 5 & 4 & 3 & 6 \\ 1 & 0 & 2 & 5 & 4 & 3 & 6 & 7 & 8 & 11 & 10 & 9 & 12 & 14 & 13 \\ 8 & 11 & 10 & 9 & 12 & 14 & 13 & 1 & 0 & 2 & 5 & 4 & 3 & 6 & 7 \\ 0 & 2 & 5 & 4 & 3 & 6 & 7 & 8 & 11 & 10 & 9 & 12 & 14 & 13 & 1 \\ 11 & 10 & 9 & 12 & 14 & 13 & 1 & 0 & 2 & 5 & 4 & 3 & 6 & 7 & 8 \\ 2 & 5 & 4 & 3 & 6 & 7 & 8 & 11 & 10 & 9 & 12 & 14 & 13 & 1 & 0 \\ 10 & 9 & 12 & 14 & 13 & 1 & 0 & 2 & 5 & 4 & 3 & 6 & 7 & 8 & 11 \\ 5 & 4 & 3 & 6 & 7 & 8 & 11 & 10 & 9 & 12 & 14 & 13 & 1 & 0 & 2 \\ 9 & 12 & 14 & 13 & 1 & 0 & 2 & 5 & 4 & 3 & 6 & 7 & 8 & 11 & 10 \end{bmatrix}. \qquad (30)$$

An ultra-magic square M_{U15} again is generated by (19) with $M_{B15} = M_{A15}R$. The foregoing method is a systematic alternative to Ollerenshaw's [8] element-exchange approach which gave a (nonregular) panmagic square for n an odd multiple of 3. Thus, an ultra-magic square of any odd order $n \geq 5$ can be constructed systematically by the methods presented in this section.

4. Doubly-Even-Order Squares

Here we develop systematic methods of constructing regular, pandiagonal, and most-perfect magic squares of doubly-even order ($n = 4, 8, \ldots$) and ultra-magic squares of doubly-even order $n \geq 8$.

Regular magic squares of doubly even-order can be constructed by a relatively simple method. To illustrate this method, we start with the standard order-8 *serial matrix*

$$M_{S8} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 \\ 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 \\ 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 \\ 48 & 49 & 50 & 51 & 52 & 53 & 54 & 55 \\ 56 & 57 & 58 & 59 & 60 & 61 & 62 & 63 \end{bmatrix}. \tag{31}$$

On reflecting the 1^{st}, 3^{rd}, 6^{th}, and 8^{th} rows of M_{S8} about its vertical centerline and then reflecting the 1^{st}, 3^{rd}, 6^{th}, and 8^{th} columns about the horizontal centerline we obtain the regular magic square

$$M_{R8} = \begin{bmatrix} 63 & 6 & 61 & 4 & 3 & 58 & 1 & 56 \\ 48 & 9 & 50 & 11 & 12 & 53 & 14 & 55 \\ 47 & 22 & 45 & 20 & 19 & 42 & 17 & 40 \\ 32 & 25 & 34 & 27 & 28 & 37 & 30 & 39 \\ 24 & 33 & 26 & 35 & 36 & 29 & 38 & 31 \\ 23 & 46 & 21 & 44 & 43 & 18 & 41 & 16 \\ 8 & 49 & 10 & 51 & 52 & 13 & 54 & 15 \\ 7 & 62 & 5 & 60 & 59 & 2 & 57 & 0 \end{bmatrix}. \tag{32}$$

By (19) and (20), the auxiliary form of this matrix is

$$M_{R8} = 8 \begin{bmatrix} 7 & 0 & 7 & 0 & 0 & 7 & 0 & 7 \\ 6 & 1 & 6 & 1 & 1 & 6 & 1 & 6 \\ 5 & 2 & 5 & 2 & 2 & 5 & 2 & 5 \\ 4 & 3 & 4 & 3 & 3 & 4 & 3 & 4 \\ 3 & 4 & 3 & 4 & 4 & 3 & 4 & 3 \\ 2 & 5 & 2 & 5 & 5 & 2 & 5 & 2 \\ 1 & 6 & 1 & 6 & 6 & 1 & 6 & 1 \\ 0 & 7 & 0 & 7 & 7 & 0 & 7 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}, \tag{33}$$

where these two auxiliary squares are orthogonal, transposed, regular, unnatural magic squares; but they are not Latin squares, nor are their elements in a knight's path. The auxiliary form of M_{R8} provides a simple method of constructing regular magic squares of any doubly-even order and shows why they are regular magic squares. Other regular magic squares of doubly-even order-n can be constructed from a serial square by reflecting any $n/2$ symmetrically-placed rows and columns about their respective centerlines. A similar method of constructing magic squares of doubly-even order is used in MATLAB [6].

The magic square M_{R8} of (32) can be transformed to a panmagic square by (14), giving

$$M_{P8} = WM_{R8}W = \begin{bmatrix} 27 & 34 & 25 & 32 & 28 & 37 & 30 & 39 \\ 20 & 45 & 22 & 47 & 19 & 42 & 17 & 40 \\ 11 & 50 & 9 & 48 & 12 & 53 & 14 & 55 \\ 4 & 61 & 6 & 63 & 3 & 58 & 1 & 56 \\ 35 & 26 & 33 & 24 & 36 & 29 & 38 & 31 \\ 44 & 21 & 46 & 23 & 43 & 18 & 41 & 16 \\ 51 & 10 & 49 & 8 & 52 & 13 & 54 & 15 \\ 60 & 5 & 62 & 7 & 59 & 2 & 57 & 0 \end{bmatrix}. \tag{34}$$

As already noted in connection with (14), M_{P8} satisfies the first most-perfect condition (12). In addition (and amazingly!), M_{P8} also satisfies the second most-perfect condition (13). By (19) and (20), the auxiliary form of $M_{P8} \equiv M_{M8}$ is

$$M_{M8} = 8\begin{bmatrix} 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 \\ 2 & 5 & 2 & 5 & 2 & 5 & 2 & 5 \\ 1 & 6 & 1 & 6 & 1 & 6 & 1 & 6 \\ 0 & 7 & 0 & 7 & 0 & 7 & 0 & 7 \\ 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 \\ 5 & 2 & 5 & 2 & 5 & 2 & 5 & 2 \\ 6 & 1 & 6 & 1 & 6 & 1 & 6 & 1 \\ 7 & 0 & 7 & 0 & 7 & 0 & 7 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 & 0 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 3 & 2 & 1 & 0 \end{bmatrix}, \tag{35}$$

where the two auxiliary squares are transposed, MP, unnatural magic squares but they are not Latin squares. The elements of M_{A8} and $M_{B8} = M_{A8}^T$ in (35) are formed by paths $[1, 4]$ and $[4, 1]$, respectively, giving $b = 15$ from (22) which makes them orthogonal under Brée's criterion. The auxiliary form of M_{M8} provides a simple method of constructing MP magic squares M_{Mn} of higher doubly-even order and shows why they are MP magic squares. The paths for the auxiliary squares of M_{Mn} are $[1, n/2]$ and $[n/2, 1]$ which again establishes orthogonality from (22) and Brée's criterion. Other panmagic squares follow from applying the transformation (14) to regular magic square formed from (31) by other reflection schemes but the resulting squares are not MP.

Ultra-magic squares of doubly-even order are more difficult to construct. Only two order-8 ultra-magic squares are posted on the Suzuki website [13] and I have not seen any of higher doubly-even order. A new method of constructing them is presented here, first for an order-8 square formed by the knight's path $[1, -2]$ with a special ordering of the elements, namely

$$M_{K8} = \begin{bmatrix} 0 & 1 & 2 & 5 & 6 & 7 & 4 & 3 \\ 2 & 5 & 6 & 7 & 4 & 3 & 0 & 1 \\ 6 & 7 & 4 & 3 & 0 & 1 & 2 & 5 \\ 4 & 3 & 0 & 1 & 2 & 5 & 6 & 7 \\ 0 & 1 & 2 & 5 & 6 & 7 & 4 & 3 \\ 2 & 5 & 6 & 7 & 4 & 3 & 0 & 1 \\ 6 & 7 & 4 & 3 & 0 & 1 & 2 & 5 \\ 4 & 3 & 0 & 1 & 2 & 5 & 6 & 7 \end{bmatrix} \qquad (36)$$

which is regular and pandiagonal and the rows add to $m = 28$. However, the odd and even numbered columns contain even and odd integers that add to 24 and 32, respectively, whence M_{K8} does not satisfy the column sum condition of (3). This column sum imbalance can be corrected by exchanging $2 \leftrightarrow 5$ and $4 \leftrightarrow 3$ throughout M_{K8} which gives the following regular, pandiagonal, unnatural magic square:

$$M_{A8} = \begin{bmatrix} 0 & 1 & 5 & 2 & 6 & 7 & 3 & 4 \\ 5 & 2 & 6 & 7 & 3 & 4 & 0 & 1 \\ 6 & 7 & 3 & 4 & 0 & 1 & 5 & 2 \\ 3 & 4 & 0 & 1 & 5 & 2 & 6 & 7 \\ 0 & 1 & 5 & 2 & 6 & 7 & 3 & 4 \\ 5 & 2 & 6 & 7 & 3 & 4 & 0 & 1 \\ 6 & 7 & 3 & 4 & 0 & 1 & 5 & 2 \\ 3 & 4 & 0 & 1 & 5 & 2 & 6 & 7 \end{bmatrix}. \qquad (37)$$

On taking $M_{B8} = M_{A8}^T$ the knight's paths for M_{A8} and M_{B8} are $[1, -2]$ and $[2, -1]$, respectively and (22) gives $b = 3$, making them orthogonal under Brée's criterion. Then, (19) gives the ultra-magic square

$$M_{U8} = \begin{bmatrix} 0 & 41 & 53 & 26 & 6 & 47 & 51 & 28 \\ 13 & 18 & 62 & 39 & 11 & 20 & 56 & 33 \\ 46 & 55 & 27 & 4 & 40 & 49 & 29 & 2 \\ 19 & 60 & 32 & 9 & 21 & 58 & 38 & 15 \\ 48 & 25 & 5 & 42 & 54 & 31 & 3 & 44 \\ 61 & 34 & 14 & 23 & 59 & 36 & 8 & 17 \\ 30 & 7 & 43 & 52 & 24 & 1 & 45 & 50 \\ 35 & 12 & 16 & 57 & 37 & 10 & 22 & 63 \end{bmatrix}. \quad (38)$$

This method can be applied to higher doubly-even order squares when n is not a multiple of $12 = 4 \times 3$ by following the same numbering scheme as in (36) for the first row, namely

$$0, 1, \ldots, n/2-2, n/2+1, n/2+2, \ldots, n-1, n/2, n/2-1 \quad (39)$$

which when replicated by the knight's path $[1, -2]$ gives a regular and pandiagonal square M_{Kn} since the last row is

$$n/2, n/2-1, 0, 1, \ldots, n/2-2, n/2+1, n/2+2, \ldots, n-1, \quad (40)$$

and thereby all other rows comply with the regularity condition (6). Again, the rows satisfy the magic sum condition (3) but the columns do not since the odd numbered columns contain the even integers $0, 2, \ldots n-2$ twice and the even numbered columns contain the odd integers $1, 3, \ldots n-1$ twice. The imbalance in column sums again can be corrected by interchanging elements, where the elements to be exchanged can be determined as follows:

Let k denote the number of exchanges. Let x and y denote the sum of the integers to be exchanged in the odd and even columns, respectively. Any two integers to be exchanged must add to $n-1$ in order to preserve regularity, thus

$$x + y = k\,(n-1)\,. \quad (41)$$

It is not difficult to determine that the imbalance for the odd/even columns is

$$y - x = n/4\,. \quad (42)$$

From the last two equations we have

$$x = \frac{k}{2}\,(n-1) - \frac{n}{8}, \quad y = \frac{k}{2}\,(n-1) + \frac{n}{8}, \quad (43)$$

where x must be an even integer since it is the sum of even integers. For $n = 8$ and $k = 2$, (43) gives $x = 6$ and $y = 8$ which includes the previous exchanges $2 \leftrightarrow 5$, $4 \leftrightarrow 3$ as well as $0 \leftrightarrow 7$, $6 \leftrightarrow 1$. For $n = 16$, the lowest suitable k is 4 for which $x = 28$ and $y = 32$ and one set of possible exchanges is $2 \leftrightarrow 13$, $4 \leftrightarrow 11$, $10 \leftrightarrow 5$, $12 \leftrightarrow 3$. The following table gives some possible exchanges for higher doubly-even orders that are not multiples of 12:

Order	k	x	Possible Exchanges
8	2	6	$2 \leftrightarrow 5,\ 4 \leftrightarrow 3$ or $0 \leftrightarrow 7,\ 6 \leftrightarrow 1$
16	4	28	$2 \leftrightarrow 13,\ 4 \leftrightarrow 11,\ 10 \leftrightarrow 5,\ 12 \leftrightarrow 3$
20	3	26	$4 \leftrightarrow 15,\ 10 \leftrightarrow 9,\ 12 \leftrightarrow 7$
28	1	10	$10 \leftrightarrow 17$

Table 1: Possible exchanges for constructing ultra-magic squares

Other exchanges that satisfy (43) also are possible. We find that k is given by

$$k = 4 - \frac{n}{4} \leq 4 \ (\text{mod}4) \tag{44}$$

and exchanges for higher orders can easily be calculated. After the exchanges we have an ultra-magic auxiliary square M_{An}, where again $M_{Bn} = M_{An}^T$ is orthogonal to M_{An} and an ultra-magic square follows from (19). For example, the following order-16 ultra-magic square was constructed by this method:

$$\begin{bmatrix}
0 & 209 & 189 & 108 & 91 & 58 & 230 & 137 & 5 & 212 & 179 & 98 & 94 & 63 & 232 & 135 \\
29 & 204 & 171 & 154 & 70 & 41 & 245 & 116 & 19 & 194 & 174 & 159 & 72 & 39 & 240 & 113 \\
219 & 186 & 102 & 89 & 53 & 228 & 131 & 2 & 222 & 191 & 104 & 87 & 48 & 225 & 141 & 12 \\
198 & 169 & 149 & 68 & 35 & 242 & 126 & 31 & 200 & 167 & 144 & 65 & 45 & 252 & 123 & 26 \\
181 & 100 & 83 & 50 & 238 & 143 & 8 & 215 & 176 & 97 & 93 & 60 & 235 & 138 & 6 & 217 \\
163 & 146 & 78 & 47 & 248 & 119 & 16 & 193 & 173 & 156 & 75 & 42 & 246 & 121 & 21 & 196 \\
110 & 95 & 56 & 231 & 128 & 1 & 221 & 188 & 107 & 90 & 54 & 233 & 133 & 4 & 211 & 178 \\
152 & 71 & 32 & 241 & 125 & 28 & 203 & 170 & 150 & 73 & 37 & 244 & 115 & 18 & 206 & 175 \\
80 & 49 & 237 & 140 & 11 & 218 & 182 & 105 & 85 & 52 & 227 & 130 & 14 & 223 & 184 & 103 \\
77 & 44 & 251 & 122 & 22 & 201 & 165 & 148 & 67 & 34 & 254 & 127 & 24 & 199 & 160 & 145 \\
59 & 234 & 134 & 9 & 213 & 180 & 99 & 82 & 62 & 239 & 136 & 7 & 208 & 177 & 109 & 92 \\
38 & 249 & 117 & 20 & 195 & 162 & 158 & 79 & 40 & 247 & 112 & 17 & 205 & 172 & 155 & 74 \\
229 & 132 & 3 & 210 & 190 & 111 & 88 & 55 & 224 & 129 & 13 & 220 & 187 & 106 & 86 & 57 \\
243 & 114 & 30 & 207 & 168 & 151 & 64 & 33 & 253 & 124 & 27 & 202 & 166 & 153 & 69 & 36 \\
142 & 15 & 216 & 183 & 96 & 81 & 61 & 236 & 139 & 10 & 214 & 185 & 101 & 84 & 51 & 226 \\
120 & 23 & 192 & 161 & 157 & 76 & 43 & 250 & 118 & 25 & 197 & 164 & 147 & 66 & 46 & 255
\end{bmatrix} . \tag{45}$$

However, if n is a multiple of 12, then the foregoing knight's path $[1,-2]$ again gives rise to repeating sets of elements on the right-sloping diagonals of M_{An}, e.g., the symbolic form of M_{A12} reads

$$M_{A12} = \begin{bmatrix} a & b & c & d & e & f & g & h & i & j & k & l \\ c & d & e & f & g & h & i & j & k & l & a & b \\ e & f & g & h & i & j & k & l & a & b & c & d \\ g & h & i & j & k & l & a & b & c & d & e & f \\ i & j & k & l & a & b & c & d & e & f & g & h \\ k & l & a & b & c & d & e & f & g & h & i & j \\ a & b & c & d & e & f & g & h & i & j & k & l \\ c & d & e & f & g & h & i & j & k & l & a & b \\ e & f & g & h & i & j & k & l & a & b & c & d \\ g & h & i & j & k & l & a & b & c & d & e & f \\ i & j & k & l & a & b & c & d & e & f & g & h \\ k & l & a & b & c & d & e & f & g & h & i & j \end{bmatrix}. \tag{46}$$

In order to make M_{A12} satisfy the row/column sum conditions (3) and the pandiagonal conditions (8) with $m = 66$, the elements of M_{A12} must satisfy

$$\begin{gathered} a+c+e+g+i+k=33, \quad b+d+f+h+j+l=33, \\ a+d+g+j=22, \quad b+e+h+k=22, \quad c+f+i+l=22, \end{gathered} \tag{47}$$

and to meet the regularity condition (6) with $r = 11$, they must satisfy

$$\begin{gathered} a+j=11, \quad b+i=11, \quad c+h=11, \\ d+g=11, \quad e+f=11, \quad k+l=11\,. \end{gathered} \tag{48}$$

The solution of (47) and (48) allows four of the elements $a, b, \ldots, l$ to be freely assigned. One solution with distinct elements results in the unnatural, ultramagic, auxiliary square

$$M_{A12} = \begin{bmatrix} 5 & 8 & 9 & 7 & 11 & 0 & 4 & 2 & 3 & 6 & 1 & 10 \\ 9 & 7 & 11 & 0 & 4 & 2 & 3 & 6 & 1 & 10 & 5 & 8 \\ 11 & 0 & 4 & 2 & 3 & 6 & 1 & 10 & 5 & 8 & 9 & 7 \\ 4 & 2 & 3 & 6 & 1 & 10 & 5 & 8 & 9 & 7 & 11 & 0 \\ 3 & 6 & 1 & 10 & 5 & 8 & 9 & 7 & 11 & 0 & 4 & 2 \\ 1 & 10 & 5 & 8 & 9 & 7 & 11 & 0 & 4 & 2 & 3 & 6 \\ 5 & 8 & 9 & 7 & 11 & 0 & 4 & 2 & 3 & 6 & 1 & 10 \\ 9 & 7 & 11 & 0 & 4 & 2 & 3 & 6 & 1 & 10 & 5 & 8 \\ 11 & 0 & 4 & 2 & 3 & 6 & 1 & 10 & 5 & 8 & 9 & 7 \\ 4 & 2 & 3 & 6 & 1 & 10 & 5 & 8 & 9 & 7 & 11 & 0 \\ 3 & 6 & 1 & 10 & 5 & 8 & 9 & 7 & 11 & 0 & 4 & 2 \\ 1 & 10 & 5 & 8 & 9 & 7 & 11 & 0 & 4 & 2 & 3 & 6 \end{bmatrix}. \tag{49}$$

On taking $M_{B12} = RM_{A12}^T$ the knight's paths are $[1, -2]$ and $[2, 1]$ for M_{A12} and M_{B12}, respectively, giving $b = 5$ from (22), whence they are orthogonal by Brée's criterion. Then, the following order-12, ultra-magic square is obtained from (19):

$$\begin{bmatrix} 125 & 104 & 93 & 7 & 35 & 72 & 124 & 98 & 87 & 6 & 25 & 82 \\ 21 & 67 & 119 & 132 & 52 & 38 & 15 & 66 & 109 & 142 & 53 & 44 \\ 83 & 120 & 100 & 86 & 3 & 30 & 73 & 130 & 101 & 92 & 9 & 31 \\ 40 & 14 & 63 & 114 & 133 & 58 & 41 & 20 & 69 & 115 & 143 & 48 \\ 27 & 78 & 121 & 106 & 89 & 8 & 33 & 79 & 131 & 96 & 88 & 2 \\ 49 & 46 & 17 & 68 & 117 & 139 & 59 & 36 & 16 & 62 & 111 & 138 \\ 5 & 32 & 81 & 127 & 107 & 84 & 4 & 26 & 75 & 126 & 97 & 94 \\ 141 & 55 & 47 & 12 & 64 & 110 & 135 & 54 & 37 & 22 & 65 & 116 \\ 95 & 0 & 28 & 74 & 123 & 102 & 85 & 10 & 29 & 80 & 129 & 103 \\ 112 & 134 & 51 & 42 & 13 & 70 & 113 & 140 & 57 & 43 & 23 & 60 \\ 99 & 90 & 1 & 34 & 77 & 128 & 105 & 91 & 11 & 24 & 76 & 122 \\ 61 & 118 & 137 & 56 & 45 & 19 & 71 & 108 & 136 & 50 & 39 & 18 \end{bmatrix}. \tag{50}$$

This approach can be extended to higher doubly-even orders that are a multiple of 12 . For $n = 24$ the top row of M_{A24} reads

$$5, 8, 9, 7, 11, 0, 4, 2, 3, 6, 13, 10, 17, 20, 21, 19, 23, 12, 16, 14, 15, 18, 1, 22 \tag{51}$$

which is obtained by taking the first 12 elements the same as the top row of M_{A12}, adding 12 to these elements to obtain the last 12 elements, and then interchanging 1 and 13 . On replicating this top row in the knight's path $[1, -2]$, we form an unnatural, ultra-magic, auxiliary square M_{A24}. A detailed analysis shows that this construction method is valid for any solution of (47) and (48).

On taking $M_{B24} = RM_{A24}^T$, (19) gives an order-24 ultra-magic square. By this same scheme, the top row of M_{A36} reads

$$5, 8, 9, 7, 11, 0, 4, 2, 3, 6, 13, 10, 17, 20, 21, 19, 23, 12, 16, 14, 15, 18, 25, 22, \\ 29, 32, 33, 31, 35, 24, 28, 26, 27, 30, 1, 34 \quad (52)$$

and that of M_{A48} reads

$$5, 8, 9, 7, 11, 0, 4, 2, 3, 6, 13, 10, 17, 20, 21, 19, 23, 12, 16, 14, 15, 18, 25, 22, \\ 29, 32, 33, 31, 35, 24, 28, 26, 27, 30, 37, 34, \quad (53) \\ 41, 44, 45, 43, 47, 36, 40, 38, 39, 42, 1, 46\,.$$

On replicating these top rows in the path $[1, -2]$ and taking $M_{B36} = RM_{A36}^T$ and $M_{B48} = RM_{A48}^T$ we obtain orthogonal auxiliary squares which again lead to ultra-magic squares from (19) as can be verified numerically. As shown in Appendix A, this method does not work for a square whose order is a multiple of $60 = 3 \times 4 \times 5$ since M_{Bn} taken as any phase of M_{An} is not orthogonal to M_{An} in this case. However, higher-order ultra-magic squares that are multiples of 12 $(n = 60, 72, \ldots)$ can be constructed by the composite method to be discussed next.

If $n = n_1 \times n_2$, where ultra-magic squares of orders n_1 and n_2 are known, then the well-known composite method based on (19) gives an ultra-magic square of order n as discussed by Pickover [10] and Ollerenshaw [8]. For example, the composite method applies to $n = 60 = 5 \times 12$, $n = 72 = 8 \times 9$, and $n = 168 = 7 \times 24 = 8 \times 21$. Thus, regular, pandiagonal, most-perfect, and ultra-magic squares of any doubly-even order $n \geq 8$ can be constructed systematically by the methods of this section. Also, the composite method can be used to construct regular and pandiagonal magic squares, but not most-perfect ones. The composite method applies for $n \geq 9$ unless n is singly-even $(10, 14, \ldots)$ or a prime.

5. Summary of Constructions

The following table summarizes the construction methods presented here for regular (Reg), pandiagonal (Pan), ultra- (Ult), and most-perfect (MP) natural magic squares:

Order n	3 factor	Method	Reg	Pan	Ult	MP
Odd	OK	de la Loubère	Y	N	N	N/A
Odd, $n \geq 5$	N	Auxiliary*	Y	Y	Y	N/A
Odd, $n \geq 9$	Y	Auxiliary*	Y	Y	Y	N/A
Doubly-Even, $n \geq 4$	OK	Serial/Reflect or Auxiliary*	Y	Y	N	Y
Doubly-Even, $n \geq 8$	N	Auxiliary*	Y	Y	Y	N
12, 24, 36, 48	Y	Auxiliary*	Y	Y	Y	N
$n \geq 9$, Restricted†	OK	Composite	P‡	P‡	P‡	N

*uses a specific form of auxiliary squares
†not singly-even or prime
‡preserves properties of auxiliary squares

Table 2: Summary of construction methods

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Appendix A: Brée's Orthogonality Criterion

Here we derive Brée's criterion (as stated by Ollerenshaw [8]) for orthogonality of two auxiliary squares that are based on knight's paths.

For a square matrix A of order n, let A_{xy} $(x, y = 1, 2, \ldots, n)$ denote the element of A at position (x, y) thought of as a cell on a $n \times n$ chess board with A_{11} in the upper left corner. A white knight starts at (x_0, y_0) and follows the path $[r, c]$, i.e., r cells in the y direction (down) and c cells in the x direction (right). A black knight starts at (x_0, y_0) and follows the path $[r', c']$. After k and l moves (with wraparound), the cells occupied by the white and black knight are

$$\begin{aligned} x_k &= (x_0 + kc)\,(\mathrm{mod} n)\,, \quad y_k = (y_0 + kr)\,(\mathrm{mod} n)\,, \quad (k = 0, 1, \ldots, n)\,, \\ x_\ell &= \left(x_0 + lc'\right)(\mathrm{mod} n)\,, \quad y_\ell = \left(y_0 + lr'\right)(\mathrm{mod} n)\,, \quad (l = 0, 1, \ldots, n)\,, \end{aligned} \tag{54}$$

respectively. Since $x_n = x_0$, $y_n = y_0$, the knights return to their starting square after n moves. The two knights occupy the same square for $k = l = 0$ and again if

$$\begin{aligned} x_k - x_\ell &= \left(kc - lc'\right)(\mathrm{mod} n) = 0, \\ y_k - y_\ell &= \left(kr - lr'\right)(\mathrm{mod} n) = 0 \end{aligned} \tag{55}$$

for some k and l, whence

$$kc - lc' = in, \quad kr - lr' = jn \tag{56}$$

for some integers i and j. On solving (56) for k and l, we find that

$$k = nb^{-1}\left(jc' - ir'\right), \quad l = nb^{-1}\left(jc - ir\right), \tag{57}$$

where

$$b = rc' - r'c. \tag{58}$$

Since $(jc' - ir')$ and $(jc - ir)$ are integers, it follows from (57) that if b does not have a common factor with n, then the integers k and l are multiples of n and therefore the knights' paths do not reintersect. Also, if $b = 1$, then the integers k and l again are multiples of n. However, if b and n have a common factor other than 1, then a second intersection occurs. For example, consider the case

$$n = 15, \quad [r, c] = [2, 1], \quad [r', c'] = [1, 2], \quad b = 3,$$

where n and b have the common factor 3 and (57) gives

$$\begin{aligned} k &= 10, \quad l = 5, \quad \text{for} \quad i = 1,\ j = 0, \\ k &= 5, \quad l = 10, \quad \text{for} \quad i = 0,\ j = -1. \end{aligned} \tag{59}$$

Thus, the knights intersect after 10 and 5 moves each and after 5 and 10 moves each, respectively, as can be verified numerically. Note that we can always make $b > 0$ by interchanging $[r, c] \leftrightarrow [r', c']$ and thus we may take

$$b = \left|rc' - r'c\right|. \tag{60}$$

The above is Brée's test for intersecting paths. To apply this test to a criterion for orthogonality of two auxiliary squares M_A and M_B whose elements $0, 1, \ldots, n-1$ are arranged in paths $[r, c]$ and $[r', c']$, respectively, let (x_0, y_0) be the element cell containing a specific ordered pair of numbers, one from each auxiliary square. If the two paths starting from this square do not reintersect, i.e., if b and n have no common factors other than 1, then there is only one instance of this ordered number pair. Since this same argument can be repeated for all n^2 cells, each ordered pair of numbers can appear only once. Thus, M_A and M_B are orthogonal and if they are unnatural magic squares, then their combination in (19) produces a natural magic square. On the other hand, if the two paths reintersect, then there is more than one instance of the ordered number pair and the two auxiliary squares are not orthogonal.

The following table gives b from (60) for various combinations of knight's paths:

Path	$[2,1]$	$[2,-1]$	$[1,2]$	$[1,-2]$
$[2,1]$	–	4	3	5
$[2,-1]$	4	–	5	3
$[1,2]$	3	5	–	4
$[1,-2]$	5	3	4	–

Table 3: b for combinations of knight's paths

If a matrix M is formed by the knight's path $[x,y]$ then the 8 phases of M are knight's paths that are related operationally by

$$\begin{aligned} RM \text{ and } MR : [x,y] \to [x,-y]\,, \quad M^T : [x,y] \to [y,x]\,, \\ RMR : [x,y] \to [x,y]\,, \quad [-x,-y] \equiv [y,x] \end{aligned} \tag{61}$$

which may be combined sequentially.

In order to check the orthogonality of two knight's-path squares that are phases of each other, the knight's paths for the phase relation is determined from (61) and b is read from Table 3. For example, suppose that M_A is formed by the path $[2,-1]$ and $M_B = M_A^T R$. Then M_B is formed by the path $[1,2]$ and $b = 5$ which can be used in Brée's orthogonality criterion.

Also, from Table 3 we see that for squares of order $60 = 3 \times 4 \times 5$ there are no pairs of knight's paths that give orthogonal auxiliary squares. Thus, the method of using a pair of phases for knight's-path auxiliary squares does not produce a magic square of order 60 or a multiple of 60. However, the composite method does apply to this case as discussed in Section 4.